Abstract: Definitions and methods for checking the identifiability of linear and nonlinear systems are now well established. However this property has not really been investigated for uncertain systems, in particular for set-membership models in a bounded-error context. In this paper, we propose two complementary definitions, set-membership identifiability and \( \mu \)-set-membership identifiability, the first one is conceptual whereas the second can be put in correspondence with existing set-membership parameter estimation methods. The links between these definitions are exhibited and two methods to check the properties are proposed, illustrated by examples.

Keywords: Identifiability; Uncertain dynamic systems; Nonlinear models; Bounded disturbances; Bounded noise

1. INTRODUCTION

Providing models representing physical systems is a common concern spread over all scientific and engineering communities. But complex systems are often subjected to uncertainties that make the modeling task awkward. It is particularly difficult to get an accurate model of the perturbations and noises acting on the system. This may turn the usual stochastic framework inappropriate and this is why stochastic models are sometimes disregarded to the benefit of set-membership models which naturally cope with uncertain and inaccurate knowledge when perturbations and noises are assumed to be bounded but otherwise unknown.

Set-membership methods are now the subject of a growing interest in various communities and are applied for many tasks Armengol et al. (2000, 2001); Guerra et al. (2006); Jaulin and Walter (1993); Kieffer et al. (2002); Lesecq et al. (2003). The literature on this topic shows interesting progress in the last five years, for example, Raisi et al. (2004). In particular, set-membership parameter estimation has been shown to be an interesting alternative to stochastic estimation methods. Set-membership estimation can be based on interval analysis that was introduced by Moore (1966) and several algorithms have been proposed: Jaulin and Walter (1993); Jaulin et al. (2001); Ribot et al. (2007); Kieffer and Walter (1998); Raisi et al. (2004). However, to our knowledge, a definition and method for the identifiability problem of error-bounded uncertain models are still missing.

This paper proposes two complementary definitions, set-membership identifiability and \( \mu \)-set-membership identifiability. The first one is conceptual whereas the second one can be put in correspondence with existing set-membership parameter estimation methods which use a specified precision threshold. One of the motivations for set-membership identifiability is that it bypasses standard identifiability and allows one to give (set) estimates of parameters that are unidentifiable. Set-membership identifiability indeed guarantees that there exists a partition of the parameter space into connected subsets so that every subset can be associated with a distinguishable output behavior. We consider the identifiability analysis of set-membership (controlled or uncontrolled) models of the following form:

\[
\Gamma^p_t = \begin{cases} 
\dot{x}(t,p) = f(x(t,p), u(t),p), \\
y(t,p) = h(x(t,p), p), \\
x(t_0,p) = x_0 \in X_0, \\
p \in \mathbb{R} \cup \mathcal{U}_p, \\
t_0 \leq t \leq T,
\end{cases}
\]

where \( x(t,p) \in \mathbb{R}^n \) and \( y(t,p) \in \mathbb{R}^m \) denote the state variables and the outputs at time \( t \) respectively, \( u(t) \in \mathbb{R}^p \) is the input vector at time \( t \). The initial conditions \( x_0 \), if any, are supposed to belong to a bounded set \( X_0 \). The functions \( f \) and \( h \) are real and analytic on \( M \), where \( M \) is an open set of \( \mathbb{R}^n \) such that \( x(t,p) \in M \) for every \( t \in [t_0, T] \) and \( p \in \mathbb{R} \). \( T \) is a finite or infinite time bound. The vector of parameters \( p \) belongs to a connected set of parameters \( P \). \( P \) is supposed to belong to \( \mathcal{U}_p \) where \( \mathcal{U}_p \) is an a priori known set of admissible parameters. \( \mathcal{U}_p \) is included in \( \mathbb{R}^p \). \( \mathcal{U}_p \) can be equal to \( \mathbb{R}^p \). In the case of uncontrolled models, \( u \) is equal to 0.

In the following, \( \Gamma^{p,x_0}_2 \) denotes a specific set-membership model of the family of models represented by (1), where \( p \in \mathbb{R} \). In this paper, we are interested in the identifiability of the parameter set \( P \). In the following, we consider controlled models, the case of uncontrolled models is considered in a later specific section.

