Discrete-time Adaptive Mixing Control with Stability-preserving Interpolation: the Output Regulation Problem

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Abstract: The main result of this paper is the extension of the Adaptive Mixing Control (AMC) approach, to the discrete-time setting. The stability and robustness properties of the adaptive mixing control scheme are analyzed. It is shown that in the ideal case, when no disturbances or unmodelled dynamics are present, the regulation error converges to zero; otherwise the mean-square regulation error is of the order of the modeling error provided the unmodelled dynamics satisfy a norm-bound condition. A stability preserving method for the synthesis of the multicontroller is also proposed, by interpolating a family of precomputed candidate controllers. A two carts system is used to show, via simulations, the effectiveness of the method.

Keywords: Robust adaptive control; Multiple model adaptive control.

1. INTRODUCTION

A practical control design must be able to maintain performance and stability robustly in the presence of uncertainties present in all real systems, due to unmodelled dynamics, unknown system parameters and disturbances. When model uncertainties are sufficiently small, robust linear time invariant (LTI) control theories, e.g., $H_\infty$ and $\mu$-synthesis, can ensure satisfactory closed-loop objectives. However, failure or degradation of components or unexpected changes typically lead to a large parametric uncertainty, with the result that a single fixed LTI controller may no longer achieve satisfactory closed-loop behavior. Adaptive control was motivated to meet the challenge of handling parametric uncertainties much larger than those that robust control can handle. Supervisory based adaptive control architecture provides an attractive framework for combining tools from adaptive control and robust non-adaptive control. In fact, the complicated relationship between plant parameters and $H_\infty$ and $\mu$-synthesis controller gains makes very challenging the use of conventional adaptive versions of these robust compensators. The supervisory architecture comprises a multicontroller consisting of a family of precomputed candidate controllers and a supervisor that influences the control by selecting the candidate controllers based on processed plant input/output data [Morse, 1995]. Another important characteristics is that, utilizing precomputed controllers, supervisory based adaptive control avoids computational and existance issues of calculating controller gains when stabilizable of the estimated plant is lost. This problem is known in literature as singularity or stabilizability problem [Goodwin and Sin, 1984, Bay and Sastry, 1987, Giri et al., 1993]. The approach taken in this work is a novel adaptive control approach called adaptive mixing control (AMC), shown in Fig. 1. Each of the $N$ candidate controllers $C_1,\ldots,C_N$ is tuned to a small subset of the parameter uncertainty. The set of candidate controllers is sufficiently rich such that for every admissible plant there exists at least one controller that achieves the performance objective. By monitoring the plant’s input/output data, the supervisor ‘mixes’ the candidate controllers. This mechanism thus achieves a smooth transition from one controller to another instead of the popular switching using hysteresis approach [Hespanha et al., 2003, Stefanovic and Safonov, 2008, Baldi et al., 2010]. The supervisor comprises two subsystems: the online parameter estimator and the mixer. The online parameter estimator generates real-time estimates $\theta_p(t)$ of the unknown parameter vector $\theta^*$, and the mixer de-

![AMC Architecture](image-url)
terminates the participation level each candidate controller based on \( \theta_p(t) \).

The AMC approach, developed in a continuous-time setting in [Kuipers and Ioannou, 2010], is here analyzed in discrete-time for the output regulation problem. We establish that in the ideal case, when no disturbances or unmodelled dynamics are present, the regulation error converges to zero; otherwise when the scheme is applied to the plants with multiplicative uncertainty and bounded disturbance, the closed-loop states remain bounded and the mean-square regulation error is of the order of the modeling error.

The paper is organized as follows: preliminary definitions are exposed in Section 2, Section 3 deals with the problem formulation. In Section 4 a stability preserving interpolation method to construct the multicontroller is exposed, while the main theorem dealing with stability and robustness of the adaptive mixing control scheme is presented in Section 5 (the proof is stated in the appendix). The key results used in the stability analysis of the adaptive scheme are omitted for lack of space, but can be found in [Desoer and Vidyasagar, 1975, Ioannou and Sun, 1996]. In Section 6 a two cars system is used to show the effectiveness of the method. The method is also compared, via simulations, with a conventional adaptive pole-placement controller (APPC).

