

A Constructive Procedure based on Dynamic Scaling for Adaptive Control of Nonlinearly Parameterized Nonlinear Systems^{*}

Xiangbin Liu^{*} Romeo Ortega^{**} Hongye Su^{***} Jian Chu^{***}

^{*} *Advanced Control Systems Laboratory of School of Electronics and Information Engineering, Beijing Jiaotong University, Beijing 100044, P. R. China. (Tel: 86-10-5168-4105; e-mail: xbnliu@yahoo.cn)*

^{**} *Laboratoire des Signaux et Systèmes, CNRS-SUPELEC, 91192, Gif-sur-Yvette, France (e-mail: ortega@lss.supelec.fr)*

^{***} *State Key Lab. of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou, 310027, P. R. China. (e-mail: hysu@csc.zju.edu.cn, chuj@csc.zju.edu.cn)*

Abstract: A new framework to design immersion and invariance adaptive controllers for nonlinearly parameterized, nonlinear systems was recently proposed by the authors. The key step is the construction of a *monotone* mapping, via a suitable selection of a controller tuning function, which has to satisfy some integrability conditions—this translates into the need to solve a partial differential equation (PDE). In this paper this result is extended providing some answers to the questions of characterization of “monotonizable” systems and solvability of the PDE. First, adding to the design a nonlinear dynamic scaling, we *obviate* the need to solve the PDE. Second, for the case of *factorizable nonlinearities*, the following results are established. (i) It is shown that the monotonicity condition is satisfied if a linear matrix inequality is feasible. (ii) Directly verifiable involutivity conditions that ensure the solution of the PDE are presented. (iii) An explicit formula for the required tuning function is given, provided the regressor matrix satisfies some rank conditions. Hence, adding a dynamic scaling, this yields a constructive solution to the problem.

Caveat This is an abridged version of the full paper, which is available upon request to the authors.

1. INTRODUCTION

In the paper we are interested in the problem of stabilization of uncertain, nonlinearly parameterized, nonlinear systems described by

$$\dot{x} = F(x, u) + \Phi(x, \theta) \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $n \geq m$, $\theta \in \mathbb{R}^q$ is a constant vector of *unknown parameters* and $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\Phi : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^n$ are *known mappings*.

Some results on this problem using standard (gradient-like) direct adaptive controllers have been reported in the literature for *convexly* parameterized systems. It was first reported in Fomin et al. [1981] (see also Ortega [1996]) that convexity is enough to ensure that the gradient search “goes in the right direction” in a *certain region* in the state space. The idea is then to apply a standard

^{*} This work was supported by the National Basic Research Program of China (973 Program 2007CB714006), the State Key Program (No.60834001) of National Natural Science Foundation of China, the National Natural Science Foundation of China(61040050) and the Fundamental Research Funds for the Central Universities(2011JBM008).

adaptive scheme in this region, while in the “bad” region either the adaptation is frozen and a robust constant parameter controller is switched-on in Fradkov et al. [2001] or, as proposed in Annaswamy et al. [1998], the adaptation is running all the time and stability is ensured with a high-gain mechanism which is suitably adjusted incorporating prior knowledge on the parameters. In Netto et al. [2000] *reparametrization* to convexify an otherwise non-convexly parameterized system is proposed. See also Netto et al. [2006] and Tyukin et al. [2003] for some recent interesting results along these lines, where the controller and the estimator switch between over/underbounding convex/concave functions. Similarly to the present paper, in Tyukin et al. [2007] monotonicity, instead of convexity, is used—but under radically different assumptions. The reader is referred to Liu et al. [2009a] for additional details on the literature review.

A new methodology to design direct and indirect adaptive controllers for systems of the form (1), called Immersion and Invariance (I&I), was recently proposed in Astolfi et al. [2003], see also Astolfi et al. [2007].¹ In Liu et al. [2009a] the methodology was shown to be successful,

¹ For the sake of brevity, in this paper attention is restricted to indirect schemes. As will become clear below, the results apply, *mutatis mutandis*, also to direct schemes.

provided there exists a mapping $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^q$ such that the following key assumption holds.

