Oscillations in a Decentralized Economic Network

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Abstract: The study of exchange markets dates back to Leon Walras’s general equilibrium theory. Since then the economic market has been studied for its’ equilibrium properties, fairness of allocations of private and public goods, and even the psychological incentives of participants. This paper studies the dynamics of an exchange economy built on a network of markets where consumers trade with suppliers to optimize utility. Viewing the market in as a decentralized network we study the system from the usual control theory point of view, evaluating the system’s dynamic performance, stability and robustness. It is shown that certain consumer demand dynamics can lead to oscillations while others can converge to optimal allocations.

1. INTRODUCTION

Exchange economies and markets have been studied in economics as far back as the 1800’s. The famous French mathematical economist Léon Walras is credited for the creation of general equilibrium theory and the theory of price adjustment, the so called tatonnement process, that brings about equilibrium. General equilibrium theory explains the conditions under which supply, demand and prices in a whole economy equilibrate. However, in many instances it is the dynamics that are the most important. Furthermore, as opposed to a singular market for a good, prices are usually set by multiple suppliers in multiple markets. In this paper we will design demand functions for agents connected to suppliers through a network of markets, with emphasis here on transient performance. The need for design was highlighted by Roth [2002].

The area of decentralized resource allocation in economics has in large part been focused on mechanism design for equilibrium outcomes, the genesis of this field could most readily be attributed to Hurwicz [1973]. From there mechanism design developed into two different fields focusing on informational aspects of markets (see the early work of Wilson [1985] and Samuelson [1984] to name a few) and game theory (for example the extension to Bayesian Nash equilibriums by Harris and Townsend [1981] and Myerson [1982]). The focus of mechanism design literature is on incentives, auction design, information and behavioural strategies. This paper will focus on dynamic performance, stability of decentralized markets. Here we include dynamics in the prices and an agent’s demand function, and then take up the issue of evaluating an agents’ demand dynamics on the basis of optimality, transient performance and robustness.

Similar work has been studied in control in the internet congestion control literature (see Kelly et al. [1998], Low and Lapsley [1999] and Low et al. [2002]). In this paper the economic network will be similar to Arrow et al. [1958] which motivated the congestion control literature in Kelly et al. [1998] and others.

2. THE MODEL

Consumers and suppliers are connected in a network where prices are set at each supplier. Each consumer has their own utility function which depends only on total demand. A consumer will determine which supplier to trade with to optimize utility while at a supplier, prices are determined by supply and demand.

2.1 Notation

Let \( x_i \) denote the total demand of consumer, \( c_i \), \( i \in [1,...,n] \), and \( y_j \) the supply of supplier, \( s_j \), \( j \in [1,...,m] \). Here supply is fixed and so it is denoted by upper case, \( Y_j \). Each consumer, \( i \), is connected to a set of suppliers \( J_i \). A consumer, \( c_i \), is connected through a link, \( l \), to supplier \( s_l \). The demand on link \( l \) is denoted by \( z_l \), \( l \in [1,...,L] \) where there are a total of \( L \) links. The consumer and supplier connected through link \( l \) are given by \( i(l) \) and \( j(l) \) respectively. Therefore, consumer \( i \)’s total demand is:

\[
x_i = \sum_{l \in i(l)} z_l. \tag{1}
\]

A consumer \( i \)’s initial wealth is denoted by \( w_i \) and the utility received by \( x_i \) is \( U_i(x_i) \). At a supplier, total demand \( \sum_{j \neq j(l)} z_l \) should be upper bounded by total supply \( Y_j \) in equilibrium,

\[
Y_j \geq \sum_{l \neq j(l)} z_l. \tag{2}
\]

Using this trade connection representation allows for a particular economy to be represented by a coupled of matrices \( A \) and \( H \). The network connecting consumers to suppliers can be represented in matrix form by two \( 0 \times 1 \) matrices, \( A = (A_{ij}, j \in [1,...,m], l \in [1,...,L]) \) where \( A_{ij} = 1 \) if link \( l \) is connected to supplier \( s_j \) and 0 if not, and \( H = (H_{il}, i \in [1,...,n], l \in [1,...,L]) \) where \( H_{il} = 1 \) if \( l \) belongs to consumer \( i \) and 0 if not.