The paper is organized as follows. Section 2 briefly presents the identifiability problem for uncertain systems. Section 3 reviews some existing identifiability definitions for models of the form \( \Gamma^{p,x_0}_2 \), in particular identifiability and interval-identifiability at a vector of parameters. Section 4 proposes the two definitions of set-membership identifiability and the links with the recalled existing definitions. Methods to analyse set-
membership identifiability are given in Section 5. Finally, some conclusions and future works are outlined in Section 6.

2. UNCERTAIN SYSTEMS IDENTIFIABILITY

Several sources of uncertainty exist and may lead to different models. Bounded uncertain parameter systems in particular fall into the following two cases:

- the case for which the system has constant parameters but the knowledge about the parameter values is uncertain. This case corresponds to the plus/minus tolerance value provided by the builder of physical device parameters. Then, the study of such systems can be brought back to the study of a family of constant parameter systems.
- the case for which parameter uncertainty comes from the fact that parameters may vary across time. This case is typical of devices that operate in different environmental conditions, which may affect parameter values.

In this paper, we are interested in the first situation.

3. EXISTING IDENTIFIABILITY DEFINITIONS

3.1 Structural local or global identifiability

An input-output approach can be used to frame identifiability and test it for some nonlinear systems. In the 90s, Fliess and Diop proposed a new approach for nonlinear observability and identifiability: Diop and Fliess (1991), based on differential algebra: Kolchin (1973). This method consists in rewriting, when it is possible, the nonlinear system $\Gamma_p^{x_0}$ as a differential polynomial system that is augmented with $\bar{p} = 0$, where $p$ is the vector of parameters of the model. The resulting system $\Gamma_p$ can be described by the following polynomial system:

$$\Gamma_p = \begin{cases} r(x, y, u, p) = 0, \\ s(x, y, p) = 0, \\ \bar{p}_i = 0, \ i = 1, \ldots, p. \end{cases}$$

(2)

The initial conditions are then $\Gamma_p$ as they are in the model $\Gamma_p$. A solution of $\Gamma_p$ is a triplet of functions $(x, y, u)$ which satisfy all the equations of the model. Thus, the solution of these equations may not be unique and some solutions may be degenerated. Therefore, the sets of non-degenerated solutions $X(p, u)$, $Y(p, u)$, corresponding to every possible initial condition, have to be taken into account in the identifiability definition. The notion of identifiability is strongly related to observability.

Here we adopt the definition introduced in Ljung and Glad (1994).

**Definition 3.1.** The model $\Gamma_p^{x_0}$ is globally (resp. locally) identifiable at $p^*$ with respect to $D_m \subseteq U_p$ if there exists a control $u$ such that, $Y(p^*, u) \neq \emptyset$ and $Y(p^*, u) \cap Y(p, u) \neq \emptyset$, $\bar{p} \in D_m \implies p^* = \bar{p}$ (resp. if there exists an open neighborhood $W$ of $p$ such that $\Gamma_{p^*}^{x_0}$ is globally identifiable at $p$ with $U_p$ restricted to $W$).

In most models, there exist atypical points in $U_p$ for which the model is unidentifiable. To account for these singularities, the previous definition can be generically extended.

**Definition 3.2.** $\Gamma_p^{x_0}$ is said to be globally (locally) structurally identifiable if it is globally (locally) identifiable at all $p \in U_p$ except at a subset of points of zero measure in $U_p$.

3.2 Case of uncontrolled models

In the case of uncontrolled models, the identifiability definitions are mainly taken from Denis-Vidal et al. (2001a). In this case, the identifiability can be stated as follows:

**Definition 3.3.** The model $\Gamma_p^{y_0}$ is globally (resp. locally) identifiable at $p^* \in U_p$ if for any $\bar{p} \in U_p$ (resp. if there exists an open neighborhood $W$ of $p^*$ such that for any $\bar{p} \in W$), $p^* \neq \bar{p}$, then the models $\Gamma_p^{y_0}$ and $\Gamma_{p^*}^{y_0}$ yield different outputs.