Assumption 1. The parameterized mapping $\mathcal{Q}^x : \mathbb{R}^q \rightarrow \mathbb{R}^q$

$$\mathcal{Q}^x(\theta) := \nabla\beta(x)\Phi(x, \theta) \quad (2)$$

is strictly monotone² for all $x \in \mathcal{X} \subset \mathbb{R}^n$. That is, for all $a, b \in \mathbb{R}^q$, $a \neq b$ and all $x \in \mathcal{X}$, the mapping satisfies

$$(a - b)^\top [\mathcal{Q}^x(a) - \mathcal{Q}^x(b)] > 0.$$

There are two difficulties for the application of this result. First, the identification of a class of “monotonizable” functions Φ , *i.e.*, those that verify Assumption 1. (Some examples may be found in Liu et al. [2009a].) Second, that the function β appears in (2) in *gradient* form. This restriction, essentially, translates into the need to *solve a PDE*. Indeed, even if there exists a mapping $\mathcal{B} : \mathbb{R}^n \rightarrow \mathbb{R}^{q \times n}$ such that the parameterized mapping $\mathcal{Q}_{\mathcal{B}}^x : \mathbb{R}^q \rightarrow \mathbb{R}^q$

$$\mathcal{Q}_{\mathcal{B}}^x(\theta) := \mathcal{B}(x)\Phi(x, \theta) \quad (3)$$

is strictly monotone for all $x \in \mathcal{X}$, to satisfy Assumption 1 the solution of the PDE

$$\nabla\beta(x) = \mathcal{B}(x), \quad (4)$$

must be found. As is well-known, even if the PDE is solvable, getting its explicit solution is a difficult problem. See Astolfi et al. [2007], Liu et al. [2009a] for further discussion, and Karagiannis et al. [2009] for a major breakthrough for the solution of this problem.

The purpose of this paper is to extend the results of Liu et al. [2009a], providing some answers to these two questions. Our main contributions are:

- 1) *Obviate* the need to solve the PDE adding to the design a variation of the nonlinear dynamic scaling proposed in Karagiannis et al. [2009]. More precisely, replace the assumption of monotonicity of \mathcal{Q}^x by monotonicity of $\mathcal{Q}_{\mathcal{B}}^x$.
- 2) For the case of *factorizable nonlinearities* of the form $\Phi(x, \theta) = \phi(x)\mu(\theta)$, where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$ and $\mu : \mathbb{R}^q \rightarrow \mathbb{R}^p$, for some $p \in \mathbb{Z}_+$, the following results are established.
 - (i) It is shown that the aforementioned monotonicity condition imposed on the mapping $\mathcal{Q}_{\mathcal{B}}^x$ is satisfied if a *linear matrix inequality* (LMI) (involving $\nabla\mu$) is feasible.
 - (ii) Directly verifiable involutivity conditions on ϕ that ensure the solution of the PDE (4) are given.
 - (iii) Imposing some rank conditions on the regressor vectors an *explicit formula* for a “monotonizing” \mathcal{B} is provided. Hence, adding dynamic scaling, a complete solution to the stabilization problem is given.³

For the sake of completeness the main result on indirect I&I adaptive control of Liu et al. [2009a] is recalled below—the interested reader is referred to this reference for further details. In particular, in Liu et al. [2009a] other (weaker) versions of Assumption 1 are imposed, requiring

² In Liu et al. [2009a] a more general property, called *P-monotonicity*, is considered.

³ The use of dynamic scaling obviates the need to solve the PDE. However, the new scheme is more complex and imposes more stringent conditions on the systems, hence it is of interest to investigate when the PDE is solvable. See Remark 1.

in this case, a kind of *persistence of excitation* condition to ensure parameter convergence.

In order to streamline the proposition we introduce the following robust stabilizability assumption.

Assumption 2. There exists a mapping $\psi : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$, such that the system

$$\dot{x} = F(x, \psi(x, \theta)) + \Phi(x, \theta) \quad (5)$$

has a globally asymptotically stable equilibrium at x_* . Moreover, the system

$$\dot{x} = F(x, \psi(x, z + \theta)) + \Phi(x, \theta),$$

with input z , is input–state stable Sontag et al. [1995].

Proposition 1. Consider system (1) and the estimator

$$\dot{\hat{\theta}} = -\nabla\beta(x) \left[F(x, u) + \Phi(x, \hat{\theta} + \beta(x)) \right], \quad (6)$$

where β is such that Assumption 1 holds with $\mathcal{X} = \mathbb{R}^n$.

- (P1) For all $(x(0), \hat{\theta}(0)) \in \mathbb{R}^n \times \mathbb{R}^q$, and $u(t)$ such that $(x(t), \hat{\theta}(t))$ exist for all $t \geq 0$, we have

$$\lim_{t \rightarrow \infty} [\hat{\theta}(t) + \beta(x(t))] = \theta.$$

- (P2) If Assumption 2 holds, then the I&I adaptive controller

$$u = \psi(x, \hat{\theta} + \beta(x))$$

guarantees that, for all $(x(0), \hat{\theta}(0)) \in \mathbb{R}^n \times \mathbb{R}^q$, the trajectories are bounded and $\lim_{t \rightarrow \infty} x(t) = x_*$.