Therefore the constraint on total demand at a supplier/node in (2) can be written in vector form as:

\[
\mathbf{Y} \geq \mathbf{Az}. \tag{3}
\]

and the total demand of each consumer in (1) can be written in vector form as:

\[
x = \mathbf{Hz}. \tag{4}
\]

Note, prices will be introduced to the system through the Lagrange multipliers of the optimization problem.

3. OBJECTIVE

The economic welfare problem to be maximized is written in terms of each consumer’s utility function \( U_i(x_i) \), given the total demand received \( x_i \) under the constraints. Consumers seek to maximize utility by adjusting demand. Total welfare of the
economy can be considered the sum of the utilities of each agent and thus, we formulate the following welfare problem:

**Problem 1 (WELFARE):** Maximize the sum of consumer utilities, \( \sum U_i(x_i) \), subject to the supply constraints \( Y \geq Az \) and consumer demand \( Hz = x \), for the economic network defined by \( A \) and \( H \):

\[
\max_{(x,z)} \sum_i U_i(x_i) \quad \text{subject to} \quad Y \geq Az, \quad Hz = x, \quad x, z \geq 0.
\]

(5)

where \( U_i(x_i) \) is a continuously differentiable, strictly concave, radially unbounded, increasing function.

This concave problem seeks the maximum achievable utility over all consumers. A maximizer to WELFARE exists since the objective function is concave, continuous and the feasible solution set is compact.

Given the economic framework of the problem, we will consider the dual problem where prices enter conveniently as the Lagrange multipliers.

Utilizing theory outlined in Arrow et al. [1958] and Boyd and Vandenberghe [2004] we will define the Lagrangian of WELFARE as

\[
L(x,z;p,\lambda) = \sum_i U_i(x_i) + p^T(Y - Az) - \lambda^T(x - Hz)
\]

(6)

where \( \lambda = [\lambda_1, \ldots, \lambda_m] \), \( p = [p_1, \ldots, p_m] \) are vectors of Lagrange multipliers and where \( \lambda \) are free while \( p \geq 0 \). The dual problem is thus **Problem 2 (DUAL):**

\[
\min_{(\lambda,p) \geq 0} \left[ \max_{(x,z) \geq 0, xz} \sum_i U_i(x_i) + p^T(Y - Az) - \lambda^T(x - Hz) \right]
\]

(7)

where the vector, \( p(j) : j \in \{1, \ldots, m\} \), represents the prices at supplier \( j \).

Now, consider an individual consumer’s optimization problem where there is a price per unit good from each supplier, \( p \), and they are free to vary their total demand \( x \). If a consumer \( c_i \) has an initial wealth \( w_i \) and is charged a price \( p_j \) at supplier \( s_j \) and receives in return an allocation of the good \( x_i \) then the utility maximization problem for consumer \( c_i \) is as follows:

**Problem 3 (CONSUMER):**

\[
\max_{(x,z)} U_i(x_i) - \sum_{l=i} p_j z_l
\]

subject to \( Hz = x, \quad x, z \geq 0. \)

(8)

A consumer wishes to maximize utility but at a cost of paying more.

**3.1 Utility function**

Consumers will choose their own utility function and therefore, will be left generic, however, it should be noted that certain utility functions can be chosen so that the equilibrium outcomes are fair. For example, it is known that logarithmic utility functions result in a proportionally fair solution to DUAL, for more details see Kelly [1997].

