An alternative solution to the weakened anti-windup problem for LFT perturbed plants

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Abstract: The problem of robust in the large anti-windup compensation (in the "weakened" anti-windup sense) is solved using a different control structure based on the two degrees-of-freedom control and focusing on general stable uncertain systems in LFT form. It is shown that, whenever a suitable "certificate" of robust stability of the plant is satisfied, then the weakened anti-windup problem is solved (even in cases where the natural anti-windup problem can not be solved). A design approach is proposed relying on µ-synthesis, thus imposing less restrictions than previous designs based on small gain reasonings for additive uncertainties.

1. INTRODUCTION

One of the most common nonlinearity in a control system is the actuator saturation, e.g. the maximum and minimum torque in an engine or the maximum and minimum voltage of an electric circuit. When applying some control design methods to the saturated closed-loop system, the limits on the actuator must be taken into account, because otherwise serious consequences, e.g. oscillations, limit cycle or even instability can be the result. These consequences are the outcome of a discrepancy between the controller output and the plant input, which is equivalent to inconsistencies in the controller states. This phenomenon, usually called "controller windup", motivates the study and application of anti-windup compensators (Kothare et al. [1994]).

Anti-windup (AW) techniques are aimed at designing an add-on compensator which, connected to a control system which works well in the absence of saturation, ensures that the windup effects are drastically reduced. A survey of early and recent contributions appeared in the vast literature in the field can be found in Tarbouriech and Turner [2000], Galeani et al. [2009]. While all AW techniques ensure some robustness for sufficiently small uncertainties, it was shown in Turner et al. [2007], Galeani and Teel [2006] that the same natural definition of the AW problem (see Teel and Kapoor [1997a]) can be incompatible with a requirement of robustness with respect to a prescribed set of (possibly large) uncertainties. In these cases, the AW requirements must be relaxed (weakened) and a robust solution can be adopted, at the price of reduced performance (due to the need of dealing with the worst case uncertainty): robust solutions were proposed in Turner et al. [2007], Marcos et al. [2007], Galeani and Paolletti [2005].

The development of the weakened anti-windup originated in Galeani and Teel [2006] from the idea of Teel and Kapoor [1997b], where two controllers are merged to satisfy local performance and global stability properties; however, the weakened AW solution in Galeani and Teel [2006] was based on the idea of combining a high performance controller for the unsaturated nominal plant with a robust global controller for the saturated uncertain plant. As a first contribution, an alternative solution to the weakened anti-windup problem is presented in this work. Instead of achieving robustification by introducing a filter F in a suitable (virtual) feedback path in order to do satisfy a suitable small gain condition, here the robustification will be obtained by a suitable choice of the feedback controller in a two-degrees-of-freedom (2DOF) control structure. The alternative solution is mainly inspired by Grimble [1994], Tyler Jr. [1964], Moore et al. [1986], where a 2DOF control is used to implement by a 2DOF structure a robust controller such that the closed loop system satisfies a model-matching requirement with respect to the nominal closed loop system which would have been obtained using a given high performance controller designed for the nominal plant (but having poor open-loop robustness properties).

Compared to previous work, this paper 1) proposes a new compensator structure (which, although shown to be essentially equivalent to the previous one, is favorable from the computational point of view); 2) considers structured, possibly real parameter uncertainties (instead of unstructured additive uncertainty); 3) exploits µ-synthesis ideas (instead of purely L2 gain ideas) to ensure the required small gain stability condition, thus being less conservative than previous designs.

The paper is structured as follows: Section 2 reviews some preliminaries about (possibly weakened) anti-windup and robust control; Section 3 describes the alternative solution to the weakened anti-windup formulation; finally Section 4 presents a simulation example.