**Definition 3.4.** The model $\Gamma_p^{y_0}$ is globally (locally) structurally identifiable if it is globally (locally) identifiable at all $p \in U_p$ except at a subset of points of zero measure in $U_p$.

Methods to check global or local (structural) identifiability in the case of controlled or uncontrolled models can be found, for example, in Denis-Vidal et al. (2001b), Denis-Vidal et al. (2001a) or in Audoly et al. (2001), Margaria et al. (2004).

3.3 Interval identifiability

The notion of interval identifiability has been introduced by DiStefanoIII (1983) for compartmental models and has been generalized by Vajda et al. (1989b). These works concern the identifiability analysis of unidentifiable linear models of the form (Walter (1994)):

$$\begin{align*}
q(t, p) &= Kq(t, p) + Bu(t), \\
y(t, p) &= Cq(t, p), \\
y(0, p) &= 0,
\end{align*}$$

(3)

where $K = [k_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ are constant; $q(t, p) \in \mathbb{R}^m$ denotes the state vector, $u(t) \in \mathbb{R}^n$ is the input, $y(t, p) \in \mathbb{R}^m$ denotes the measured output, and $p \in \mathbb{R}^p$ is the unknown parameter vector of the model. The entries of $K$ (fractional transfer rates when $q_i$’s are masses) satisfy the following compartmental constraint relations for $i, j = 1, \ldots, n$:

$$k_{ij} \geq 0, \ i \neq j,$$

$$-k_{0i} = \sum_{j=1}^n k_{ij} \leq 0.$$

The model parameter vector $p$ is defined as all the unknown $k_{0i}$ and $k_{ij}, i \neq j$ and all unknown $b_{ij}$ and $c_{ij}$. The $k_{ij}$ components of $p$ are nonnegative, $b_{ij}$ and $c_{ij}$ are also usually nonnegative. Thus the a priori parameter domain $\mathbb{R}_+^p$ is defined by: $\mathbb{R}_+^p = \{ p : p_i \geq 0, \ i = 1, 2, \ldots, v \}$.

To consider the identifiability properties of $p$ at any particular (nominal) parameter value $p^* \in \mathbb{R}_+^p$, the dependence of the output $y$ on $p$ and the input $u$, denoted $y(t, p, u)$ is emphasized. If the model is unidentifiable, then for almost any $p^* \in \mathbb{R}_+^p$ there exists an uncountable subset $\Omega(p^*) \subset \mathbb{R}_+^p$ such that every $p \in \Omega(p^*)$ generates the same input-output behaviour, that is $y(t, \bar{p}, u) = y(t, p^*, u)$. The parameter interval strategy is based on the idea that in most compartmental models, the set $\Omega(p^*)$ is bounded for almost every $p^* \in \mathbb{R}_+^p$ where:

$$\Omega(p^*) = \{ \bar{p} \in \mathbb{R}_+^p, y(t, \bar{p}, u) = y(t, p^*, u) \ \forall u \}.$$  

(4)

Moreover, the inequalities thus implied for all components of $p$ localize unidentifiable parameters within finite intervals. The following definitions were proposed by Vajda et al. (1989b) to account for the above concept.

---

1 Without changing the definitions, their names have been adapted to be consistent with the other definitions of the paper.
Definition 3.5. The model described by (3) is interval-
identifiable at \( p^* \in \mathbb{R}^v_+ \) if it is unidentifiable at \( p^* \) and the set \( \Omega(p^*) \) defined by (4) is bounded.

Definition 3.6. The model described by (3) is (structurally) interval-identifiable if it is interval-identifiable for (almost) all \( p^* \in \mathbb{R}^v_+ \).

A bounded set \( \Omega(p^*) \) yields lower and upper bounds \( \underline{p}_i^{u_{\text{min}}} \) and \( \bar{p}_i^{u_{\text{max}}} \) on each \( \bar{p}_i \).