**3.2 Equilibrium conditions**

The objective function in WELFARE is differentiable and strictly concave and the feasible region \( Hz = x, \ Y \geq Az \) over \( x, z \geq 0 \) is compact. Hence, there exists a maximizing value of \((x,z)\). Since the objective function in WELFARE is strictly concave in \( x \) there is a unique optimum \( x^* \), however, it is not unique in \( z \). We will now outline the Lagrangian conditions that the dynamic agent models must meet in equilibrium in order to achieve optimality of the constrained objective, WELFARE.

**K.K.T. conditions**

Consider the Lagrangian in (6)

\[
L(x,z;p,\lambda) = \sum_i U_i(x_i) + p^T(Y - Az) - \lambda^T(x - Hz)
\]

\[
= \sum_i (U_i(x_i) - \lambda_i x_i) + \sum_i \tilde{z}_i (\lambda_i(t) - p_{j,f}) + \sum_j p_j Y_j.
\]

(9)

and note that the partial derivatives of the Lagrangian w.r.t. \( x_i \), \( z_i \) are given by

\[
\frac{\partial L}{\partial x_i} = U'_i(x_i) - \lambda_i, \quad \frac{\partial L}{\partial z_i} = \lambda_i(t) - p_{j,f}.
\]

It can be shown from the KKT conditions for the WELFARE problem, we will consider the dual problem where prices enter conveniently as the Lagrange multipliers.

Utilizing theory outlined in Arrow et al. [1958] and Boyd and Vandenberghe [2004] that if \((x^*, z^*)\) is a maximizing solution of (5) there exist \( p^*, \lambda^* \) s.t. \( \frac{\partial L}{\partial x_i} \leq 0, \frac{\partial L}{\partial z_i} \leq 0 \) with the partial derivatives being equal to zero if the positivity constraints on \( x, z \) are not active, i.e.

\[
\lambda_i = U'_i(x_i) \text{ if } x_i > 0 \quad \lambda_i(t) = p_{j,f} \text{ if } z_i > 0.
\]

(10)

The KKT conditions additionally require primal and dual feasibility and corresponding conditions on the dual variables \( \lambda, p \), i.e.

\[ Hz^* = x^*, \quad Az \leq Y, \quad p^T(Y - Az^*) = 0, \quad p^* \geq 0, \quad z^* \geq 0. \]

(11)

Note that (10)-(11) are necessary and sufficient conditions for optimality in (5) since this is a convex optimization problem with linear constraints.

From (10) it is interesting to note that if a consumer \( i \) trades with two different suppliers on links \( l \) and \( l^* \), so that \( z_l > 0 \) and \( z_{l^*} > 0 \), then

\[ U'_i(x_i) = \lambda_i = \begin{cases} \lambda_i(t) = p_{j,f} & \text{if } z_t > 0 \\ \lambda_i(t) = p_{j,f} & \text{if } z_l > 0 \end{cases} \]

(12)

and therefore the prices at suppliers who trade with the same consumer, \( z_l > 0 \) and \( z_{l^*} > 0 \), are equal in equilibrium. This is consistent with economic intuition.

Consider now the CONSUMER problem in (8), and define the Lagrangian

\[ L_{\lambda}(\tilde{x}_i;\tilde{z};\mu) = U_i(x_i) - \sum_{l=i} p_{j,f} \tilde{z}_l - \lambda^T(x - Hz) \]

(13)

Analogous KKT conditions hold at the optimum points \((\tilde{x}, \tilde{z})\) for some \( \tilde{\mu} \) i.e.

\[
\frac{\partial L_{\lambda}}{\partial x_i} = U'_i(\tilde{x}_i) - \tilde{\mu}_i = \begin{cases} 0 & \text{if } x_i > 0 \\ \leq 0 & \text{if } x_i = 0 \end{cases}
\]

\[
\frac{\partial L_{\lambda}}{\partial z_l} = \tilde{\mu}_i - p_{j,f}(l) = \begin{cases} 0 & \text{if } z_l > 0 \\ \leq 0 & \text{if } z_l = 0 \end{cases}
\]

(14)

Thus, a solution \((\lambda^*, p^*, x^*, z^*)\) that satisfies (10), (11) also satisfies (14).