Notation Given \( w \in \mathbb{R}^p \), \( \text{diag}(w) \) is the diagonal matrix with the elements of \( w \) on the diagonal. Vector inequalities are meant componentwise, i.e., \( w > v \) means \( w_i > v_i \) for all \( i = 1, \ldots, p \). The saturation function of level \( w \in \mathbb{R}^p_{>0} \) is defined by saying that its \( i \)-th component

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is \( \text{sat}_w(v_i) = \text{sat}_w(v_i) = \text{sign}(v_i) \min(|w_i|, |v_i|) \)

\( i = 1, \ldots, p \). The corresponding deadzone function is \( \text{dz}_w(v) := v - \text{sat}_w(v) \). The conjugate transpose of matrix \( M \) is denoted \( M^\dagger \).

The Euclidean norm of \( v \) is denoted \( |v| = \sqrt{v^\dagger v} \).

Given a signal \( q(\cdot) \), let \( \|q\|_{2,t}^2 := \int_0^t |q(\tau)|^2 d\tau \) and \( \|q\|_2^2 := \lim_{t \to \infty} \|q(\cdot)\|_{2,t}^2 \).

Then \( q(\cdot) \in \mathcal{L}_2 \) if \( \|q\|_2^2 < \infty \) for all \( t \geq 0 \), and \( q(\cdot) \in \mathcal{L}_2 \) if \( \|q\|_2^2 < \infty \).

Abusing notation, the same symbol indicates a signal and its Laplace transform (or a system, its impulse response and its transfer function), the exact meaning being clear from the context; in particular \( F \circ q \) is the response to input \( q(\cdot) \) of a system with transfer function \( F(s) \) and zero initial condition.

A function \( \pi : \mathbb{R} \to \mathbb{R} \) is right continuous if \( \pi(t) = \pi(t^+) \), \( \forall t \in \mathbb{R} \), where \( \pi(t^+) = \lim_{\tau \to t^+, \tau > t} \pi(\tau) \). Given a square matrix \( X \), \( \text{He}(X) := X + X^\dagger \).

For a partitioned matrix \( M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \) and matrices \( D_1, D_2 \) of suitable dimensions, define the upper Linear Fractional Transformation (LFT) \( \mathcal{F}_u(M, D_1) := (M_{22} + M_{21}(I - M_{11}D_1)^{-1}M_{12}) \)

and the lower Linear Fractional Transformation \( \mathcal{F}_l(M, D_2) := (M_{11} + M_{12}(I - M_{22}D_2)^{-1}M_{21}) \).

2. PRELIMINARIES

2.1 The unconstrained nominal closed-loop

A linear nominal plant is given as

\[
\dot{x} = Ax + Bu, \\
y = C_2x + D_{2d}u + D_{2y}u,
\]

where \( x \in \mathbb{R}^n \) is the plant state, \( u \in \mathbb{R}^p \) is the control input, \( y \in \mathbb{R}^q \) is the measurement output, \( d \) is the exogenous disturbance, \( z \) is the performance output. In order to give global stability results with bounded controls, the following assumption is made.

Assumption 1. A is Hurwitz.

A linear controller \( K_M \) is also given which induces desirable performance (i.e. a desirable \((r, d)\) to \( z \) response) when connected to \( P \) without saturation:

\[
\dot{x}_c = A_c x_c + B_{cu} u_c + E_c r, \\
y_c = C_c x_c + D_{cd}u + D_{cy} u_c,
\]

where \( r \) is the reference signal, \( u_c \in \mathbb{R}^p \) is the feedback signal, and \( y_c \in \mathbb{R}^q \) is the controller output. Let \( \Sigma_U \) be the unconstrained closed-loop system formed by \( K_M \) and \( P \), namely \( \Sigma_U \) is given by (1), (2) and

\[
u = y_c, \quad u_c = y
\]

Since \( K_M \) induces desirable performance on \( P \), it is reasonable to assume the following.

Assumption 2. \( \Sigma_U \) is well-posed and internally stable.

2.2 Model recovery (\( \mathcal{L}_2 \)) anti-windup compensation

Since the input \( u \) to the plant is affected by the saturation nonlinearity an add-on AW compensator \( K_{AW} \) is to be designed and suitably connected to \( P \) and \( K_M \) via the AW interconnection

\[
u_c = y + v_c, \quad u = \text{sat}_m(y_c + v_1)
\]

1 Since (1) is linear and asymptotically stable by Assumption 1, there is no loss of generality in considering the disturbance \( d \) as not affecting (1a); see Galeani and Paoletti [2005].