Local and global interval identifiability can be differentiated in the same way as local and global identifiability. Two methods to analyse (local) interval-identifiability can be found in Vajda et al. (1989b) or DiStefanoIII (1983). They are based on a transfer function approach or on a similarity transformation approach. In the following, \( \Omega(p^*) \) is supposed to be given by a finite union of connected sets.

4. SET-MEMBERSHIP IDENTIFIABILITY

This section proposes a formulation of the set-membership identifiability problem.

4.1 Definitions

Two definitions of global set-membership identifiability are provided, as well as their local counterpart. The first one is a conceptual definition, whereas the second one, relying on the definition of a measure \( \mu \), can be put in correspondence with operational set-membership estimation methods.

In these definitions, \( P \) remains a connected set of \( \mathbb{R}^p \) and \( Y(P; u) \) (respectively \( Y(P) \)) denotes the set of outputs, solution of \( \Gamma^p \) with the input \( u \) (resp. when \( u = 0 \)). In the case of controlled systems, the following definitions are given:

Definition 4.1. The model \( \Gamma^p \) given by (1) is globally set-membership identifiable for \( P^* \neq \emptyset \), \( P^* \subset U_P \) if there exists an input \( u \) such that \( Y(P^*, u) \neq \emptyset \) and \( Y(P^*, u) \cap Y(P, u) \neq \emptyset \), \( P \subset U_P \implies P^* \cap P \neq \emptyset \).

Let us now consider a bounded set \( \Pi \) of \( \mathbb{R}^p \), we then propose to use the diameter of \( \Pi \), noted \( \mu(\Pi) \). The diameter of \( \Pi \) is given by the least upper bound of \( \{d(\pi_1, \pi_2), \pi_1, \pi_2 \in \Pi\} \) and \( d \) is a classical metric on \( \mathbb{R}^p \) (Bourbaki (1989)). If \( \Pi \) is not bounded, we define \( \mu(\Pi) = +\infty \).

In the following definition, the set \( P^* \) is supposed bounded and \( \mu(P^*) \) is said as small as desired when it may be taken as tending to zero.

Definition 4.2. The model \( \Gamma^p \) given by (1) is globally \( \mu \)-set-membership identifiable for \( P^* \neq \emptyset \) with \( \mu(P^*) \) as small as desired, if there exists an input \( u \) such that \( Y(P^*, u) \neq \emptyset \) and \( Y(P^*, u) \cap Y(P, u) \neq \emptyset \), \( P \subset U_P \implies P^* \cap P \neq \emptyset \).

The above definition differs from definition 4.1 in the sense that the set \( P^* \) can be taken as small as desired. If the diameter of \( P^* \) remains at least equal to \( \varepsilon \), i.e., \( \mu(P^*) \geq \varepsilon \), then we refer to \( \varepsilon \)-set-membership identifiability. This definition will be shown to have practical importance in section 4.4.

To account for possible singularities in \( U_P \), \( \mu \)-set-membership identifiability can be generically extended into structural \( \mu \)-set-membership identifiability, which means that the model \( \Gamma^p_\mu \) is \( \mu \)-set-membership identifiable for all \( P \in U_P \) except at a subset of points of zero measure in \( U_P \).

4.2 Case of uncontrolled models

In the case of uncontrolled systems, the following definitions can be given.

Definition 4.5. The model \( \Gamma^p_1 \) given by (1) is globally set-membership identifiable for \( P^* \neq \emptyset \), \( P^* \subset U_P \) if \( Y(P^*) \neq \emptyset \) and \( Y(P^*) \cap Y(P) \neq \emptyset \), \( P \subset U_P \implies P^* \cap P \neq \emptyset \) (resp. if there exists an open neighbourhood \( W \) of \( P^* \) such that \( \Gamma^p_1 \) is globally set-membership identifiable for \( P^* \) with \( U_P \) restricted to \( W \)).