2. NOTATION AND PRELIMINARIES

Given the vector-valued time sequence \( v \in \mathbb{R}^n \), \( \Delta v(k) := v(k+1) - v(k) \) denotes its increment, and \( v_k \) the time truncation of the sequence \( v \) up to time \( k \), i.e., \( v_k = \{v(0), \ldots, v(k)\} \); then the \( L_2 \) norm of \( v_k \) is:

\[
\|v_k\|_{L_2} := \left( \sum_{i=0}^{k} \delta^{k-i} |v(i)|^2 \right)^{1/2}
\]

where \( 0 < \delta < 1 \) is a constant, and \( |v| \) the Euclidean norm.

By \( \|v\|_{L_2} \) we mean \( \|v\|_{L_2}^1 \) with \( \delta = 1 \), and we say that \( v \in L_2^1 \) if \( \|v\|_{L_2} \) exists for all \( k \in \mathbb{Z}^+ \). Let \( v \in L_2^1 \), and consider the set

\[
\mathcal{S}(\mu) = \left\{ v : \sum_{i=k}^{k+q-1} |v(i)| \leq c_0 q + c_1, \forall k, q \geq 0 \right\}
\]

for a given constant \( \mu \), where \( c_0, c_1 \geq 0 \) are some finite constants independent of \( \mu \). We say that \( v \) is \( \mu \)-small in the mean square sense (m.s.s.) if \( v \in \mathcal{S}(\mu) \). Let \( H(z) \) be the transfer function of some LTI system. If \( H(z) \) is a proper transfer function and analytic in \( |z| \geq \sqrt{\delta} \) for some \( 0 \leq \delta < 1 \) then the \( \|H\|_{L_2} \) system norm of \( H(z) \) is defined as

\[
\|H\|_{L_2} := \left( \frac{2\pi}{\sqrt{\delta}} \right)^{1/2} \int_0^{2\pi} |H(\sqrt{\delta} e^{i\omega})|^2 \, d\omega
\]

3. PROBLEM FORMULATION

The objective is to design a controller for the uncertain plant

\[
y_p = G(z, \theta^*)(u_p + d)
\]

\[
G(z, \theta^*) = G_0(z, \theta^*)(1 + M_\Theta(z))
\]

\[
G_0(z, \theta^*) = N_0(z, \theta^*) = \frac{\theta^T \alpha_m(z)}{D_0(z, \theta^*)} = \frac{\theta^T \alpha_m(z)}{z^n + \theta^T \alpha_{n-1}(z)}
\]

where \( G_0(z, \theta^*) \) represents the nominal plant; the vector \( \theta^* := [\theta^T, \theta^T, \ldots, \theta^T]^T \in \Omega \subseteq \mathbb{R}^{n+m+1} \) contains the unknown parameters of \( G_0(z, \theta^*) \); \( \alpha_{n-1}(z) := \{z^{n-1} \ldots 1\} \); \( M_\Theta(z) \) is an unknown multiplicative perturbation; and \( d \) is a bounded disturbance, i.e., \( |d(k)| \leq d_0, \forall k \in \mathbb{Z}^+ \). We make the following plant assumptions:

P1. The degree \( n \) of \( D_0(z, \theta^*) \) is known.

P2. \( m \leq n - 1 \).

P3. \( M_\Theta(z) \) is proper, rational, and analytic in \( |z| \geq \sqrt{\delta_0} \) for some known \( 0 < \delta_0 < 1 \).

P4. \( \theta^* \in \Omega \) for some known compact convex set \( \Omega \subseteq \mathbb{R}^{n+m+1} \).

Requirements P1-P4 are the same usually required in adaptive pole-placement control and include both unstable and nonminimum phase plants.