2 Such a restriction is necessary; see Teel and Kapoor [1997a].

3 Note that different “hats” denote a closed-loop system and its signals, e.g. the state of \( P \) as a subsystem of \( \Sigma_{SAW} \) is denoted by \( \hat{z} \).
subject to an uncertain feedback
\[ y_\Delta = \Delta \circ u_\Delta, \quad \Delta \in \mathcal{S}, \]  
where \( \mathcal{S} \) is the uncertainty set. For future use, the matrix transfer function of (7) will be partitioned as
\[ \begin{bmatrix} u_\Delta \\ y \end{bmatrix} = \begin{bmatrix} P_{\Delta y/\Delta u} & P_{\Delta d/\Delta u} \\ P_{\Delta y/\Delta d} & P_{\Delta d/\Delta d} \end{bmatrix} \begin{bmatrix} u_\Delta \\ d \end{bmatrix}; \]  
using the LFT notation and considering the feedback \( y_\Delta = \Delta \circ u_\Delta \) around the first input-output channel of \( P_0 \) as an upper LFT, it holds that \( P_\Delta = F_2(P_0, \Delta) \). As for the structure of uncertainties in \( \mathcal{S} \), using the standard LFT uncertainty representation Zhou et al. [1995], Dullerud and Paganini [2000], the considered case is
\[ \mathcal{S} = \Delta_{s,f} := \{ \text{diag}(\Delta_1, \ldots, \Delta_s, \Delta_{s+1}, \ldots, \Delta_{s+f}) : \Delta_i = \delta_i I, |\delta_i| \leq 1, \delta_i \in \mathbb{R}, i = 1, \ldots, s, \Delta_i \in \mathcal{L}_2, ||\Delta_i||_{\infty} \leq 1, i = s + 1, \ldots, s + f) \}, \]  
where the \( \delta_i \), \( i = 1, \ldots, s \) correspond to (normalized) real uncertain parameters and the blocks \( \Delta_i, i = s + 1, \ldots, s + f \) correspond to (normalized) uncertain dynamic subsystems, and the non negative integers \( s \) and \( f \) denote, respectively, the number of uncertain real parameters and uncertain dynamic subsystems; note that associated to each block \( \Delta_i \) there is a dimension (assuming, for simplicity, that each block is square) which will not be explicitly indicated in the following.

Classical results in robust control theory exploit the structured singular value (or \( \mu \)) associated with the uncertainty structure \( \Delta_{s,f} \), to show that \( P_\Delta \) is robustly stable for all \( \Delta \in \mathcal{S} \) if and only if
\[ \mu_{\Delta_{s,f}}(P_{\Delta y/\Delta u}(\omega)) < 1, \forall \omega \in \mathbb{R}, \]  
where by definition
\[ \mu(\Delta)(M) := \frac{1}{\inf_{\Delta \in \mathcal{C}\Delta} \{ \sigma(\Delta) : \det(I - M(\Delta)) = 0 \}} \]  
(12)

(where \( \mu(\Delta)(M) = 0 \) if the minimum does not exist) and \( \mathcal{C}\Delta := \{ \alpha : \Delta \in \Delta, \alpha \in \mathbb{R} \} \) (usually \( \Delta \) is a normalized set with radius 1, and then \( \mathcal{C}\Delta \) is the corresponding unbounded cone).

