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On indefinite damping and gyroscopic stabilization

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Abstract:
We consider linear vibrational systems with positive definite stiffness matrix K and indefinite damping matrix D. For the system to be stabilizable by gyroscopic forces it is necessary that both the trace of D and the trace of $K^{-1}D$ is positive. In the present note we discuss sufficiency of this condition.

1. INTRODUCTION

Consider a vibrational mechanical system of the form

$$\ddot{x} + D\dot{x} + Kx = 0\,, \tag{1}$$

with indefinite damping matrix D and positive definite stiffness matrix K. In general, this system possesses both stable and unstable modes and thus is unstable. It is the idea of *gyroscopic stabilization* to mix these modes so that the system is stabilized, without introducing further dissipation. This is done by adding gyroscopic forces $G\dot{x}$ with a suitable skew-symmetric matrix G to the left-hand side (see e.g. L. Barkwell [1992]).
Negative damping appears in models with friction induced vibrations, see Brommundt [1995]) or Popp and Rudolph [2003]). A good overview on mechanisms that create friction induced vibrations and shows counter measures to avoid these vibrations can be found in Popp, Rudolph, Kroeger and Lindner [2004]).

Definition 1. We call $G = -G^T \in \mathbb{R}^{n\times n}$ a *gyroscopic stabilizer* for system (1), if

$$\ddot{x} + (D+G)\dot{x} + Kx = 0 \tag{2}$$

is asymptotically stable. In this case the system is *gyroscopically stabilizable.*

A well-known necessary condition for gyroscopic stabilizability (which we recall below) is that the traces of D and $K^{-1}D$ are both positive. In the recent paper Kliem and Pommer [2009], a Lyapunov matrix equation approach ist used. The authors ask whether this condition is also sufficient. While in the case $n = 2$ they give an affirmative answer, for $n > 2$ gyroscopic stabilizability so far has only been shown under additional conditions.

2. PRELIMINARIES: PROJECTIONS AND TRACES

By $\operatorname{tr} A$ we denote the trace of a square matrix. It is well-known that $\operatorname{tr} BC = \operatorname{tr} CB$ if the product BC is a square matrix. Hence, if $U = [u_1, \dots, u_n] \in \mathbb{C}^{n\times n}$ is unitary then $\operatorname{tr} A = \operatorname{tr} AUU^* = \operatorname{tr} U^*AU = \sum_{j=1}^n u_j^* A u_j$.

More generally, if $U = [u_1, \dots, u_k] \in \mathbb{C}^{n\times k}$ has orthonormal columns, we write $\mathcal{U} = \operatorname{Span}\{u_1, \dots, u_k\}$ and $P_{\mathcal{U}} = UU^*$ for the orthogonal projection onto $\mathcal{U}$. Then $P_U A P_U$ is the projection of A to $\mathcal{U}$ and

$$\operatorname{tr}_{\mathcal{U}} A := \operatorname{tr}(P_U A P_U) = \operatorname{tr}(U^*AU) = \sum_{j=1}^k u_j^* A u_j$$

is the trace of the projected matrix. It is important that $\operatorname{tr}_{\mathcal{U}}$ depends continuously on $\mathcal{U}$, or, equivalently, on the orthogonal projector $P_{\mathcal{U}}$.

A matrix $D = D^T \in \mathbb{R}^{n\times n}$ is positive definite, if $\operatorname{tr}_{\mathcal{U}} D > 0$ for all non-zero subspaces $\mathcal{U} \subset \mathbb{C}^n$. If D is indefinite, there exists a vector $u \in \mathbb{R}^n$ with $u^T D u = 0$.

If G is skew-symmetric then $\operatorname{tr} G = 0$. Moreover, if $P = QQ^T$ is positive definite then also $\operatorname{tr} PG = \operatorname{tr} Q^T G Q = 0$.