**4. MARKET DYNAMICS**

The traditional representation for price dynamics is the tatonnement process where prices adjust to supply and demand in the following way:

\[
\frac{dp_j}{dt} = \left( \frac{k_{pj}}{\sum_{l=j} z_l(t) - Y_j} \right)^+ p_j
\]

(15)

where \([\cdot]^+\) denotes positive projection if \( p_j < 0 \) i.e. it ensures \( p_j \) is non-negative.
5. DYNAMIC CONSUMER CHOICE MODEL

Now consider a basic dynamic demand model that in equilibrium solves WELFARE and therefore satisfies the Lagrangian conditions (10). We consider dynamics which are analogous to the primal/dual algorithms in Kelly et al. [1998] and appropriate extensions are then considered in section 6 by making use of the Arrow-Hurwicz gradient method in Arrow et al. [1958]. Consider the demand adjustment process on link $l$ by consumer $i(l)$

$$\frac{dz_i}{dt} = \left[ k_i x_i \left( U_i(x_i(t)) - p_j(t) \right) \right]_{z_i}^+$$

where $p_j$ is the price at supplier $j$, and $[\cdot]^+$ ensures $z_i(t)$ stays non-negative.

Equation (16) describes the dynamics for a consumer’s demand based on utility verses price, sometimes referred to as the primal dynamics. If the marginal utility received from an extra unit of $z_i$ is greater than the price paid, then $z_i$ is increased. Though these dynamics satisfy the Lagrangian conditions (10) in equilibrium and avoid flipping between suppliers which are present in a static choice model they display oscillations. Consider the network in Figure 1, where $Y_1 = 0.8$ and $Y_2 = 1.2$. As shown in Figure 2, oscillations are present.

Fig. 1. Two supplier market, one consumer

6. LOWPASS DYNAMICS

We modify the dynamic demand model (16) by introducing a lowpass filter given by (17)-(18). In economic terms the lowpass filter can be thought of as accounting for some level of reluctance to changing suppliers too quickly. In control terms, they act like a high frequency pole reducing bandwidth and limiting oscillations.

$$\frac{dz_i}{dt} = \left[ \left( U_i(x_i(t)) - p_j(t) \right) + \tilde{k}(z_i - \bar{z}_i) \right]_{z_i}^+$$

(17)

$$\frac{dp_j}{dt} = \left[ \left( \sum_{l: i = j(l)} z_i(t) - Y_j \right) \right]_{p_j}^+.$$ (18)

Figure 3 gives the time path of the link demands and prices for the network in Figure 1. Clearly, a promising feature of the lowpass dynamics is that the oscillations have disappeared. Thus it is now important to determine if the dynamics (17)-(18) are stable and if they optimize the DUAL problem in equilibrium.

6.1 Modified objective

Consider the following modified objective function:

$$f(z, \bar{z}) = \sum_i U_i(x_i) - \frac{1}{2} \sum_i \sum_{l: i = j(l)} \tilde{k}(z_i - \bar{z}_i)^2$$

(19)

where $\bar{z}_i$ is an augmented variable.

Note that if $(z^*, \bar{z}^*)$ is a maximizer of (19), subject to constraints $H z^* = x^*$ and $Y \geq A z^*$ and $(z^*, \bar{z}^*) \geq 0$, then $z^*$ must also be an optimal solution of the original WELFARE problem (5). This is because at optimality $z^* = \bar{z}^*$, thus the added non-positive term is zero. Furthermore if $(z^*, \bar{z}^*)$ is an equilibrium point of (17) then it maximizes $^1$ (19) hence also solving WELFARE.