Since the condition \( \mu_{\Delta_{s,f}}(M) < 1 \) is difficult to check exactly, upper bounds are used to guarantee it. Let \( \mathcal{M}_{s,f} \) denote the set of multipliers used by pairs of block diagonal matrices \((\Gamma, G)\) whose \( i \)-th block has the same dimension as the corresponding block of \( \Delta \in \Delta_{s,f} \), and such that
\[ \Gamma := \{ \text{diag}(\Gamma_1, \ldots, \Gamma_s, \gamma_{s+1} I, \ldots, \gamma_{s+f} I) : \Gamma_i = \Gamma_i^* > 0, i = 1, \ldots, s, \gamma_i > 0, i = s + 1, \ldots, s + f) \}; \]  
\[ G := \{ \text{diag}(G_1, G_s, 0, \ldots, 0) : G_i = G_i^* > 0, i = 1, \ldots, s \}. \]  
(13a, 13b)

A sufficient condition for \( \mu_{\Delta_{s,f}}(M) < 1 \) is then given by the existence of \( (\Gamma, G) \in \mathcal{M}_{s,f} \) satisfying
\[ M'TM - \Gamma + j(M'G - GM) < 0. \]  
(14)

2.4 Weakened AW and robustification

It is assumed that the plant \( P_0 \) and the set \( \mathcal{S} \) satisfy the following assumption, which amounts to robust stability of \( P_\Delta \) for \( \Delta \in \mathcal{S} \), thus implying Assumption 1.

**Assumption 4.** \( \mu_{\Delta}(P_{\Delta y/\Delta u}) < 1. \) \( \square \)

From the point of view of computations, it is more useful to assume the following assumption (implying Assumption 4), which states that a certificate for robust stability of \( P_\Delta \) for \( \Delta \in \mathcal{S} \) can be exhibited in the form of a suitable pair of multipliers \( \Gamma, G \).

**Assumption 5.** \( \exists (\Gamma, G) \in \mathcal{M}_{s,f} \) such that (14) holds. \( \square \)

It is shown in Galeani and Teel [2006] that the only way to ensure robust-in-the-large stability is to relax the requirements in Definition 1 at least as much as in the following Definition 2. In Definition 2, the unconstrained AW closed loop \( \Sigma_{\text{UAW}} \) denotes the interconnection of (7), (8), (2), and (5) by \( u = y + p_2, u = y + \nu_1 \).

**Definition 2.** The weakened \( \mathcal{L}_2 \) AW problem with domain of robustness \( \mathcal{S} \) is to find an AW compensator \( K_{\text{AW}} \) such that \( \Sigma_{\text{SAW}} \) is well-posed and

(1) for \( \Psi = 0 \) and \( d = 0 \), \( x_{aw} > 0 \), \( x_{aw}(0) = x_{aw}^0 \) and \( \bar{u} (\cdot) \equiv \text{sat}(\bar{u}(\cdot)) \), then \( \bar{z}(\cdot) \equiv \bar{z}(\cdot) \);
(2) \( \Sigma_{\text{UAW}} \) is well-posed and internally stable, \( \forall \Psi \in \mathcal{S} \);
(3) if \( d_{aw}(\bar{u}(\cdot)) \equiv \bar{z}_{aw}(\cdot) \) then \( (\bar{z} - \bar{z})(\cdot) \in \mathcal{L}_2, \forall \Psi \in \mathcal{S} \).

The usual and the weakened AW problem are compared in Galeani and Teel [2006], Galeani and Paoletti [2005], where solutions for special classes of norm bounded uncertainties are given; instead in this paper the general class of norm bounded, structured LFT uncertainties is dealt with.

**Remark 2.** Assumption 2 requires \( \Sigma_{U} \) to be only nominally stable, so \( K_{M} \) may be designed to boost nominal performance (\( \Psi = 0 \)). Hence, the weakened AW can also be seen as an a posteriori fix for a high performance control system endowed with a high performance AW solution in cases where the interaction between uncertainties and saturation requires an additional robustification in the presence of very large (but infrequent) uncertainties. \( \square \)

3. ALTERNATIVE WEAKENED ANTI-WINDUP FORMULATION

In Galeani and Teel [2006], Galeani et al. [2005], solutions of the weakened anti-windup problem in Definition 2 were proposed in Galeani and Paoletti [2005] by using a filter \( F \) ensuring a small gain condition in the interconnection of two closed loop systems (the unconstrained a priori given closed loop and a bounded input uncertain closed loop). In this work, an alternative solution to the weakened anti-windup problem is presented (see Fig. 2) (already used in a preliminary form in Bruckner et al. [2010a]) which is based on a two-degrees-of-freedom (2DOF) control structure. The main reason for the new architecture is that the feedback controller \( K_{2}\text{aw} \) of the 2DOF control structure (see Fig. 2) will be used for robustification.