The adaptive mixing law approach replaces \( \theta^* \) with its estimate \( \theta_p \). Because of the presence of multiplicative uncertainty \( M_\Theta(z) \) and disturbance \( d \), we use a robust online parameter estimator. A gradient law based on integral cost with dynamic normalization signal and parameter projection [Ioannou and Sun, 1996] is used:

\[
\theta_p(k) = \text{Pr}(\theta_p(k-1) - \Gamma \{R(k)\theta_p(k-1) + Q(k)\})
\]

\[
R(k+1) = \lambda R(k) + \frac{\phi(k)^T \phi(k)}{m^2(u)}
\]

\[
Q(k+1) = \lambda Q(k) - \frac{\zeta(k)^T \zeta(k)}{m^2(u)}
\]

\[
\epsilon(k) = \frac{\zeta(k) - \theta^T \phi(k)}{m^2(u)}
\]

\[
\sum_{k=0}^{k_0} \epsilon^2(k) = 1 + \phi^T(k)\phi(k) + n_d(k)
\]

\[
n_d(k+1) = \delta n_d(k) + u^2(k) + y^2(k), \quad n_d(0) = 0
\]

where \( \theta_p(0) \in \Omega \), \( \text{Pr} \) stands for the projection operator that forces the estimated parameters to stay within the specified convex set, \( 0 < \lambda < 1 \), \( 0 < \Gamma < 2 \), \( \zeta(k) = \frac{z^n \phi^2(k) y_p(k)}{\lambda \theta_p(k) \theta^T_p(k)} \), \( \phi(k) = \left[ \delta^2 \frac{\alpha_m(z)}{\alpha_{n-1}(z)} y_p(k) - \delta^2 \frac{\alpha_{n-1}(z)}{\alpha_{n-1}(z)} y_p(k) \right] \), and \( \lambda_p \) is a Hurwitz polynomial of degree \( n \). The adaptive law (6)-(11) guarantees:

E1. \( \epsilon(k), (k)\phi_m(k), \delta \phi_p(k) \in \mathcal{S} \left( \frac{\alpha}{\sqrt{\delta}} \right) \) if \( M_\Theta, d \neq 0 \).

E2. \( \epsilon(k), (k)\phi_m(k), \delta \phi_p(k) \in L_2 \) if \( M_\Theta, d = 0 \).

The control objective is to choose the plant input \( u_p \) so that the plant output \( y_p \) is regulated close to zero. The parameter set \( \Omega \) is divided into \( N \) smaller not disjointive subsets \( \Omega_1, \ldots, \Omega_N \). The parameter partition \( \Omega_i \subseteq \mathbb{R}^{n+m+1} \), \( i \in \mathcal{I} \), \( \mathcal{I} \) denotes the index set \{1, \ldots, N\} is developed such that each parameter subset \( \Omega_i \) is compact and \( \Omega \subseteq \bigcup_{i \in \mathcal{I}} \Omega_i \). For each subset \( \Omega_i \), a LTI controller with rational transfer function \( C_i(z) \) is synthesized such that the control law \( u_p = -C_i(z)y_p \) yields a stable closed-loop system that meets the performance requirements in the subset \( \Omega_i \). Given the family of candidate controllers \( \mathcal{C} := \{C_i(z)\}_{i \in \mathcal{I}} \), a multicontroller \( C(\beta) \) is constructed from \( \mathcal{C} \). The multicontroller is a dynamical system capable of generating each candidate control laws, as well as a mix of candidate control laws. Construction of the multicontroller involves interpolating the candidate
controllers over the parameter overlaps. Numerous controller interpolation approaches have been proposed in the context of gain scheduling. These methods interpolate controller poles, zeros, and gains [Nichols et al., 1993]; solutions of the Riccati equations for an $H_\infty$ design [Reichert, 1992]; state and observer gains [Stiwell and Rugh, 2000]; controller output blending [Fekri et al., 2006], i.e., $u = \sum_{i=1}^{N} \beta_i u_i$, where $u_i = -C(z)\gamma_i$ is the output of the $i$-th controller. The multicontroller depends on the mixing signal $\beta = [\beta_1, \ldots, \beta_N]^T$ which determines the participation level of the candidate controllers. For fixed values of $\beta$ the multicontroller $u_\beta = -C(z)\gamma_\beta$ has the transfer function:

$$\hat{P}(z;\beta) = \frac{p_0(\beta)z^r + \bar{p}(\beta)\alpha_{r-1}(z)}{z^r + \bar{T}(\beta)\alpha_{r-1}(z)}$$ (12)