6.2 Stability

In order to prove stability we determine a Lyapunov function for the system of differential equations (17)-(18) by noticing that these dynamics fit a continuous-time version of the Arrow-Hurwicz gradient method Arrow et al. [1958], where the state vector is augmented to include $\bar{z}$. Consider the modified objective function $f(z, \bar{z})$ in (19) subject to the constraint $Y - A z \geq 0$, $(z, \bar{z}) \geq 0$ and $x = H z$ i.e.

$$\max_{(z, \bar{z})} f(z, \bar{z})$$

subject to $g(z) \geq 0$, $z, \bar{z} \geq 0$

$$\text{where } f(z, \bar{z}) = \sum_i U_i \left( \sum_{l: i = j(l)} \bar{z}_i - \frac{1}{2} \sum_{l: i = j(l)} \tilde{k}(z_i - \bar{z}_i)^2 \right)$$

(20)

$$g(z) = Y - A z \geq 0.$$

From the Kuhn-Tucker Theorem, a vector $v^* = (z^*, \bar{z}^*)$ is an optimum of the problem (20) if and only if there is an $m$-vector $p^*$ such that $(v^*, p^*)$ is a saddle-point of the Lagrangian:

$$\phi(v, p) = f(v) + p \cdot g(v)$$

(21)

where $v = (z, \bar{z})$ and $p \geq 0$, i.e.,

$$\phi(v^*, p^*) = \max_{v \geq 0} \phi(v, p) = \min_{p \geq 0} \phi(v, p).$$

(22)

Hence, solving the maximization problem (20) is reduced to finding a saddle-point of the Lagrangian (21). Let $V^*$ and $P^*$ denote the sets of the $v$ and $p$ components of the saddle-points of $\phi$ respectively. Note also that the KKT conditions for optimality in (20), (22) are given by

$^1$ This follows by noting that the equilibrium points of (17), (18) satisfy the KKT conditions for the constraint maximization of (19) (this is also discussed in the next section).
\[ U_{d_l}(z_l) = p_j(t) - \bar{k}(z_l - \bar{z}_l) \begin{cases} = 0 & \text{if } z_l > 0 \\ \leq 0 & \text{if } z_l = 0 \end{cases} \]
\[ z_l = \bar{z}_l \]
\[ p^T(Y - Ax) = 0, \ Y - Az \geq 0, \ p \geq 0, \ z \geq 0. \]

**Proposition 6.1.** Let \( E \) be the set of equilibrium points of the system (17)-(18). Then for any initial conditions \([v, p] = [z(0), \bar{z}(0), p(0)]\) the system, (17)-(18), approaches \( E \) as \( t \to \infty \).

**Proof.** Consider the Lyapunov function candidate \( \psi(v(t), p(t)) \) defined as:
\[ \psi(v(t), p(t)) = \left\| \left( \frac{v(t) - v^*}{p(t) - p^*} \right) \right\|_2 \]
\[ = \left( \sum_t (z(t) - \bar{z}_t)^2 + \sum_t (\bar{z}_t - z^*_t)^2 + \sum_j (p_j(t) - p^*_j)^2 \right) \]

where \( v(t) = [z(t), \bar{z}(t)] \) and \( v^* \in V^* \) and \( p^* \in P^* \), and \( \psi : \mathbb{R}^{L \times L \times m} \to \mathbb{R} \) is a continuously differentiable function that is radially unbounded.

The derivative with respect to \( t \) of \( \psi \) along the system trajectories are given by
\[ \frac{d \psi(v, p)}{dt} = d \left( \sqrt{\sum_t (z(t) - \bar{z}_t)^2 + \sum_t (\bar{z}_t - z^*_t)^2 + \sum_j (p_j(t) - p^*_j)^2} \right) \]
\[ = \left( \sum_t (z(t) - \bar{z}_t) \frac{dz_t}{dt} + \sum_t (\bar{z}_t - z^*_t) \frac{d\bar{z}_t}{dt} + \sum_j (p_j(t) - p^*_j) \frac{dp_j}{dt} \right) \]

where we must show that \( \frac{d \psi(v, p)}{dt} \leq 0 \).