3.1 2DOF controllers and WAW internal structure

A parameterization of stabilizing 2DOF controllers was given in Moore et al. [1986], where it is shown that any
Fig. 2. Block structure of the weakened anti-windup in the two degrees-of-freedom framework

Fig. 3. Analysis block structure of the weakened anti-windup in Fig. 2 (the block AW is essentially a copy of $P_{yy}$, whereas $P_{yy}$ is $P$ with the block $P_{yy}$ removed). Given 2DOF controller $K_M$ in (2) mapping $(r, y)$ to $y_c$, as

$$
y_c = [K_r \ K_y] \begin{bmatrix} r \\ y \end{bmatrix}
$$

(15)

and stabilizing a plant $P_{yy}$, can be implemented as:

$$
y_c = K_{f_f} r + K_{f_b} (K_{f_f} r - y) 
$$

(16a)

$$
K_{f_f} = \begin{bmatrix} K_{f_1} \\ K_{f_2} \end{bmatrix} = \begin{bmatrix} I \\ P_{yy} \end{bmatrix} (I - K_y P_{yy})^{-1} \in \mathcal{RH}_\infty 
$$

(16b)

$$
K_{f_b} = K_R
$$

(16c)

where $R$ is any controller stabilizing $P_{uu}$ without modifying the nominal input/output response induced by $K_M$ but achieving the robust stability induced by $K_R$.

Based on this idea, the internal structure of the proposed WAW compensator $K_{W_AW}$ (having state $\tilde{x}_{aw}$, input $\tilde{u}_{aw} = (y_c, u, y)$ and output $\tilde{y}_{aw} = (v_1, v_2)$) is chosen in such a way to realize the closed loop system Fig. 2, where the block AW corresponds to the AW compensator designed for the nominal plant (e.g. described by (5)) and the remaining blocks are add-ons to ensure the achievement of the requirements in Definition 2. The detailed expression of the state space description of $K_{W_AW}$ as

$$
\begin{bmatrix} A_{aw} & B_{aw,1} \\ C_{aw} & D_{aw} \end{bmatrix} = \begin{bmatrix} A_{aw} & B_{aw,1} \\ C_{aw,1} & D_{aw,1} \end{bmatrix} \begin{bmatrix} D_{aw,1,1} & D_{aw,1,2} & D_{aw,1,3} \\ C_{aw,2,1} & D_{aw,2,1} & D_{aw,2,2} \end{bmatrix} \begin{bmatrix} B_{aw,2} \\ B_{aw,3} \end{bmatrix}
$$

in terms of the state space description $(A_{F}, B_{F}, C_{F}, D_{F})$ of $K_R$ in Fig. 2 and the (necessarily nonsingular) matrix $\Delta_p = (I + D_{F} D_{22})^{-1}$ can be obtained by simple (though lengthy) algebra, and it is given in (17).

The first three blocks from the left in Fig. 2 (that is $K_M$, $P_{M}$ and $K_R$) correspond to a 2DOF implementation of the a priori given controller $K_M$ by a 2DOF controller composed of a feedforward part $K_{ff}$ and a feedback part $K_{fb}$ such that the issues of nominal response and robust stability are completely decoupled between $K_{ff}$ and $K_{fb}$; in particular, the feedback part $K_{ff}$ ensures the unconstrained, unaltered closed-loop response of system $\Sigma_U$ (i.e. the response obtained applying directly $K_M$ to the nominal plant $P_0$), while the robust stability of the overall closed loop system depends only on the feedback part $K_{fb}$.