Note that the eigenvalues of a skew-symmetric matrix G are either zero or complex conjugate pairs of purely imaginary numbers and a set of eigenvectors of a skew-symmetric matrix can be chosen as an orthonormal basis of $\mathbb{C}^n$, where the complex eigenvectors come in conjugate pairs as well. If v, w is any pair of normalized complex conjugate orthogonal vectors, we have

$$\begin{aligned} v^*Dv = w^*Dw &= \frac{1}{2}\operatorname{tr}_{\operatorname{Span}\{v,w\}} D \\ &= \frac{1}{2}\operatorname{tr}_{\operatorname{Span}\{b_1,b_2\}} D = \frac{1}{2}(b_1^*Db_1 + b_2^*Db_2) \end{aligned} \tag{3}$$

for any orthonormal basis $\{b_1, b_2\}$ of $\operatorname{Span}\{v, w\}$ and any matrix D of suitable size, a fact we use freqently.

3. A NECESSARY CONDITION

The second-order system (2) can be written in first order form as

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D-G \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A_G \begin{bmatrix} x \\ \dot{x} \end{bmatrix}\,. \tag{4}$$

It is asymptotically stable if and only if $\sigma(A_G) \subset \mathbb{C}_-$, which implies $\operatorname{tr} A_G < 0$, i.e. $\operatorname{tr} D > 0$. Moreover, since $\sigma(A_G) \subset \mathbb{C}_-$ if and only if $\sigma(A_G^{-1}) \subset \mathbb{C}_-$, where

$$A_G^{-1} = \begin{bmatrix} -K^{-1}(D+G) & -K^{-1} \\ I & 0 \end{bmatrix},$$

we conclude that also $0 < \operatorname{tr} K^{-1}(D+G)$. Let Q denote the positive definite square root of K^{-1}. Then $\operatorname{tr} K^{-1}G = \operatorname{tr} QGQ = 0$, since QGQ is skew-symmetric. It follows $0 < \operatorname{tr} K^{-1}D$. These *necessary* criteria are well known (e.g. Kliem and Müller [1997], Müller [1971]). To analyze *sufficiency* we first reformulate the gyroscopic stabilization problem as an inverse eigenvector problem.

4. AN INVERSE EIGENVECTOR PROBLEM

The matrix A_G is asymptotically stable if and only if the matrix

$$-A_G - A_G^{-1} = \begin{bmatrix} K^{-1}(D+G) & K^{-1} - I \\ K - I & D+G \end{bmatrix}$$

is positive stable (i.e. has all eigenvalues in $\mathbb{C}_+$). A similarity transformation with $T = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$ brings $-(A_G + A_G^{-1})$ to the form

$$-T^{-1}(A_G + A_G^{-1})T = \begin{bmatrix} Q(D+G)Q & Q - Q^{-1} \\ Q^{-1} - Q & D+G \end{bmatrix}$$
$$= \begin{bmatrix} QGQ & 0 \\ 0 & G \end{bmatrix} + \begin{bmatrix} QDQ & Q - Q^{-1} \\ Q^{-1} - Q & D \end{bmatrix} \tag{5}$$

By a perturbation argument we can formulate a stabilizability criterion as an inverse eigenvector problem.

Proposition 2. Let $\tau > 0$ and $D_\tau = D - \tau I$.
For system (2) to be gyroscopically stabilizable it is sufficient that there exists a skew-symmetric matrix $G = -G^T$ with the following properties:

(a) Both G and QGQ only have simple eigenvalues.
(b) If v is an eigenvector of G then $v^* D_\tau v \geq 0$.
(c) If w is an eigenvector of QGQ then $w^* Q D_\tau Q w \geq 0$.

Proof. Instead of (5) consider the matrix

$$M_\varepsilon = \begin{bmatrix} QGQ & 0 \\ 0 & G \end{bmatrix} + \varepsilon \begin{bmatrix} QDQ & Q - Q^{-1} \\ Q^{-1} - Q & D \end{bmatrix}.$$

We show that for small $\varepsilon > 0$ this matrix is positive stable, which implies that $\varepsilon^{-1}G$ is a gyroscopic stabilizer.