The mixer implements the mapping $\beta : \Omega \mapsto B_{\theta_\beta}$, where $B_{\theta_\beta}$ is the set of admissible mixing values, in such a way that $C(z;e_i) = C_i(z)$, where $e_i \in \mathbb{R}^N$ is the $i$-th standard basis vector. This can be achieved by defining the set of all admissible mixing values in $\theta_\beta \in \Omega$

$B_{\theta_\beta} = \{ \beta \in \mathbb{R}^N : \sum_{i=1}^N \beta_i \theta_i = 1 ; \beta_i \geq 0 ; \beta_i = 0 \text{ if } \theta_i \notin \Omega \}$ (13)

The following properties of $\beta(\cdot)$ and of the multicontroller $C(\beta)$ are assumed

**M1.** $\beta(\cdot)$ is Lipschitz in $\Omega$.

**C1.** The elements $p_0(\cdot)$, $\bar{p}(\cdot)$, and $\bar{I}(\cdot)$ are Lipschitz in $\mathbb{R}^N$.

**C2.** For all $\theta^* \in \Omega$, let $\beta^{*} := \beta(\theta^*)$; then $C(z;\beta^{*})$ interna lly stabilizes the plant $G(z;\theta^*)$.

Property M1, together with C1 ensures that if $\theta_\beta$ is tuned slowly (thanks to E1-E2) then the closed-loop system will vary slowly. Property C2 ensures that $C(\beta)$ is a certainty equivalence stabilizing controller.

## 4. STABILITY PRESERVING INTERPOLATION

As in gain scheduling, interpolation methods may not satisfy the point-wise stability requirement C2 (cf. the counter example of [Stiwell and Rugh, 2000]) that should be previously verified. Otherwise, there also exist theoretically justified methods, which can be used to construct the multicontroller in order to assure property C2. The following result, mutuated from [Stiwell and Rugh, 2000], whose proof is omitted for lack of space, allows the construction of a stability preserving multicontroller which guarantees the certainty equivalence property C2:

**Theorem 1.** Consider the nominal plant given by $G_0(z, \theta_\beta) = N_0(z, \theta_\beta)D_0(z, \theta_\beta)$, depending on $\theta_\beta$ smoothly. Let the stable transfer functions $X(z, \theta_\beta)$, $Y(z, \theta_\beta)$ depend on $\theta_\beta$ smoothly and satisfy the Bezout identity, $\forall \theta_\beta \in \Omega$

$$X(z, \theta_\beta)D_0(z, \theta_\beta) + Y(z, \theta_\beta)N_0(z, \theta_\beta) = I$$ (14)

Let each candidate controller be given as $C_i(z) = S_i(z)R_i^{-1}(z)$, $i \in \overline{N}$. If the multicontroller is given by

$$C(z, \beta) = S(z, \beta)R^{-1}(z, \beta)$$ (18)

where

$$S(z, \beta) = \sum_{i=1}^N \beta_i S_i = Y + D_0 \sum_{i=1}^N \beta_i Q_i$$ (19)

$$R(z, \beta) = \sum_{i=1}^N \beta_i R_i = X - N_0 \sum_{i=1}^N \beta_i Q_i$$ (20)

and $\beta = [\beta_1, \ldots, \beta_N]^T \in B_{\theta_\beta}$, then $C(z, \beta)$ internally stabilizes $G_0(z, \theta_\beta)$, for all $\theta_\beta \in \Omega$.

## 5. STABILITY AND ROBUSTNESS OF AMC

The following theorem establishing the stability and robustness properties of the AMC scheme can be stated:

**Theorem 2.** Let the unknown plant given by (3)-(5) satisfy the plant assumptions P1-P4. Consider the adaptive mixing controller with the multicontroller $C(\beta)$ given by (12) and satisfying assumptions C1-C2; mixer satisfying M1; and robust adaptive law given by (6)-(11). Then the following results hold

1. If $M_\Delta, d = 0$, then all closed-loop signals are bounded, i.e., $\phi$, $u_\beta$, $y_\beta \in L_\infty$; furthermore $u_\beta(k), y_\beta(k) \to 0$ as $k \to \infty$.