Remark 3. It is worth noticing that a 2DOF controller (16) robustly stabilizes if and only if $K_{fb} = K_R$ is robustly stabilizing for the plant, and $K_{ff}$ is essentially $\Sigma_U$, so its stability is assured in the present context by Assumption 2, i.e. due to the fact that the unconstrained nominal controller $K_M$ is designed such that the unconstrained closed-loop $\Sigma_U$ is well-posed and internally stable.

Remark 4. The structures in Galeani and Paoletti [2005] and Fig. 2 yield the same nominal $r$ to $z$ closed loop response, and can be made to yield the same $d$ to $z$ response by noting that in the two cases such response is respectively given by

$$
W_{zd}^F = P_{zd} + P_{zu} (I - K_y P_{yu})^{-1} K_y F P_{yd}, 
$$

(18a)

$$
W_{zd} = P_{zd} + P_{zu} (I - K_R P_{yu})^{-1} K_R P_{yd},
$$

(18b)

so that the choice

$$
K_R = (I - K_y (I - F) P_{yu})^{-1} K_y F
$$

(19)

achieves coincidence between the two quantities. Hence, from the point of view of the induced nominal responses considered in Galeani and Teel [2006], the two schemes are essentially equivalent. However, from the design point of view, some of the computations described in the following are easier with the structure in Fig. 2.

3.2 A family of solutions

Due to the discussion in Section 3.1, the robust stability of the overall closed loop system in Fig. 3 (and then of the equivalent system in Fig. 2) is equivalent to the stability of the closed loop containing $K_R$, which can be guaranteed if $K_R$ ensures a loop gain less than $\gamma_{\infty,NL}$ (recall Assumption 3) between the output $y_{NL}$ and the input $u_{NL}$ of the nonlinear loop. In order to achieve this objective, a reformulation as a $\mu$-synthesis problem is now proposed.

With the notation in (9), define the augmented plant mapping $[y'_{\Delta} y'_{NL} y'_{K_R}]$ to $[u'_{\Delta} y_{NL} u_{NL} y'_{K_R}]$ as

$$
P_a = \begin{bmatrix} P_{u_{\Delta}} & P_{u_{\Delta} u_{NL}} & 0 \\ 0 & 0 & \gamma_{\infty,NL} \\ -P_{y_{\Delta} y_{NL}} & 0 & -P_{y_{\Delta} y_{NL}} \end{bmatrix}
$$

(20)

and the augmented uncertainty mapping $[u'_{\Delta} y_{NL} u_{NL}]$ to $[y'_{\Delta} y_{NL}]$ as

$$
\Delta_a = \begin{bmatrix} \Delta \\ 0 \\ \Delta_p \end{bmatrix}, ||\Delta_a||_{\infty} \leq 1.
$$

(21)

Note that $\Delta_a$ has the same structure of $\Delta$ plus an additional full block; hence $\Delta_a \in \mathcal{S}_a := [\Delta_{1,1}]$ and the corresponding class of multipliers is $M_{a,f+1}$. Let $M$ denote the closed loop system formed by attaching $K_R$ on the third input/output pair of $P_a$, and note that the conditions

$$
\mu_{\Delta} (M_{11}) < 1, \max_{\Delta \in \mathcal{S}_a} \mu_{\Delta} (F_u (M, \Delta)) < 1,
$$

(22)

are equivalent, respectively, to robust stability and robust performance of $M$ over $\mathcal{S}$. The main loop theorem Zhou et al. [1995] establishes the equivalence of the above two conditions (22) with the single condition

$$
\mu_{\Delta} (M) < 1;
$$

(23)

hence, the selection of a suitable $K_R$ can be recast as a $\mu$-synthesis problem for the above plant $P_a$.

The main result of this paper is the following, which is analogous to the main result in Galeani and Teel [2006], Galeani et al. [2005] for the case of LFT uncertainties considered here. The key point in the theorem is that whenever robust stabilization of the underlying plant
is possible, then the weakened anti-windup problem is solvable.