Note that all eigenvalues of M_ε are perturbations of the (imaginary) eigenvalues of M_0. We will show that for each eigenvalue $\lambda_0 \in \sigma(M_0) = \sigma(G) \cup \sigma(QGQ) \subset i\,\mathbb{R}$ of multiplicity k the perturbed matrix M_ε has k eigenvalues with positive real part in a neighbourhood of λ_0.

Case 1 Assume $\lambda_0 \in \sigma(G) \setminus \sigma(QGQ)$.

Then G has an eigenvector $v \in \mathbb{C}^n$, so that $\|v\| = 1$ and $Gv = \lambda_0 v$. Condition (a) implies that λ_0 is a simple eigenvalue of M_0. A unit eigenvector of M_0 is given by $v_0 = [0, v]^T$. For small $\varepsilon > 0$ a standard perturbation result (e.g. [Stewart and Sun, 1990, Thm. IV 2.3]) gives that M_ε has a simple eigenvalue $\lambda_\varepsilon = \lambda_0 + \varepsilon v^* D v + \mathcal{O}(\varepsilon^2)$. Since $v^* D v > 0$ by (b), we have $\lambda_\varepsilon \in \mathbb{C}_+$.

Case 2 Assume $\lambda_0 \in \sigma(QGQ) \setminus \sigma(G)$.

For a corresponding unit eigenvector $w_0 = [w, 0]^T$ of M_0, an analogous argument as in the first case shows that M_ε has a simple eigenvalue $\lambda_\varepsilon = \lambda_0 + \varepsilon w^* QDQw + \mathcal{O}(\varepsilon^2) \in \mathbb{C}_+$.

Case 3 Assume $\lambda_0 \in \sigma(QGQ) \cap \sigma(G)$.

Then λ_0 is a double eigenvalue of M_0. The corresponding two-dimensional invariant subspace is spanned by vectors v_0 and w_0 as in the first two cases. For small $\varepsilon \geq 0$ the perturbed matrix M_ε also has a two-dimensional invariant subspace, which depends smoothly on ε and coincides with $\operatorname{Span}\{v_0, w_0\}$ for $\varepsilon = 0$. The restriction of M_ε to this subspace has the representation (e.g. [Stewart and Sun, 1990, Thm. V 2.8])

$$[v_0, w_0]^* M_\varepsilon [v_0, w_0] + \mathcal{O}(\varepsilon^2)$$
$$= \lambda_0 I + \begin{bmatrix} v^* D v & v^*(Q^{-1} - Q)w \\ w^*(Q - Q^{-1})v & w^* QDQw \end{bmatrix} + \mathcal{O}(\varepsilon^2).$$

The 2×2-matrix in the previous term is positive stable, since it has positive trace and positive determinant. Thus M_ε has two positive stable eigenvalues (counting multiplicity) in a neighbourhood of λ_0. □

If $\frac{\operatorname{tr} D}{n} \leq \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$ we consider $\tau = \frac{\operatorname{tr} D}{n}$, so that $\operatorname{tr} D_\tau = 0$ and $\operatorname{tr} QD_\tau Q \geq 0$. Otherwise, let $\tau = \frac{\operatorname{tr} QDQ}{\operatorname{tr} Q^2}$, and we have $\operatorname{tr} D_\tau \geq 0$ and $\operatorname{tr} QD_\tau Q = 0$.
In the latter case, we put $\tilde{D}_\tau = QD_\tau Q$ and $\tilde{Q} = Q^{-1}$. Suppose there exists a skew-symmetric $\tilde{G}$ such that $\tilde{v}_i^* \tilde{D}_\tau \tilde{v}_i = 0$ for any orthonormal set of pairwise complex conjugate eigenvectors $\tilde{v}_i$ of $\tilde{G}$ and $\tilde{w}_i^* \tilde{Q} \tilde{D}_\tau \tilde{Q} \tilde{w}_i \geq 0$ for any orthonormal set of pairwise complex conjugate eigenvectors $\tilde{w}_i$ of $\tilde{Q}\tilde{D}\tilde{Q}$.
Then for the skew-symmetric $G = Q^{-1}\tilde{G}Q^{-1}$ we have $v_i^* D v_i \geq 0$ for any orthonormal set of pairwise complex conjugate eigenvectors $v_i = \tilde{w}_i$ of G and $w_i^* QDQ w_i \geq 0$ for any orthonormal set of pairwise complex conjugate eigenvectors $w_i = \tilde{v}_i$ of QDQ.
Hence it suffices to consider $\operatorname{tr} D_\tau = 0$ and $\operatorname{tr} QD_\tau Q \geq 0$.