2. If $M_\Delta, d \neq 0$, then there exists $\mu^* > 0$ such that, if $c \Xi_1^2 < \mu^*$ where $\Xi_1 = \|N_0 \Lambda_{M_\Delta} \|_{\infty}$ and $c > 0$ a finite constant, then the adaptive mixing control scheme guarantees $\phi$, $u_\beta$, $y_\beta \in L_\infty$ and

$$\sum_{i=0}^{k-1} |y_{\beta}(i)|^2 \leq c_0 \mu^2 k + c_1$$ (21)

where $\mu^2 = c(\Xi_1^2 + \Xi_2^2)$, and $\Xi_2 = \|N_0 \Lambda_{M_\Delta} \|_{\infty}$. 

**Proof -** See the Appendix

**Remark 3.** The stability and robustness results as well as the proofs are conceptually similar to those in robust adaptive control, like model reference or pole-placement adaptive control. The advantage is that the proposed scheme does not suffer from the drawback of singularity in the calculus of the control law and in addition it allows well developed results from robust control to be incorporated in the design. In fact, thanks to modularity, the analysis of the overall system relies on certain properties of its individual parts, so that AMC can handle controllers that are not directly parameterized by the unknown plant parameters, e.g. $H_\infty$ or $\mu$-synthesized robust controllers: this extend the control objective beyond the model-reference or pole-placement control objectives typically found in adaptive control.

## 6. NUMERICAL EXAMPLE

A two-carts system is used as an example to demonstrate the results developed in the previous sections. The system, shown in Fig. 2, is composed of two masses $M$ moving along a straight line on a horizontal plane with a known...
dynamic friction coefficient \( b \), and coupled with an elastic force proportional to the deformation of the spring through an unknown stiffness constant \( \theta^* \) [Mosca and Agnoloni, 2001]. The model was discretized and used in the context of switching supervisory control in [Angeli and Mosca, 2002]. In the simulations reported hereafter we consider a sample time \( T_s = 0.1 \) s, \( M = 1 \) kg and \( b = 0 \) Ns/m. The unknown constant stiffness \( \theta^* \) is restricted to the compact set \( \Omega = \{ \theta : 0.25 \leq \theta \leq 1.5 \) N/m\}. The only measurement available is the second cart displacement, corrupted by a bounded sensor noise \( d \). The control \( u_p(t) \) is applied to the first cart through a control channel with a time delay of \( \tau \), that represents the unmodelled dynamics. The nominal plant is given by

\[
G_0(z, \theta^*) = \frac{\theta^* N_0^U(z)}{D_0^U(z) + \theta^* D_0^b(z)}  \tag{22}
\]

where \( N_0^U(z) = T_1^4 z^3 \), \( D_0^U(z) = (z - 1)^2 \), \( M = M(z + M)^2 \), \( D_0^b(z) = 2T_2^2 z(z - 1)((-bT_2 - M)z + M) \). The control objective is to keep the displacement of the second cart close to zero by applying a force to the first cart, in front of a nonzero initial condition and of the sensor noise. We consider the family of \( N = 3 \) candidate controllers \( \{C_i(z) = S_i(z)/R_i(z)\}_{i \in \mathbb{Z}} \) considered in [Mosca and Agnoloni, 2001], and designed respectively on nominal models with stiffness \( \theta^* = 0.3, 0.5, 1.0 \). The coefficients of \( S_i(z) = \sum_{n=0}^{3} s_{in} z^{3-n} \) and \( R_i(d) = z^3 + \sum_{n=1}^{3} r_{in} z^{3-n} \) are reported in Table 1.