Theorem 5. Under Assumption 2, Assumption 3 and Assumption 4, the weakened anti-windup problem in Definition 2 is always solvable. Moreover, if Assumption 5 holds, then a certificate of stability can be given in the form of a multiplier $(\Gamma, G) \in \mathcal{M}_{s,f+1}$. □

3.3 Optimal design of the robustifying controller $K_R$

While all the controllers in the family characterized in Section 3.2 solve the problem in Definition 2, it is desirable to obtain the “best” possible controller. It was argued in Galeani and Teel [2006] that the goal of optimization in weakened anti-windup is to make “as little weakened as possible”, in the sense of relaxing as little as possible the model matching requirement on the $d$ to $z$ response of the anti-windup closed loop with respect to the original response of $\Sigma_d$, which in the notations (9) is given by:

$$W_d = F_d(P, K_y) = P_{sd} + P_{u}(I - K_y P_{yu})^{-1} K_y P_{yd}. \quad (24)$$

Hence, $W_d$ is the objective to be matched by the design of $K_R$, i.e. the optimal $K_R$ should yield a $d$ to $z$ response as close as possible to $W_d$ under the stability constraint. Since achieving such model matching requirement is usually more important at certain frequencies (i.e. the frequency content of $d$ is found) than at other (where possibly the frequency content of measurement noise is found, and then just attenuation is desired), it is useful to introduce a performance weight $w_p(s) \in \mathcal{R}H_{\infty}$ such that $|w_p(j\omega)| \gg 1$ at frequencies $\omega$ where a closer match is desired, and $|w_p(j\omega)| \ll 1$ otherwise, and require the minimization of the $H_{\infty}$ norm of the difference $W_d - W_d$ and the achieved response, weighted by $w_p(j\omega)$.

The optimization problem above can be cast as a $\mu$-synthesis problem with an additional full (performance) block (normalized by introducing a further scalar weight $\gamma > 0$) for the augmented plant $P_{aa}$ mapping $[\Delta \gamma y NL \ d' y'k_{r}]'$ to $[\Delta \gamma y NL y'k_{r}']'$ as

$$P_{aa} = \begin{bmatrix} \frac{P_{u}}{s} & 0 & 0 & 0 \\ 0 & \frac{P_{u}}{s} & 0 & 0 \\ \gamma w_p P_{za} & 0 & 0 & 0 \\ -\gamma w_p P_{za} & 0 & 0 & 0 \\ \frac{P_{u}}{s} P_{wa} & 0 & 0 & 0 \\ -\gamma w_p P_{za} & 0 & 0 & 0 \end{bmatrix} \gamma NL P_{I}$$

(25)

and the augmented uncertainty $\Delta_{aa} \in \Delta_{s,f+2}$ mapping $[\Delta \gamma y NL y'k_{r}']'$ to $[\Delta \gamma y NL d' y'k_{r}']'$ as

$$\Delta_{aa} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \gamma NL P_{I} \gg 1, \|\Delta_{pp}\|_{\infty} \leq 1, \|\Delta_{pp}\|_{\infty} \leq 1. \quad (26)$$

Solving the $\mu$-synthesis problem for (25) actually amounts to minimize the worst case mismatch over the considered uncertainty set. However, as was pointed out in Galeani and Teel [2006], one of the main motivations for the weakened anti-windup problem is given by the scenario where large deviations of the parameters are possible but infrequent (whereas the plant operates most of the time very close to the nominal conditions), so that robust stability must be guaranteed over a large uncertainty set, although it is enough to achieve performance in the nominal parameters. In such a case, a modified $\mu$-synthesis problem should be addressed, for the same uncertainties (26) but for the modified augmented plant $P_{aa}$ obtained by replacing the block $\gamma w_p P_{yu}$ in position (3,1) of $P_{aa}$ by a zero block. Similarly to Theorem 5, it can be proven that $\mu$ ensuring that $\mu_{\Delta_{s,f+2}}(P_{aa}) < 1$ (or $\mu_{\Delta_{s,f+2}}(P_{aa}) < 1$) can be found whenever $\mu_{\Delta_{s,f+2}}(P_{0}) < 1$, and that a multiplier in $\mathcal{M}_{s,f+2}$ for the former inequality can be exhibited whenever a corresponding multiplier in $\mathcal{M}_{s,f+1}$ is available for proving the latter. Note that due to the normalization of the $\Delta_{pp}$ obtained by introducing $\gamma > 0$, the achieved model matching quality is measured by $\gamma^{-1}$, so that $\gamma$ should be possibly maximized; a lower bound on the achievable result is also given next.