The remaining of the paper is organized as follows. In Section 2 the I&I adaptive controller with dynamic scaling is presented. In Section 3 the results pertaining to systems with factorizable nonlinearities are given. In Section 4 an example is given. The paper is wrapped up with some concluding remarks and open problems in Section 5.

2. ADAPTIVE CONTROL WITH DYNAMIC SCALING

In this section we propose a procedure to replace Assumption 1 by the following one.

Assumption 3. There exists a mapping $\mathcal{B} : \mathbb{R}^n \rightarrow \mathbb{R}^{q \times n}$ such that the parameterized mapping (3) is *strongly* monotone for all $x \in \mathcal{X}$. That is, for all $a, b \in \mathbb{R}^q$, $a \neq b$ and all $x \in \mathcal{X}$, the mapping satisfies

$$(a - b)^\top [\mathcal{Q}_{\mathcal{B}}^x(a) - \mathcal{Q}_{\mathcal{B}}^x(b)] > c|a - b|^2,$$

for some $c > 0$, where $|\cdot|$ is the Euclidean norm.

If the monotonicity condition of Assumption 3 is relaxed, from strong to strict, Assumption 1 implies Assumption 3. On the other hand, the converse implication presumes the solvability of the PDE (4). In the recent interesting paper Karagiannis et al. [2009] this difficulty is removed, for linearly parameterized systems, introducing a nonlinear *dynamic scaling*. Unfortunately, the construction critically relies on linearity. In this section it is shown that the technique is still applicable using the “stability margin” provided by *strong*—instead of *strict*—monotonicity and adding a *parameter projection*.

To streamline the presentation of the main result define the mapping $\beta_a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^q$ as

$$\beta_a(x, \hat{x}) := \mathcal{B}_1(\hat{x})x_1 + \cdots + \mathcal{B}_n(\hat{x})x_n \quad (7)$$

where $\mathcal{B}_i : \mathbb{R}^n \rightarrow \mathbb{R}^q$ are the columns of the mapping \mathcal{B} that satisfies Assumption 3, that is,

$$\mathcal{B}(x) = [\mathcal{B}_1(x) \cdots \mathcal{B}_n(x)],$$

and $\hat{x} \in \mathbb{R}^n$ is defined below.

To make the following Proposition concise, let $z_a := \frac{1}{c}(\hat{\theta} + \beta_a - \theta)$, $\Delta(x, \hat{x}, e)$ and $\ell(x)$ are defined in (8) and (9),

$$\Delta(x, \hat{x}, e) = \mathcal{B}(x) - \nabla_x \beta_a, \quad (8)$$

with $\Delta(x, \hat{x}, 0) = 0, \quad \forall x, \hat{x}, e \in \mathbb{R}^n,$

$$|\Phi(x, \theta) - \Phi(x, \hat{\theta} + \beta_a)| \leq \ell(x)|\hat{\theta} + \beta_a - \theta| = \ell(x)r|z_a|, \quad (9)$$

Proposition 2. Consider system (1) and assume $\theta \in \Theta \subset \mathbb{R}^q$, where Θ is known, *compact* and convex with known boundary $\partial\Theta$. Let β_a be given by (7) with \mathcal{B} satisfying Assumption 3 with $\mathcal{X} = \mathbb{R}^n$ and

$$\dot{\hat{x}} = F(x, u) + \Phi(x, \hat{\theta} + \beta_a) - K(x, e, r)e, \quad (10)$$

where the error

$$e := \hat{x} - x \quad (11)$$

the dynamic scaling function is given by

$$\dot{r} = -\frac{c}{4}(r-1) + \frac{r}{2c}\ell^2(x)\|\Delta(x, \hat{x}, e)\|^2, \quad r(0) > 1, \quad (12)$$

with $c > 0$, and $K : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is a function satisfying the bound as

$$K(x, e, r) \geq r^2\ell^2(x)\left(\frac{2}{c} + \|\bar{\Delta}(x, \hat{x}, e)\|^2\right). \quad (13)$$

Define the estimator

$$\hat{\theta} = \text{Pr} \left\{ -\nabla_x \beta_a \left[F(x, u) + \Phi(x, \hat{\theta} + \beta_a) \right] - \nabla_x \beta_a \hat{x} \right\} \quad (14)$$

with $\text{Pr}\{\cdot\}$ a smooth projection⁴ that ensures $\hat{\theta}(t) + \beta_a(x(t)) \in \Theta - \partial\Theta$.