Definition 4.6. The model \( \Gamma^p_1 \) given by (1) is globally \( \mu \)-set-membership identifiable for \( P^* \neq \emptyset \) and \( \mu(P^*) \) as small as desired, if \( Y(P^*) \neq \emptyset \) and \( Y(P^*) \cap Y(P) \neq \emptyset \), \( P \subset U_P \implies P^* \cap P \neq \emptyset \) (resp. if there exists an open neighbourhood \( W \) of \( P^* \) such that \( \Gamma^p_1 \) is globally \( \mu \)-set-membership identifiable for \( P^* \) with \( U_P \) restricted to \( W \)).

4.3 Links between the different identifiability definitions

In this section, we give implications between the existing identifiability definitions recalled in section 3 and set-membership identifiability.
Proposition 4.2. If $\Gamma_{p,x}^{2}$ is (structurally) globally identifiable at $p^* \in P$ a connected set of UP then there exists a connected set $P^* \subset P$ belonging $p^*$ such that $\Gamma_{p}^{2}$ is (structurally) globally $\mu$-set-membership identifiable for $P^* \subset U_P$.

Proof – If the model $\Gamma_{p,x}^{2}$ is (structurally) globally identifiable at $p^*$ in $P^*$, there exists a control $u$ such that for any $\bar{p}$ different from $p^*$ in $U_P$, trajectories $Y(p^*, u)$ and $Y(\bar{p}, u)$ are different. Thus it is possible to choose a set of parameters $P^*, p^* \subset P^*$ with $\mu(P^*)$ as small as desired and such that for all set $\bar{P}$ belonging $\bar{p}$, we have $P^* \cap \bar{P} = \emptyset$ yielding to $Y(P^*, u) \cap Y(\bar{P}, u) = \emptyset$. Thus $\Gamma_{p,x}^{2}$ is (structurally) globally identifiable at $p^*$.

The same proposition holds for local properties.

Proposition 4.4. Local interval-identifiability of $\Gamma_{p}^{2}$ at $p^* \subset U_P$ implies non set-membership identifiability of $\Gamma_{p}^{2}$ for $P^* \subset U_P$, with $p^* \in P^*$.

Idea of proof – If the model $\Gamma_{p,x}^{2}$ is locally interval-identifiable at $p^* \in U_P$, it is always possible to choose two sets $P^*$ and $\bar{P}$ in $\Omega(p^*)$, with $p^* \subset P^*$ such that $P^* \cap \bar{P} = \emptyset$ and $Y(P^*, u) \cap Y(\bar{P}, u) \neq \emptyset$ by definition of the set $\Omega(P^*)$, which gives non set-membership identifiability.

Proposition 4.5. Interval-identifiability of $\Gamma_{p,x}^{2}$ does not guarantee set-membership identifiability of $\Gamma_{p}^{2}$ for $P^*$.

Idea of proof – The proof is a direct extension of the previous proposition. □

4.4 Correspondence with operational set-membership estimation methods

In this section, we first give a brief description of the algorithm SIVIA (Set Inversion Via Interval Analysis), which is at the basis of a classical set-membership parameter estimation method Jaulin and Walter (1993). We then exhibit an interpretation of $\mu$-set-membership identifiability. In SIVIA, the following tests are used: inclusion test and contractor. Given a subset $S$ of $\mathbb{R}^n$, we test if $[x]$ belongs to $S$, more precisely if $[x] \subset S$ or $[x] \cap S = \emptyset$. These tests are used to prove that all points in a given box satisfy a given property or to prove that none of them does.

The contraction of $[x]$ with respect to $S$ means that we search a smaller box $[\bar{x}]$ such that $[\bar{x}] \cap S = [\bar{x}] \cap S$. If $S$ is the feasibility set of a problem and $[\bar{x}]$ turns out empty, then the box $[x]$ may not contain the solution (Jaulin and Walter (1993)).

These operations are used to test if a box can or cannot be removed from the solution set. When no conclusion can be drawn, the box may be bisected and each of the sub-boxes can be tested in turn (this corresponds to branch-and-bound algorithms).