Now, consider the modified Lagrangian in (21) given by
\[ \phi(v, p) = f(v) + p \cdot g(v) \]
\[ = \sum_t U_l(\sum_{i=1}^L z_{i,l}) - \frac{1}{2} \sum_t \sum_{i=1}^L \sum_{j=1}^L \bar{k}(z_{i,l} - \bar{z}_{i,l}) + p^T(Y - Az). \]

Let \((z^*, \bar{z}) \in V^* \) and \( p^* \in P^* \), then by concavity of \( \phi(v, p) \) in \( v \)
\[ (z - z^*) \cdot \phi_z + (\bar{z} - \bar{z}^*) \cdot \phi_{\bar{z}} + (p - p^*) \cdot \phi_p \leq 0 \]

for \((z, \bar{z}) \notin V^* \) or \( p \notin P^* \) where \( \phi_z = \left[ \frac{\partial \phi}{\partial z_1}, \ldots, \frac{\partial \phi}{\partial z_L} \right]^T \), \( \phi_{\bar{z}} = \left[ \frac{\partial \phi}{\partial \bar{z}_1}, \ldots, \frac{\partial \phi}{\partial \bar{z}_L} \right]^T \), and \( \phi_p = \left[ \frac{\partial \phi}{\partial p_1}, \ldots, \frac{\partial \phi}{\partial p_m} \right]^T \) stand for the partial derivatives of \( \phi \) with respect to \( z, \bar{z} \) and \( p \), respectively.

Evaluating the partial derivatives gives
\[ \phi_{z_l} = U_{d_l}(z_l) - \bar{k}(z_l - \bar{z}_l) - p_j(t) = \frac{dz_l}{dt} \]
\[ \phi_{\bar{z}_l} = \bar{k}(z_l - \bar{z}_l) = \frac{d\bar{z}_l}{dt} \]
\[ \phi_{p_j} = Y_j - \sum_{t=1}^L z_t = -\frac{dp_j}{dt} \]

for \( l \in \{1, \ldots, L\} \) and \( j \in \{1, \ldots, m\} \) from (17)-(18) if \( z \) and \( p \) are positive. Hence, the inequality in (27) can be written as
\[ (z - z^*) \cdot \phi_z + (\bar{z} - \bar{z}^*) \cdot \phi_{\bar{z}} + (p - p^*) \cdot \phi_p \leq 0 \]
\[ \sum_t (z(t) - z^*_t) \frac{dz_t}{dt} + \sum_t (\bar{z}_t - \bar{z}^*_t) \frac{d\bar{z}_t}{dt} + \sum_j (p_j(t) - p^*_j) \frac{dp_j}{dt} \leq 0 \]

therefore, \( \frac{d \psi(v, p)}{dt} \leq 0 \).
norm and stability margins of the lowpass system, then compare performance to an $H_\infty$ loop-shaping controller. To do this, we must first determine an operating point and find a linearized model.

7.1 Linear model

In matrix form the linearization gives

$$\Delta \dot{x} = - (W(X^2)^{-1}(H^T H \Delta x)) + \bar{K}(\Delta z - \Delta \bar{z}) - A^T \Delta p$$

and

$$\Delta \dot{p}_1 = A \Delta z$$

where $\bar{K} = \text{diag}(\bar{k}) \in \mathbb{R}^{l \times L}$, $W = \text{diag}(w_i(l))$ and $X = \text{diag}(x_{ij})$.