The mixer is constructed on the basis of the smooth bump function \( \varphi(x) = e^{1-x^2} \) if \( |x| < 1 \), and \( \varphi(x) = 0 \) otherwise. Consider the pre-normalized weights \( \beta_i(\theta_p) = \varphi_i^z(U_i - L_i) \), \( i = 1, 2, 3 \), where \( U_i \) and \( L_i \) are the upper bound and the lower bound, respectively, of the subset \( \Omega_i = \{ \theta_p : L_i \leq \theta_p \leq U_i \} \). The mixing signal \( \hat{\beta}(\cdot) \) is generated by normalizing \( \hat{\beta} = [\hat{\beta}_1 \ldots \hat{\beta}_3]^T \), i.e., \( i = 1, 2, 3 \)

\[
\hat{\beta}_i(\theta_p) = \frac{\beta_i(\theta_p)}{\sum_1^3 \beta_i(\theta_p)}  \tag{23}
\]

The small subsets are: \( \Omega_1 = [0.25, 0.44] \), \( \Omega_2 = [0.36, 0.80] \), \( \Omega_3 = [0.66, 1.5] \). The subsets have been found by a trial and error procedure. The multicontroller has been constructed using output blending, since this approach has been verified to satisfy assumption C2. The mixing strategy (23) is shown in Fig. 3.

The design parameters have been chosen, via a trial and error procedure to optimize the estimation speed, as follows: \( \Gamma = 1 \), \( \delta_0 = 0.999 \), \( \lambda = 0.5 \), \( \lambda_p(z) = ((z - 0.9)/0.1)^4 \). The AMC scheme is compared with an APPC scheme having the same robust estimator of the AMC scheme. Fig. 4 and 5(a) shows the output response and, respectively, the estimate of the unknown stiffness value, for the two adaptive scheme, for \( \theta^* = 0.5 \) in the ideal case, i.e., with no unmodelled dynamics (\( \tau = 0 \) s)

![Fig. 3. Mixing strategy](image)

![Fig. 4. Ideal case, \( \theta^* = 0.5 \), output response \( y_p(k) \): AMC (solid), APPC (dashed)](image)

![Fig. 5. Ideal case, \( \theta^* = 0.5 \): AMC (solid), APPC (dashed)](image)

![Fig. 6. Unmodelled dynamics, \( \theta^* = 0.3 \), output response \( y_p(k) \): AMC (solid), APPC (dashed)](image)
Table 1. Controller coefficients

<table>
<thead>
<tr>
<th></th>
<th>$s_{p1}$</th>
<th>$s_{p2}$</th>
<th>$s_{p3}$</th>
<th>$r_{c1}$</th>
<th>$r_{c2}$</th>
<th>$r_{c3}$</th>
</tr>
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<tr>
<td>$C_1$</td>
<td>24.1101</td>
<td>-72.0092</td>
<td>71.8009</td>
<td>-23.935</td>
<td>1.9918</td>
<td>-0.5455</td>
</tr>
<tr>
<td>$C_2$</td>
<td>16.8809</td>
<td>-50.5859</td>
<td>50.6613</td>
<td>-16.9539</td>
<td>-2.409</td>
<td>1.9556</td>
</tr>
<tr>
<td>$C_3$</td>
<td>13.3599</td>
<td>-40.6273</td>
<td>41.389</td>
<td>-14.1167</td>
<td>-2.3203</td>
<td>-0.4842</td>
</tr>
</tbody>
</table>

Fig. 7. Unmodelled dynamics, $\theta^* = 0.3$: AMC (solid), APPC (dashed)

Fig. 8. Ideal case, $\theta^* = 0.5$, output response $y_p(k)$: Q-blending AMC (solid), Output-blending AMC (dashed)

Fig. 9. Ideal case, $\theta^* = 0.5$, AMC controller weights $\beta(k)$: Q-blending AMC (solid), Output-blending AMC (dashed)

7. CONCLUSIONS

We have presented the adaptive mixing control approach in a discrete-time setting. A key contribution is the stability and analysis of the adaptive mixing scheme. For the nominal and noiseless case, it was shown that the adaptive mixing control scheme drives the plant states to zero. In the presence of unmodeled dynamics and bounded disturbances that satisfy specified bounds, it was shown that the closed-loop state remains bounded and the regulation error is of the order of the modeling error. A stability preserving method for the synthesis of the multicontroller was also proposed, by interpolating a family of candidate controllers. A two carts system was used to show, via simulations, the effectiveness of the method.

APPENDIX: PROOF OF THEOREM 2

For lack of space, only the main points of the proof are given.