Description of SIVIA: consider the problem of determining a solution set for the unknown quantities $u$ defined by

\[
S = \{u \in U | \Phi(u) \in [y]\},
\]

where $[y]$ is known a priori, $U$ is an a priori search set for $u$ and $\Phi$ a nonlinear function not necessarily invertible in the classical sense. (5) involves computing the reciprocal image of $\Phi$. This can be solved using the algorithm SIVIA, which is a recursive algorithm that explores all the search space without loosing any solution. This algorithm makes it possible to derive a guaranteed enclosure of the solution set $S$ as follows:

\[
S \subseteq S \subseteq S.
\]

The inner enclosure $S$ is composed of the boxes that have been proved feasible. To prove that a box $[u]$ is feasible it is sufficient to prove that $\Phi([u]) \subseteq [y]$. Reversely, if it can be proved that $\Phi([u]) \cap [y] = \emptyset$, then the box $[u]$ is unfeasible. Otherwise, no conclusion can be reached and the box $[u]$ is said undetermined. The latter is then bisected in two sub-boxes that are tested until their size reaches a user-specified precision threshold $\epsilon > 0$. Such a termination criterion ensures that SIVIA terminates after a finite number of iterations.

Set-membership identifiability does not provides the means to control the set of interest $P^*$. This outlines the interest of the $\mu$-set-membership identifiability which is defined through a measure $\mu(\cdot)$, which allows one to control the diameter of $P^*$. In particular, $\mu$-set-membership identifiability is defined for the diameter of $P^*$ at least equal to $\epsilon$, i.e. $\mu(P^*) \geq \epsilon$. The diameter of $P^*$ can hence be put in correspondence with the user-specified precision threshold of SIVIA. Consequently, $\mu$-set-membership identifiability provides the means to guarantee that the estimate provided by SIVIA consists of a connected set when the precision threshold is taken equal to $\epsilon$.

5. METHODS TO ANALYSE SET-MEMBERSHIP IDENTIFIABILITY

In the literature, different approaches have been proposed for studying the global identifiability of nonlinear systems like the revisited Taylor Series approach: Joly-Blanchard and Denis-Vidal (1998), the local state isomorphism theorem: Walter and Lecourtier (1982), Chappell and Godfrey (1992), Chapman et al. (2003), Denis-Vidal and Joly-Blanchard (1996) and the differential algebra: Ljung and Glad (1994), Audoly et al. (2001), Saccomani et al. (2004), Verdière et al. (2005). Most of them can be adapted in order to test the global set-membership identifiability of a system. For example, two methods are proposed. The first one is based on the power series expansion of the solution. The second one is based on differential algebra. As it will be seen, some numerical methods can be deduced in order to obtain $P$ the set of parameters.

5.1 A sufficient condition to analyse set-membership identifiability for linear models

The first method is inspired of Pohjanpalo (1978) which studies the identifiability of linear and nonlinear models owing to the Taylor series expansion of the solution. In this paper, only linear models are considered. In this approach, the outputs are expanded in a Taylor series about $t = 0^+$ and the successive terms of the expansion being expressed as functions of the model unknowns. Successive derivatives at $t = 0^+$ are supposed to
be measurable and contain informations about the parameters to be identified. Afterwards, $y$ is supposed to be analytical and $y$ and its derivatives at 0 are supposed to be in a box.

The following theorem gives a necessary condition for having the global set-membership identifiability. Thus, consider now the case when $\Gamma^P_1$ is linear and given by the form:

$$
\Gamma^P_1 = \left\{ \begin{array}{ll}
\dot{y}(t,p) = A(p)x(t,p) + B(p)u(t), \\
y(t,p) = C(p)x(t,p) + D(p)u(t), \\
x(0,p) = x_0 \in X_0,
\end{array} \right.
\quad \forall p \in P \subset U_p, \\
0 \leq t \leq T,
$$

(7)

where $A(p)$, $B(p)$, $C(p)$ and $D(p)$ are matrices depending on $p$. $y$ is supposed analytical thus $y$ is entirely characterized by the value of its derivatives at 0 and the following theorem is obtained.