Problem 3. For symmetric matrices $D, Q \in \mathbb{R}^{n \times n}$ satisfying $Q > 0$, $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$ find $G = G^T$ so that (a), (b), (c) in Prop. 2 hold for $\tau = 0$.

Note that the condition $\operatorname{tr} D = 0$ implies the existence of a unit vector $u \in \mathbb{R}^n$ satisfying $u^T D u = 0$, a fact, we will use repeatedly. Note further that in the case that Q is the identity matrix Problem 3 has been solved e.g. in Crauel et al. [2007]. Hence, from now on we assume that Q is not a multiple of the unit matrix.

5. THE 3-DIMENSIONAL CASE

Proposition 4. Let $D, Q \in \mathbb{R}^{3 \times 3}$ with $Q > 0$ and $\operatorname{tr} D = 0$. Choose $\omega \in \mathbb{R} \setminus \{0\}$ and an orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ so that $u_1^* D u_1 = 0$. Then

$$G = \begin{bmatrix} u_1, u_2, u_3 \end{bmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix} \begin{bmatrix} u_1, u_2, u_3 \end{bmatrix}^T$$

solves Problem 3.

Proof. By construction, $v_1 = u_1$ is an eigenvector for the eigenvalue 0 of G. Since G is skew-symmetric, the eigenvectors are orthogonal and the eigenvectors v_2, v_3 for

the imaginary eigenvalues can be assumed to be complex conjugate and normalized.
By (3) it suffices to show that $\operatorname{tr}_{\operatorname{Span}\{v_2,v_3\}} D \geq 0$. From $\operatorname{Span}\{v_2, v_3\} = \operatorname{Span}\{u_2, u_3\}$ it follows that

$$\begin{aligned}\operatorname{tr}_{\operatorname{Span}\{v_2,v_3\}} D &= \operatorname{tr}_{\operatorname{Span}\{u_2,u_3\}} D \\ &= \operatorname{tr} D - \operatorname{tr}_{\operatorname{Span}\{u_1\}} D = 0 \ .\end{aligned}$$

Since $QGQQ^{-1}v_1 = 0$, it follows that $w_1 = \frac{Q^{-1}v_1}{\|Q^{-1}v_1\|}$ is an eigenvector for the eigenvalue 0 of QGQ. Since $w_1^* QDQw_1 = 0$, we have $\operatorname{tr}_{\operatorname{Span} w_1} D = 0$. Let w_2, w_3 denote the other eigenvectors of QDQ. We use (3) again and get

$$\begin{aligned}\operatorname{tr}_{\operatorname{Span}\{w_2,w_3\}} QDQ &= \operatorname{tr} QDQ - \operatorname{tr}_{w_1} QDQ \\ &= \operatorname{tr} QDQ \quad \geq 0 \ ,\end{aligned}$$

which completes the proof. □

6. THE 4-DIMENSIONAL CASE

In the 3-dimensional case, we exploited the fact that the skew-symmetric matrices $G, QGQ \in \mathbb{R}^{3\times 3}$ both have a zero eigenvalue, and the corresponding eigenvectors are related via multiplication with Q^{-1}. Now we construct $G \in \mathbb{R}^{4\times 4}$ with a double eigenvalue zero, allowing us to identify spaces containing eigenvectors of QGQ. Then we use a perturbation argument to move the zero eigenvalues along $i\,\mathbb{R}$.