Case (1): Step 1. Following the guidelines used in [Ioannou and Sun, 1996, Section 7.7.1] we can manipulate the control law and the normalized estimation error equations to obtain

\[
x(k + 1) = A(k)x(k) + b_1(k)\epsilon_m^2(k) \\
\]

where $\lambda_0 \in (0, 1)$ is the exponential convergence rate of the homogeneous part of (1).

Step 2. It can be demonstrated that for each frozen time $k$

\[
det(zI - A(k)) = \hat{R}_p\hat{L} + \hat{P}\hat{Z}_p = A^*(z, k) \\
\]

where $A^*(z, k)$ is the characteristic polynomial of the closed-loop formed by the estimated plant and the controller. Thanks to C2, $A^*(z, k)$ is Hurwitz at each frozen time $k$, so that the matrix $A(k)$ has stable eigenvalues at each frozen time $k$. Let us note that $p_0$, $\bar{p}$, $\bar{l}$ are function of $\beta$, $\theta_p$, which are function of time $k$

\[
\Delta p_0 = \frac{\Delta p_0}{\Delta \beta} \frac{\Delta \beta}{\Delta \theta_p}, \\
\]

and analogously for $\Delta \bar{p}$ and $\Delta \bar{l}$. The first factors in $l_\infty$ if C1 holds, $\Delta \beta/\Delta \theta_p \in l_\infty$ if M1 holds, and $\Delta \theta_p \in l_2$ thanks to C2. We conclude that $\Delta p_0$, $\Delta \bar{p}$, $\Delta \bar{l} \in l_2$. In addition $p_0$, $\bar{p}$, $\bar{l} \in l_\infty$. This imply $\|\Delta A\| \in l_2$ and, using [Ioannou and Sun, 1996, Theorem 3.4.11], we conclude that the homogeneous part of (1) is exponentially stable (e.s.), i.e., its state transition matrix $\Phi(k_2, k_1)$ satisfies $\|\Phi(k_2, k_1)\| \leq \beta_0 \lambda_{k_2-k_1}$ for some $\beta_0 > 0$, $0 < \lambda_0 < 1$.

Step 3. For ease of exposition let us denote any positive nonzero constant whose actual value does not affect stability with the same symbol $c$. From [Ioannou and Sun, 1996, Lemma 3.3.3] we have, considering (1)

\[
\|y_p(k)\|_{25} \leq c \|\epsilon_m^2(k)\|_{25} + c \\
\|u_p(k)\|_{25} \leq c \|\epsilon_m^2(k)\|_{25} + c \\
\]

for some $\delta > \min \{\lambda_{k_2}, \delta_0\}$, where $\lambda_0 \in (0, 1)$ is the exponential convergence rate of the homogeneous part of (1).
We define the fictitious signal \( m_2^f(k) := 1 + \phi^T(k)\phi(k) + \|\{y_p\}^{k-1}_{k=1}\|_2^2 + \|\{u_p\}^{k-1}_{k=1}\|_2^2. \) Since \( \delta > \delta_0 \) we can verify that \( |\phi|, |m_2| < m_f. \) Besides using (6)

\[
m_2^f \leq c + c \|\{m_2^f\}^{k-1}_{k=1}\|_2^2 \leq c + c \|\{m_2\}^{k-1}_{k=1}\|_2^2 \quad \forall k \geq 0
\]

Using the Discrete-Time Bellman-Gronwall Lemma [Desoer and Vidyasagar, 1975, p.254] on (8),

\[
m_2^f \leq c + c \sum_{i=0}^{k-1} \left( \prod_{i<j<k} \left( 1 + \frac{1}{\lambda - i} \right) \right)
\]

Using the fact that geometric mean is less than arithmetic mean, and \( m_2 \in L_2 \), we obtain

\[
m_2^f \leq c + c \sum_{i=0}^{k-1} \left( \delta^{i-1} \epsilon m_2^2(i) \right) \left( 1 + \frac{c}{k - 1} \right) \]

\[
\leq c + c e \sum_{i=0}^{k-1} \delta^{k-1-i} \epsilon m_2^2(i) \leq c
\]

We conclude that \( m_2 \in L_\infty \) and \( em_2 \in L_2 \). Using the boundedness of \( m_2 \), we can establish the boundedness of all signals in the closed-loop plant: \( \phi, u_p, y_p \in L_\infty \).