**Theorem 1.** $\Gamma^P_1$ is globally set-membership identifiable for $P^* \neq \emptyset$ if the system:

$$
\left\{ \begin{array}{l}
C(p)x_0 + D(p)u(0) = y(0,0), \\
C(p)(A^k(p)x_0 + \sum_{i=1}^{k} A^k(p)B(p)u^{i-1}(0)) + D(p)u^{(k)}(0) \\
= y^{(k)}(0,p), \quad k = 1, \ldots, +\infty
\end{array} \right.
$$

(8)

admits for only solution the connected set $P^*$.

**Proof.** Suppose that $P \subset U_p$ verifies $Y(P^*, u) \cap Y(P, u) = \emptyset$. Then there exists a trajectory $y \in Y(P^*, u) \cap Y(P, u)$. In particular, there exists $P^* \in P^*$, $p \in P$ solutions of (8). Since the solution set is a connected set, we have $P^* \cap P = \emptyset$. □

**Example 1:** Consider the following system taken from Vajda et al. (1989a):

$$
\begin{align*}
\dot{x}_1 &= -(k_{21} + k_{31})x_1 + u, \quad x_1(0) = 1, \\
\dot{x}_2 &= k_{21}x_1 - x_2, \quad x_2(0) = x_20, \\
\dot{x}_3 &= k_{31}x_1 - c_{13}x_3, \quad x_3(0) = 1, \\
y &= x_2 + c_{13}x_3,
\end{align*}
$$

(9)

where the unknown parameters are $k_{21}$, $k_{31}$, $c_{13}$ and $U_p = \mathbb{R}^3$.

In this example, from (8), one can deduce the following system:

$$
\begin{align*}
x_{20} + c_{13}y &= y(0), \\
k_{21} - x_{20} + c_{13}k_{21} - c_{13}x_1 &= y(0), \\
k_{21}(k_{21} - k_{31}) + x_{20} + c_{13}(k_{31}(k_{21} - k_{31}) - c_{13}k_{31}) \\
+ c_{13}^2 + c_{13}y &= y(0).
\end{align*}
$$

(10)

In substituting $k_{21}$ obtained with the second equation in the third equation, one gets:

$$
\begin{align*}
(x_{20} + c_{13}^2 + y(0))(c_{13} - 1) - c_{13}(-x_{20} - c_{13}y - y(0)) &+ c_{13}(-x_{20} + c_{13}^2 + y(0) - c_{13} + 1)k_{31} \\
+ (x_{20} + c_{13}^2 + y(0))(x_{20} + c_{13}y - y(0)) &+ x_{20} + c_{13}y - y(0).
\end{align*}
$$

(11)

For example, if $x_{20}$ is supposed to be in an interval and $0 \not\in (x_{20} + c_{13}^2 + y(0))(c_{13} - 1) - c_{13}(-x_{20} - c_{13}y - y(0)) + c_{13}(-x_{20} + c_{13}^2 + y(0) - c_{13} + 1))$, the system admits for solution an only connected set $P^*$ and according to the theorem, it is globally set-membership identifiable for $P^*$.

5.2 A sufficient condition to analyse set-membership identifiability for nonlinear models

In Verdière et al. (2005), the authors prove that in chosen the appropriate elimination order $\{p\} < \{y, u\} < \{x\}$ (which consists in eliminating unobservable state variables), the differential algebraic approach (Kolchin (1973)) allows to obtain relations between outputs and parameters. These relations can be expressed as:

$$
R_i(y, u, p) = \theta_0(y, u) + \sum_{k=1}^{n_i} \theta^*_k(p)m_k(y, u), \quad i = 1, \ldots, m,
$$

(12)

where $\theta^*_1 \ldots \emptyset \subset \emptyset$ are rational in $p$, $\theta^*_k \neq \emptyset_k$ ($u \neq v$), $(m_k)_1 \subset \emptyset$ are differential polynomials with respect to $y$ and $u$ and $\theta_0 \neq 0$.

The size of the system is the number of observations. For simplicity, we suppose that $i = 1$, that is there is one observation. Recall the definition of partial injectivity of a function, given in Lagrange et al. (2007).