Proposition 5. For $\delta \in \mathbb{R}$ and an orthogonal matrix $Z = [z_1, z_2, z_3, z_4] \in \mathbb{R}^{4\times 4}$ set

$$G_\delta = Z \begin{pmatrix} 0 & \delta & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} Z^T \qquad (6)$$

If (b), (c) in Prop. 2 hold for $G = G_0$ and some $\tau > 0$, then there exists $\delta \neq 0$ so that (a), (b), and (c) also hold for G_δ and $\tau/2$.

Proof. By continuity of eigenvalues, it is clear that (a) holds for small $|\delta| \neq 0$.
Using (3) and the structure of G_δ we conclude that assumption (b) is equivalent to $\operatorname{tr}_{\operatorname{Span}\{z_1,z_2\}} D_\tau \geq 0$ and $\operatorname{tr}_{\operatorname{Span}\{z_3,z_4\}} D_\tau \geq 0$, independently of $\delta \in \mathbb{R}$.
To verify (c), note that $QG_\delta Q$ has two conjugate pairs of imaginary eigenvalues, which we denote by $\pm\lambda_\delta$ and $\pm\mu_\delta$. These depend continuously on δ (where $\lambda_0 = 0$ and $\mu_0 = i$). The same is true (e.g. Stewart and Sun [1990]) for the invariant subspaces

$$\begin{aligned}\mathcal{V}_\lambda(\delta) &:= \operatorname{Ker}\left((QG_\delta Q)^2 + |\lambda_\delta|^2 I\right) \ , \\ \mathcal{V}_\mu(\delta) &:= \operatorname{Ker}\left((QG_\delta Q)^2 + |\mu_\delta|^2 I\right) \ .\end{aligned}$$

By assumption for $\delta = 0$ and $\eta = \tau$ we have

$$\begin{aligned}\operatorname{tr}_{\mathcal{V}_\lambda(\delta)} QDQ &\geq \eta \operatorname{tr}_{\mathcal{V}_\lambda(\delta)} Q^2 > 0 \ , \\ \operatorname{tr}_{\mathcal{V}_\mu(\delta)} QDQ &\geq \eta \operatorname{tr}_{\mathcal{V}_\mu(\delta)} Q^2 > 0 \ .\end{aligned}$$

By continuity, the same holds for $\eta = \tau/2$ and sufficiently small δ. Together with (3) this completes the proof. □

Thus, we can relax the conditions in Problem 3 slightly.

Problem 6. For symmetric matrices $D, Q \in \mathbb{R}^{4\times 4}$ satisfying $Q > 0$, $\operatorname{tr} D = 0$ and $\operatorname{tr} QDQ \geq 0$, find G_0 as in (6) so that $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$ and $\operatorname{tr} QDQ \geq \operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ \geq 0$.

Let $\{u_1, u_2, u_3, u_4\}$ denote an orthonormal set of eigenvectors of Q with corresponding eigenvalues $\lambda_k > 0$. We consider the numbers $u_i^* Du_i$. Since $\operatorname{tr} D = 0$ we either have $u_i^* Du_i = 0$ for all i or some of these numbers are positive and some are negative. In the following propositions, we make a complete distinction between all possible cases.

Proposition 7. Assume that for some ordering of the u_i the matrices

$$K_1 = \begin{pmatrix} u_1^* Du_1 & u_1^* Du_2 \\ u_2^* Du_1 & u_2^* Du_2 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} u_3^* Du_3 & u_3^* Du_4 \\ u_4^* Du_3 & u_4^* Du_4 \end{pmatrix}$$

are each either indefinite or singular. Then there exists a skew-symmetric G_0 solving Problem 6.