7.2 Stability margins of lowpass dynamics

A well known measure of stability margins for multi-input multi-output systems is

$$b(P(s), K(s)) := \left\| \frac{I}{K(s)} \right\| \left\| (I - P(s)K(s))^{-1} [P(s)I] \right\|^{-1}$$

which is related to robust stability with respect to the normalized coprime factorization uncertainty for a perturbed plant $(N + \Delta N)(M + \Delta M)^{-1}$. Where $NM^{-1} = P$ is a normalized coprime factorization, such that $N(jo)^*N(jo) + M(jo)^*M(jo) = I$, where we will drop the dependence on $s$ from now on, for the plant $P$ and the controller $K$. In fact, from the small gain theorem (e.g. Zhou et al. [1996]) the perturbed system is internally stable for all $\Delta N, \Delta M$ with

$$\left\| \begin{bmatrix} \Delta N \\ \Delta M \end{bmatrix} \right\| < \gamma \text{ iff } b(P, K) \geq \gamma.$$  

Figure 5 investigates the relationship between $\bar{k}$ and the inverse of the $H_\infty$ norm of the disturbance rejection transfer function matrix $\|T_{d,\Delta z}\|_\infty$ and $b(P,K)$. $\bar{K}$ is the lowpass linear controller added to the primal dynamics (32). In the frequency domain $K$ is

$$\bar{K}(s) = \begin{bmatrix} -\bar{k} & 0 \\ s + \bar{k} & 0 \\ 0 & \bar{k} \end{bmatrix}.$$  

Fig. 5. $\|T_{d,\Delta z}\|_\infty^{-1}$ and $b(P, \bar{K})$ vs $\bar{k}$

Clearly, Figure 5 shows there is a tradeoff between $\|T_{d,\Delta z}\|_\infty^{-1}$ and $b(P, \bar{K})$ vs $\bar{k}$. Due to a resonance in the plant which is shown later in Figure 7, high-gain is required in the region of the unstable pole in order to achieve an encirclement by the Nyquist plot of the loop-gain. However, larger $\bar{k}$ reduces disturbance rejection. The region around $\bar{k} \approx 1$ is where performance is best balanced out. Looking back at Figure 4 we can map quite closely the transient performance of the nonlinear system to a deterioration in either $\|T_{d,\Delta z}\|_\infty^{-1}$ or $b(P, \bar{K})$ and therefore, it is desirable to have both large $\|T_{d,\Delta z}\|_\infty^{-1}$ and good stability margins, $b(P, \bar{K}) > 0.25 - 0.3$.

7.3 $H_\infty$ loop shaping

Here we investigate the $H_\infty$ loop shaping controller design problem which gives a centralized controller that gives good disturbance rejection and stability margins, Zhou et al. [1996]. Consider the system in Figure 6, where $P_B = W_2P_1W_1$ and $P_L$ is the transfer function from $[d, u] \to \epsilon$.

![Fig. 6. Plant and $H_\infty$ loop shaping controller](image)

The first step is to choose a pre-compensator $W_1$ and a post-compensator $W_2$ and the second step is to design a feedback controller, $K_B$, to maximize

$$b(W_2P_1W_1, K_B) := \|T_{CL}\|_\infty^{-1}$$

Loop shaping design The plant here fits the framework in Figure 6, since $B_1 = B_2, C_1 = C_2$ and $D_{ij} = 0$. We can therefore look at the loop shaping problem for the plant:

$$P_L = \begin{bmatrix} s^3 + 25s^2 + s \\ s^3 + 50s^3 + 25s^2 + 50s + 1 \\ -25s^2 \\ s^3 + 50s^3 + 25s^2 + 50s + 1 \\ s^3 + 50s^3 + 25s^2 + 50s + 1 \end{bmatrix}$$

where $w = 100$ and $x = 2$. The zero at $s = 0$ causes low gain at low frequencies which creates a poor loop shape. Therefore, the pre-compensator $W_1(s)$ should shape the loop-gain at low frequencies by cancelling the zero at $s = 0$. To avoid an unstable pole-zero cancelation, the pole is placed arbitrarily close to $s = 0$, hence

$$W_1(s) = \begin{bmatrix} 125(\tau)^2 \cdot (s + 5) \cdot (s + 5) \\ 0 \\ 125(\tau)^2 \cdot (s + 5) \cdot (s + 5) \end{bmatrix}$$

where $\tau = 10^3$. The post-compensator $W_2$ is chosen so that the loop-gain is bounded at high frequencies and chosen to be
Fig. 7. Singular values plot all in one

$$W_2(s) = \begin{bmatrix} s + 10 & 0 \\ (s + 100) & s + 10 \end{bmatrix} .$$  \hspace{1cm} (41)