**Step 4.** Recall that \( A(k) \) is e.s. and \( b_1(k)em_2 \in L_2 \) (because \( \|b_1(k)\| \|L_\infty \) and \( em_2 \in L_2 \)). Applying [Ioannou and Sun, 1996, Lemma 3.3.3] to (1.1), we conclude \( x \in L_2 \) and \( u_p, y_p \in L_2 \), which implies that \( u_p, y_p \to 0 \) as \( t \to \infty \).

**Case (2):** **Step 1.** Following the same steps of the ideal case, we can arrive to expressions (1.1)-(3.1). The modeling errors due to \( M_{\Delta u}, d \) do not appear explicitly in (1.1)-(3.1). The difference with respect to the ideal case is that

\[
c, em_2, \Delta \theta_p \in S(\frac{\eta^2}{m_2^2})
\]

**Step 2.** Like the ideal case \( A(k) \) has stable eigenvalues at each frozen time \( k \). Besides, \( |\Delta A| \in S(\frac{\eta^2}{m_2^2}) \). From the form of \( \eta(k) \) it follows that \( |\eta(k)| \leq \|z_1\| \|\{y_p\}^{k-1}_{k=1}\|_2 + \|z_2\| \|\{u_p\}^{k-1}_{k=1}\|_2 \), with \( z_1, z_2 \) as defined in Theorem 2. Because \( \overline{\|\eta\|} \leq \|z_1\| + \|z_2\| \leq 2 \|z\| \), we have \( c, em_2, \Delta \theta_p, |\Delta A| \in S(\|z\| + \|z\|) \).

If we apply [Ioannou and Sun, 1996, Lemma 3.3.3] to the homogeneous part of (1.1), we can establish that \( A(k) \) is exponentially stable provided

\[
c \left( \frac{\eta^2}{m_2^2} \right) \leq \mu^*
\]

for some \( \mu^* > 0 \). Condition (10) may not be satisfied, even for small \( \|z\| \), unless \( d_0 \) is small enough. We can deal with the disturbance using the argument in [Ioannou and Sun, 1996, Section 9.9.1] to establish that when \( m_2^2 \) grows large over an interval of time \( I = [k, k+1] \), the state-transition matrix \( \Phi(k, \tau) \) of \( A(k) \) satisfies \( \Phi(k, \tau) = \beta_0 \lambda_0^k \tau \), for all \( k \geq t \) and \( k, t \in I \). This property of \( A(k) \) (when \( m_2^2 \) grows large) is used to contradict the hypothesis that \( m_2^2 \) could grow unbounded and conclude boundedness. The condition for exponential stability becomes

\[
c \leq \min \left\{ \lambda^2_0 \frac{1}{2}, \frac{\delta}{4} \right\} =: \mu^*
\]

for some finite positive constants \( \lambda, \delta \). We continue supposing the condition (1.10) satisfied, so that \( \Phi(k_1, k_2) \leq \beta_0 \lambda_0^{k_2-k_1} \) for some \( \beta_0 > 0, 0 < \lambda_0 < 1 \) and \( k_2 \geq k_1 \).

**Step 3.** As in the ideal case, let \( \delta > \min \{\lambda^2_0, \delta_0\} \): we can show, using the Bellman-Gronwall Lemma and the fact that \( em_2 \in S(\mu^2) \cap L_\infty \), that for \( \delta < 1/(1 + \mu^4) \) we can again arrive to (8), and conclude that \( m_f, \phi, u_p, y_p \in L_\infty \) and \( em_2^2 \in S(\mu^2) \cap L_\infty \).

**Step 4.** Recall that \( A(k) \) is e.s. and \( b_1(k)em_2 \in S(\mu^2) \cap L_\infty \), we can conclude \( x \in S(\mu^2) \cap L_\infty \), and \( u_p, y_p \in S(\mu^2) \cap L_\infty \) and (21) is valid.

**REFERENCES**