Proof. By our assumptions on K_1 and K_2, there exist normalized vectors $z_1 \in \operatorname{Span}\{u_1, u_2\}$ and $z_2 \in \operatorname{Span}\{u_3, u_4\}$ with $z_1^* Dz_1 = z_2^* Dz_2 = 0$. Let $Z = \left[z_1, z_2, z_3, z_4\right] \in \mathbb{R}^{4\times 4}$ be orthogonal and define G_0 as in (6). Then $\operatorname{Span}\{z_1, z_2\} = \operatorname{Ker} G_0$ and $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$.

Again by construction, $\{Q^{-1}z_1, Q^{-1}z_2\}$ is an orthogonal basis of $\operatorname{Ker} QG_0Q$ and

$$z_1^* Q^{-1} QDQQ^{-1} z_1 = z_2^* Q^{-1} QDQQ^{-1} z_2 = 0 \ .$$

Hence $\operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ = 0$, i.e. G_0 solves Problem 6. □

Note, that the assumptions of Prop. 7 may only fail, if three of the numbers $u_i^* Du_i$ are positive and one is negative, or vice versa.

Proposition 8. Assume that $u_i^* Du_i < 0$ for exactly one fixed $i \in \{1, 2, 3, 4\}$ and $u_k^* Du_k > 0$ for all $k \neq i$.

Assume further that

$$K_{mn} = \begin{pmatrix} u_m^* Du_m & u_m^* Du_n \\ u_n^* Du_m & u_n^* Du_n \end{pmatrix}$$

be nonnegative definite for every choice of distinct $m, n \in \{1, 2, 3, 4\}\backslash\{i\}$.
Then there exists a skew-symmetric G_0 solving Problem 6.

Remark 9. We can fix j, m, n arbitrarily provided that $\{i, j, m, n\} = \{1, 2, 3, 4\}$. Thus we may assume $\lambda_j, \lambda_m, \lambda_n$ to be ordered arbitrarily.
For any such choice, $K_{ij} = \begin{pmatrix} u_i^* Du_i & u_i^* Du_j \\ u_j^* Du_i & u_j^* Du_j \end{pmatrix}$ necessarily is indefinite. If K_{mn} was not nonnegative definite, then it would be indefinite or singular, and we could apply Prop. 7.

Proof. We denote the eigenvalues of K_{ij} by μ_i, μ_j and those of K_{mn} by μ_m and μ_n, where in accordance with our assumptions $\mu_i < 0$, $\mu_j > 0$, and $\mu_n \geq \mu_m \geq 0$. Since $\operatorname{tr} D = \mu_i + \mu_j + \mu_m + \mu_n = 0$, we have $\mu_i \leq -\mu_m - \mu_n$. Thus $[\mu_j, \mu_i] \supset [-\mu_m, -\mu_n]$, i.e. (e.g. [Bhatia, 1997, Ex. I.2.9])

$$\{x_1^* K_{ij} x_1 \mid \|x_1\| = 1\} \supset \{-x_2^* K_{mn} x_2 \mid \|x_2\| = 1\} \ .$$

Thus, for each normalized $z_2 \in \operatorname{Span}\{u_m, u_n\}$ there is a normalized $z_1 = z_1(z_2) \in \operatorname{Span}\{u_i, u_j\}$ so that

$$z_1^* D z_1 = -z_2^* D z_2 \ . \tag{7}$$

We can choose $z_1 = f(\alpha) = \cos(\alpha)\tilde{u}_i + \sin(\alpha)\tilde{u}_j$ with $\alpha \in [0, \pi/2]$, where $\tilde{u}_i, \tilde{u}_j$ are orthonormal and $[\tilde{u}_i\, \tilde{u}_j]^* D [\tilde{u}_i\, \tilde{u}_j] = \operatorname{diag}(\mu_i\, \mu_j)$. Then the mapping $g : \alpha \mapsto f(\alpha)^* D f(\alpha)$ is continuous and strictly monotonically increasing and therefore continuously invertible. Since the mapping $z_2 \mapsto z_2^* D z_2$ is also continuous, we can assume the mapping $z_2 \mapsto z_1(z_2) = z_1(g^{-1}(z_2^* D z_2))$ to be continuous.