For all $(x(0), \hat{x}(0), \hat{\theta}(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \Theta$, and $u(t)$ such that $(x(t), \hat{x}(t), \hat{\theta}(t))$ exist for all $t \geq 0$, we have boundedness of all signals,

$$\lim_{t \rightarrow \infty} [\hat{\theta}(t) + \beta_a(x(t), \hat{x}(t))] = \theta,$$

and the convergence is exponentially fast.

Remark 1. The controller with dynamic scaling replaces Assumption 1 by Assumption 3, obviating the solution of a PDE. On the other hand, Assumption 3 requires a more stringent monotonicity condition and assumes some prior knowledge on the systems parameters to implement the projection. Moreover, the resulting scheme is considerably more complicated and it relies on the injection of *high gain*, that may be detrimental to the performance. For all these reasons, it is still of interest to study, as done in the next section, the I&I controller that relies on Assumption 1.

3. SYSTEMS WITH SEPARABLE NONLINEARITIES

It is clear from Proposition 1 that the key function in indirect I&I adaptive control is Φ . Consistent parameter estimators and stabilizing controllers can be designed if Φ is “monotonizable”, *i.e.*, if a (state-weighted) linear combination of the elements of this function yields a monotonic mapping—see Assumptions 1 and 3. To sharpen this general result a “more structured class” of systems is considered in this section. Namely, those when the uncertain

⁴ See Section 2.2 of Yao et al. [1997] for an explicit procedure to implement the projection.

term Φ is *factorizable*, that is, when it can be written in the form

$$\Phi(x, \theta) = \phi(x)\mu(\theta) \quad (15)$$

where, $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$ and $\mu : \mathbb{R}^q \rightarrow \mathbb{R}^p$. Notice that, with a suitable redefinition of F and Φ in (1), we can assume

$$\mu_i(0) = 0, \quad i = 1, \dots, p, \quad (16)$$

without loss of generality. Moreover, it is assumed that ϕ is full rank (uniformly in x).

The following results are established in this section. The first two are related with the basic PDE-based design of Proposition 1, while the last one is concerned with the dynamic scaling scheme of Proposition 2.

- (S1) Proof that Φ is “monotonizable” if a *parameterized LMI*—involving the image of the mapping $\nabla\mu$ —is feasible, and a standard linear PDE is solved. Moreover, *for all nonlinearities* μ , the feasibility test can be carried out, *locally*, with a finite dimensional LMI—provided the unknown parameters live in a known polytope.
- (S2) Proof that the PDE is solvable provided the columns of the matrix ϕ satisfy some, directly verifiable, involutivity conditions.
- (S3) Imposing some rank conditions on the regressor matrix an *explicit formula* for a “monotonizing” \mathcal{B} is provided. Hence, adding dynamic scaling, a complete solution to the stabilization problem is given.

3.1 Feasibility of an LMI and a PDE imply monotonicity

In this and the next subsection result (S1) above is established. To state the result the following is needed.

Assumption 4. Define the set of constant matrices

$$\mathcal{R} := \{\nabla\mu(\theta), \theta \in \mathbb{R}^q\} \subset \mathbb{R}^{p \times q},$$

i.e., the image of the mapping $\nabla\mu : \mathbb{R}^q \rightarrow \mathbb{R}^{p \times q}$. There exists a constant matrix $D \in \mathbb{R}^{q \times p}$ such that the following (parameterized) LMI is feasible

$$DR + (DR)^\top < 0, \quad \forall R \in \mathcal{R}. \quad (17)$$

Proposition 3. Consider the factorizable nonlinear term (15), for which Assumption 4 holds. For all solutions β of the linear PDE

$$\nabla\beta(x)\phi(x) = -D, \quad (18)$$

the mapping \mathcal{Q}^x , defined in (2), satisfies Assumption 1.

Remark 2. A necessary condition for Assumption 4 to hold is that the number of uncertain functions is not smaller than the number of parameters, *i.e.*, $p \geq q$. If this is not the case, however, a non-strict inequality would insure (plain) monotonicity, which might be enough to carry out the stability analysis. See Remark R5 in Liu et al. [2009a].