Theorem 6. Under Assumption 2, Assumption 3 and Assumption 5, then a certificate for $\mu_{\Delta_{s,f+2}}(P_{aa}) < 1$ can be given in the form of a multiplier in $\mathcal{M}_{s,f+2}$. Moreover, $\gamma \geq \|w_p(P_{sd} - W_d^2)\|_{\infty}$. □

4. A NUMERICAL EXAMPLE

Consider the damped-mass-spring system:

$$\begin{align*}
\dot{x} &= Ax + Bu = \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \quad (28a) \\
y &= Cx = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x; \quad (28b)
\end{align*}$$

with state $x = [q \dot{q}]^T$, where $q$ is the position, $\dot{q}$ the velocity, $m = 0.05$ the mass, $k = 1$ the elastic constant of the spring, $f = 0.06$ the damping coefficient and $u$ the force exerted on the mass. The uncertainty is a real actuator dynamics $V(s) = \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2}$, $\omega_0 \in [26, 200]$. The unconstrained controller $y_c(s) = C(s)C_f(s)Y(s) - u(s)$, with $C_f(s) = 200(s + 5)^2$ and $C_f = 5(s + 5)$ is given and guarantees fast response with zero steady-state error and rejection of step disturbances, despite the presence of very underdamped poles in the open loop system. As shown in Galeani and Paoletti [2005], in the presence of saturation no AW compensator can robustly solve the natural AW problem with this data, and then the requirements must be relaxed and a robust, weakened AW compensator can be derived. It is now shown that the proposed solution yields good performance, comparable to the unperturbed and unconstrained solution.

The uncertainty $V(s)$ can be represented by the set of additive perturbations $(\Psi(s) : \Psi(s) = P(s) \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2}$, $\omega_n \in [26, 200]) \cup \{0\}$, where $P(s) \cong C_2(I - A)^{-1}B_2$, which in turn is included in the set $\Delta \cong (\Psi(s) = W(s)\Delta(s)) : \Delta(s) \in \mathcal{R}H_{\infty} \cup \{0\}$ where $W(s) := P(s) \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2}$ (for scalar uncertainty transfer functions, a single weight $W(s)$ is sufficient). In a first step the gains $K_1, L_1, i = 1, 2, ..., \bar{N} + 1$ of the $L_2$ switching AW (5) are obtained by using the recipe proposed in Galeani et al. [2007].

4 For the scenario where large uncertainties are actually frequent, although their variations are infrequent, an adaptive solution should be preferred; see Bruckner et al. [2010b,a].
Using the additive uncertainty description a plant according to $P_{u}$ in (9) can be derived. Then, in order to perform a robustifying task with respect to (22) and the augmented plant $P_{a}$, a post-evaluation of the $L_2$ gain $\gamma_{NL}$ (Bruckner et al. [2010b]) for the nonlinear loop is necessary.

Choosing the weights $\omega_p(s)=\omega_4$ of the optimization problem, and performing a Gramian-based input/output balancing with a subsequent model reduction of the resulting controller, the following robustifying controller $K_R$ can be derived (27).

The simulation scenario can be specified by the following parameters $\omega_0(t)=\omega_0(0)=26$ and the output disturbance $d=\text{Asin}(2\pi ft); A=-0.3; f=0.1Hz$. It has to be mentioned that for a parameter value of $\omega_0=116$ the unconstrained nominal controller becomes unstable. A comparison with the unconstrained and unperturbed controller is depicted in Fig. 4. The result is quite impressive since it is almost possible to recover the unconstrained control performance.

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