We now consider three different cases.
(i) Assume $\lambda_i = \max_k \lambda_k$. Since

$$\begin{aligned} 0 \le \operatorname{tr} QDQ &= \sum_{k=1}^{4} u_k^* QDQ u_k \\ &= \sum_{k=1}^{4} \lambda_k^2 u_k^* D u_k \le \lambda_i^2 \sum_{k=1}^{4} u_k^* D u_k = 0 \ , \end{aligned}$$

it follows $\lambda_i = \lambda_k$ for all k, i.e. $Q = \lambda_k I$. But this case was excluded at the end of Sec. 4.
(ii) Let $\min_k \lambda_k < \lambda_i < \max_k \lambda_k$ and assume without loss of generality that $\lambda_m \le \lambda_i, \lambda_j \le \lambda_n$. Then

$$\begin{aligned} \lambda_m^{-1} &= \|Q^{-1} u_m\| \ge \|Q^{-1} z_1(u_m)\| \ , \\ \lambda_n^{-1} &= \|Q^{-1} u_n\| \le \|Q^{-1} z_1(u_n)\|. \end{aligned}$$

By the mean value theorem there exists a normalized $z_2 = \cos(\beta) u_m + \sin(\beta) u_n \in \operatorname{Span}\{u_m, u_n\}$ so that

$$\|Q^{-1} z_2\| = \|Q^{-1} z_1(z_2)\| \ . \tag{8}$$

We extend $z_1 = z_1(z_2)$ and z_2 to an orthogonal matrix $Z = [z_1, \ldots, z_4]$ and define G_0 as in (6).
Then $\operatorname{Span}\{z_1, z_2\} = \operatorname{Ker} G_0$ and $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$. Moreover, $\{Q^{-1} z_1, Q^{-1} z_2\}$ is an orthogonal basis of $\operatorname{Ker} QG_0Q$ and (using (7) and (8)) we have

$$\operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} = 0 \ .$$

Hence G_0 solves Problem 6.

(iii) Let $\lambda_i = \min_k \lambda_k$ and assume $\lambda_m \ge \lambda_j$.
Let $z_2 = u_m$ and $z_1 = z_1(z_2) \in \operatorname{Span}\{u_i, u_j\}$. Then

$$\lambda_i^{-2} \ge \|Q^{-1} z_1\|^2 \ge \lambda_j^{-2} \ge \lambda_m^{-2} = \|Q^{-1} z_2\|^2 \ .$$

With G_0 again as in (6), we have $\operatorname{tr}_{\operatorname{Ker} G_0} D = 0$ and

$$\operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} \ge 0 \ ,$$

because $-z_1^* D z_1 = z_2^* D z_2 = u_m^* D u_m$.

On the other hand

$$\begin{aligned} \operatorname{tr}_{\operatorname{Ker} QG_0Q} QDQ &= \tfrac{z_1^* D z_1}{\|Q^{-1} z_1\|} + \tfrac{z_2^* D z_2}{\|Q^{-1} z_2\|} \\ &\le u_m^* D u_m (\lambda_m^2 - \lambda_i^2) \\ &\le \sum_{k=1}^{4} \lambda_i^2 u_k^* D u_k = \operatorname{tr} QDQ \ . \end{aligned}$$

Again G_0 solves Problem 6. □

Finally we consider the case, where three of the numbers $u_i^* D u_i$ are negative and one is positive.

Proposition 10. Assume that $u_i^* D u_i > 0$ for exactly one fixed $i \in \{1, 2, 3, 4\}$ and $u_k^* D u_k < 0$ for all $k \ne i$.