The gain, \( k_w \), is chosen to be 100 so as to increase the bandwidth of the system, since there is a resonant peak in \( P_L \) at \( \omega = 1.414 \text{rad/} \text{sec} \). This pushes the peak before the gain-crossover frequency, and \( W_2 \) adds phase-lead near the gain-crossover frequency. A singular values plot of \( P_L \) can be found in Figure 7. Using standard \( \mathcal{H}_\infty \) controller design methods, McFarlane and Glover [1992], function ncfsyn, which is part of the Robust Control Toolbox, the optimal \( \mathcal{H}_\infty \) loop shaping controller \( K_b \) is determined. Figure 7 is a singular values plot of \( P_L W_1 K_1 W_2, P_L, W_1 W_2, W_1, W_2 \) and the closed-loop matrix outputted from ncfsyn CL, which is the transfer function matrix \( T_{CL} \) defined in (38).

The coprime factorization stability margin is

$$b(P_L, K_b) = 0.7556; \text{ it is commonly understood that a stability margin of}$$

$$b > 0.25 - 0.3 \text{ is satisfactory. Further, the } \mathcal{H}_\infty \text{ norm of the}$$

$$\text{transfer function from } d \text{ to } z \text{ in Figure 6 is } \|T_{K_b d \rightarrow z}\|_\infty = \| (I - PK_b)^{-1} P \|_\infty = 0.0092 \text{ therefore } \|T_{K_b d \rightarrow z}\|_\infty = 108.2696.$$ 

Clearly \( \|T_{K_b d \rightarrow z}\|_\infty \) is much larger than when using the lowpass dynamics in (17). Comparing \( K(s) \) with the \( \mathcal{H}_\infty \) loop shaping controller \( K_{ls} = W_1 K_1 W_2 \), which gave \( \|T_{K_{ls} d \rightarrow z}\|_\infty = 108.2696 \) and \( b(P, K_b) = 0.7556 \), the loop shaping controller \( K_{ls} \) performs considerably better on the linear system, however, we have yet to see how it compares on the nonlinear system.

**K_b on the nonlinear system**

Figure 8 gives the response of the lowpass dynamics versus the \( \mathcal{H}_\infty \) loop shaping controller \( K_{ls} \) for nonlinear system in Figure 1. Where the \( \mathcal{H}_\infty \) loop shaping controller is added to the original dynamic demand differential equation in (16) with price determined by (15). Clearly the \( \mathcal{H}_\infty \) loop shaping controller is much faster than the lowpass dynamics within the linear region. However, the lowpass dynamics are stable for any initial conditions and any network, and as long as \( K \) is chosen appropriately, the transient response is slow but smooth. Furthermore, the lowpass dynamics are much simpler than the \( \mathcal{H}_\infty \) loop shaping dynamics, and they are decentralized while still providing an adequate level of robustness, \( b(P, K) = 0.4366 \).

8. SUMMARY

We have shown in this paper that control theory tools can be useful in designing demand dynamics that stabilize economic networks. Simple primal algorithms such as dynamic choices based on the lowest price did not clear markets, and instead caused oscillations in prices. However, with intuition borrowed from control theory demand dynamics can be designed to clear markets and give good transient performance by adding a lowpass filter. Finally, centralized robust controllers where determined and compared to the lowpass decentralized dynamics.

**References**