**Definition 5.1.** Consider a function $f : A \rightarrow B$ and any set $A_1 \subseteq A$. The function $f$ is said to be a partial injection of $A_1$ over $A$, noted $(A_1, A)$-injective, if $\forall a \in A_1, \forall a \in A$.

$$
a_1 \neq a \Rightarrow f(a_1) \neq f(a).
$$

$f$ is said to be $A$-injective if it is $(A, A)$-injective.

The following theorem which was inspired by one found in Denis-Vidal et al. (2001b) allows to obtain sufficient conditions for global set-membership identifiability for $P^*$. 

**Theorem 2.** If $\forall (y, u), \quad \Delta(R)(y, u) = det(m_k(y, u), k = 1, \ldots, n) \neq 0$, then $\Gamma^P_1$ is globally set-membership identifiable for $P^*$ if the function $\Phi : p \rightarrow (\theta_1(p), \ldots, \theta_n(p))$ is $(P^*, P^*)$-injective.

**Proof.** Suppose there exists an input $u^*$ such that $Y(P^*, u^*) \neq \emptyset$ and $y^* \in Y(P^*, u^*) \cap Y(P, u^*)$ for a cartesian product of intervals $P \subset U_p$. Thus, one gets:

$$
\exists p^* \in P^*, \exists p \in P, \quad R(y^*, u^*, p^*) = R(y^*, u^*, p).
$$

Denote $Q(y^*, u^*) = R(y^*, u^*, p^*) - R(y^*, u^*, p)$. Since $det(Q)(y^*, u^*) = det(m_k(y^*, u^*), l = 1, \ldots, n) = \Delta(R)(y^*, u^*)$ is not equal to zero, $\theta_k(p^*) = \theta_k(p)$ for $k = 1, \ldots, n$. Besides, the function $\Phi$ is supposed to be $(P^*, P^*)$-injective. Thus, $P^* \cap P = \emptyset$. □

**Example 2:** Consider the model:

$$
\begin{align*}
\dot{x}_1 &= p_2x_1 + p_1x_2, \\
\dot{x}_2 &= p_1p_2x_1 + 2u, \\
y &= x_2.
\end{align*}
$$

(13)

According to the theorem 2, this model is set-membership identifiable for $P^* = [0, +\infty[$. Indeed, the package diffalg of Maple gives the following input-output polynomial:

$$
R(y, u) = y^3 - y^2 + yu - y^2 + p_1^2p_2 y^3 + p_2(yu - y^2).
$$

Consider the functional determinant:

$$
\Delta(R)(y, u) = y^3 - 3yp_2y^2 - y^2 - yu + y^2 \neq 0.
$$

(14)

We obtain $\Delta(R)(y, u) = 2y^3 - 2y^3yu - y^3 + y^2u$. This determinant is not identically equal to zero. Thus the first hypothesis is checked. Furthermore, one can easily verify that the function $\Phi : (p_1, p_2) \rightarrow (p_1^2p_2, p_2)$ is $(P^*, P^*)$-injective. Finally, one can deduce that the model is set-membership identifiable for $P^*$.

6. CONCLUSION

This paper introduces the original notion of set-membership identifiability and its operational counterpart of $\mu$-set-member-
ship identifiability. These notions provide a way to study different aspects of identifiability for uncertain bounded-error systems, in particular systems that represent an infinite family of nonlinear systems. Existing links between these two definitions and the other identifiability definitions of the literature have been provided. One of the advantages of set-membership identifiability is that it allows one to bypass standard identifiability and to give (set) estimates of parameters that are unidentifiable. Set-membership identifiability indeed guarantees that there exists a partition of the parameter space into connected subsets so that every subset can be associated with a distinguishable output behavior. The study of identifiability for uncertain bounded-error systems has a significant impact in practice. It has a role to play in many practical problems, for example related to diagnosis and prognosis in uncertain environments. Identifiability is indeed closely related to diagnosability as it provides the guaranty that two situations corresponding to different parameterized settings are distinguishable. Referring to prognosis, parameter estimation for models of wear is a real challenge when uncertainties are in the middle. Error-bounded methods have considerable power in this context. These two applications are in our intended scope for future work.

REFERENCES