Assume further that

$$K_{mn} = \begin{pmatrix} u_m^* D u_m & u_m^* D u_n \\ u_n^* D u_m & u_n^* D u_n \end{pmatrix}$$

be nonpositive definite for every choice of distinct $m, n \in \{1, 2, 3, 4\} \backslash \{i\}$.
Then there is a skew-symmetric G_0 solving Problem 6.

Proof. The proof is analogous to the proof of Prop. 8 and omitted for brevity. □

Since our distinction of cases is complete we have proven the following result.

Theorem 11. Let D and $Q > 0$ be in $\mathbb{R}^{4\times 4}$ with $\operatorname{tr} D, \operatorname{tr} QDQ > 0$. Then there exists a gyroscopic stabilizer.

7. NUMERICAL EXAMPLE

Consider the system given by

$$0 = \ddot{x} + \begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{x} + \begin{pmatrix} 4 & -1 & -3 \\ -1 & 6 & 1 \\ -3 & 1 & 4 \end{pmatrix}^{-2} x \ .$$

The system is unstable as can be seen by the eigenvalues of the first order representation as in (4), which are $-4.9936, 1.7451, 0.1156 \pm i0.7483, 0.0228, -0.0055$. As pointed out in section 3, we perform the diagonal shift $D \mapsto \tilde{D} = D - I$, compute the positive definite square root Q of K^{-1}, and construct a matrix G as in proposition 4. A suitable choice for a vector u_1 with $u_1^* D u_1 = 0$ is $u_1 = \sqrt{\frac{12}{7}} (\frac{1}{\sqrt{3}}\ \frac{1}{2}\ 0)^T$. We complete u_1 to an orthonormal basis of $\mathbb{R}^3$ and define with

$$Z = \begin{pmatrix} \frac{2}{\sqrt{7}} & \sqrt{\frac{3}{7}} & 0 \\ \sqrt{\frac{3}{7}} & -\frac{2}{\sqrt{7}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad G_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

the matrix $G_1 = Z G_0 Z'$. Now G_1 solves Problem 3, and for a small ϵ the system

$$\ddot{x} + (D + \tfrac{1}{\epsilon} G_1) \dot{x} + Kx = 0 \tag{9}$$

is stable. With $\frac{1}{\epsilon} = 10$, the system has eigenvalues $-0.9161 \pm i9.3691, -0.7954, -0.3692, -0.0016 \pm i0.0046$.
Alternatively, if we put

$$\hat{Z} = \begin{pmatrix} 0 & \frac{2}{\sqrt{7}} & \sqrt{\frac{3}{7}} \\ 0 & \sqrt{\frac{3}{7}} & -\frac{2}{\sqrt{7}} \\ 1 & 0 & 0 \end{pmatrix} \ ,$$

then with $G_2 = \hat{Z} G_0 \hat{Z}'$ and the same epsilon as before, the system has eigenvalues $-1.4861 \pm i9.3944, -0.0130 \pm i0.7139, -0.0009 \pm i0.0036$.
With G_1, the largest real part of the eigenvalues of system (9) is smaller than with G_2. This suggests that, even

though the eigenvalues of G_1 and G_2 are identical, there is a qualitative disparity between them. Also a change in the factor ϵ results in different eigenvalues of the system. A definition of optimality for a pair (ϵ, G) and the formulation of a criterion for optimality remains an open problem for further investigation.

8. CONCLUSION

Using perturbation arguments on eigenvalues, we showed that in $\mathbb{R}^3$ and $\mathbb{R}^4$ the conditions $\operatorname{tr} D > 0$ and $\operatorname{tr} K^{-1}D > 0$ are not only necessary, as pointed out by Kliem and Pommer [2009], but also sufficient for gyroscopic stabilizability of the system

$$\ddot{x} + D\dot{x} + Kx = 0 \ .$$

Our hope is that the methods we developed can inductively be extended in order to show sufficiency in higher space dimensions.

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