3.2 A class of systems verifying the conditions of

Proposition 3

To render Proposition 3 an effective synthesis tool, we should give explicit *conditions on the data* of the problem, *i.e.*, ϕ and μ , to verify Assumption 4 and to solve the PDE (18). Regarding the former, condition (17) is a parameterized (infinite dimensional) LMI that is very difficult to solve Apkarian et al. [2000]. It can be transformed into a

standard LMI if the set \mathcal{R} is a polytope, for instance, the convex hull of a finite set, that is, if

$$\mathcal{R} = \text{conv}\{R_1, \dots, R_k\}, \quad R_i \in \mathbb{R}^{p \times q}, \quad k \in \mathbb{Z}_+. \quad (19)$$

In this case, the parameterized LMI (17) is equivalent to

$$DR_i + (DR_i)^\top < 0, \quad i = 1, \dots, k,$$

whose feasibility can be easily checked. This raises the question of characterizing mappings μ such that \mathcal{R} can be embedded into such a polytope. In this respect, we recall the following two facts.

Fact 1 The second order approximation—around an arbitrary vector $\theta_* \in \mathbb{R}^q$ —of *any nonlinear*, twice differentiable function μ_i verifying (16) is of the form

$$\mu_i = \frac{1}{2}\theta^\top A_i \theta + b_i^\top \theta, \quad (20)$$

with $A_i = A_i^\top \in \mathbb{R}^{q \times q}$, $b_i \in \mathbb{R}^q$. Thus, $\nabla \mu_i$ can be locally approximated by

$$\nabla \mu_i = A_i \theta + b_i, \quad (21)$$

which is an affine mapping.

Fact 2 The image of a convex set under the action of an affine mapping is convex, *cf.*, Section 2.3.2 of Boyd et al. [2004].

From these two facts, and a simple continuity argument, we conclude that—under some reasonable conditions of parameter prior knowledge—Assumption 4 can always be, *locally*, verified with a standard LMI. More precisely, we have the following. (See also the discussion in Section 5.)

Proposition 4. Assume $\theta \in \Theta \subset \mathbb{R}^q$, with

$$\Theta := \{\theta \in \mathbb{R}^q \mid \theta_i \in [\theta_i^m, \theta_i^M] \subset \mathbb{R}\}.$$

For *any nonlinear*, twice differentiable mapping μ , verifying (16), its quadratic approximation (20) verifies the infinite dimensional LMI (17) if and only if the standard LMI

$$D\nabla \mu(v_i) + [D\nabla \mu(v_i)]^\top < 0, \quad i = 1, \dots, (2^q)^q$$

is feasible, where the vectors $v_i \in \Theta$ are computable from the vertices of Θ .

To verify Assumption 4, the monotonicity of (some of) the functions μ_i can be exploited to define a suitable matrix D —to which no condition (positivity or other) is imposed. Let us first illustrate this with the simplest case of $q = 1$. Assume that, at least one of the functions μ_i has a non-zero derivative. Clearly, the row vector $D \in \mathbb{R}^{1 \times p}$ with elements

$$D_i = \begin{cases} -\kappa \text{sign}(\mu'_i), & \text{if } \mu_i \text{ is monotone} \\ 0, & \text{if } \mu_i \text{ is arbitrary} \end{cases} \quad (22)$$

with $\kappa > 0$, satisfies Assumption 4. To construct D when $q > 1$ the “good” elements of the matrix $\nabla \mu$, that is, those verifying

$$\nabla_{\theta_j} \mu_i(\theta) \neq 0, \quad (i, j) \in \{1, \dots, p\} \times \{1, \dots, q\},$$

are first identified. Then, we try to “recombine” these terms to ensure the required positivity, by pre-multiplying by a matrix D . It should be underscored that the derivation above proves that to identify one parameter it is enough to have one monotonic function μ_i , and all the other functions are *arbitrary*. Similar conclusions are applicable for $q > 1$.

3.3 Conditions for solvability of the PDE

Clearly, solvability of the PDE (18) is not *a priori* guaranteed. Let us now proceed to study this question here and, in particular to establish the result (S2) above. For, we invoke the results of Cheng et al. [2005], where necessary and sufficient conditions for solvability of nonhomogeneous linear PDEs are given. To avoid cluttering notation, we present the results only for the simplest case $q = 1$ and $p = 2$. As will become clear below, more general cases can be treated *mutatis mutandis*.

First, recall that a necessary condition for the solvability of the PDE, provided by Frobenius’ theorem Isidori [1999], is regularity and involutivity of the distribution spanned by the columns of ϕ , denoted $\phi_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$. In our case the condition reduces to

$$\text{rank } \phi = \text{rank} \left[\phi : [\phi_1, \phi_2] \right] = 2,$$

where $[\cdot, \cdot] : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the standard Lie bracket, that is,

$$[\phi_1, \phi_2] := (\nabla \phi_2)\phi_1 - (\nabla \phi_1)\phi_2.$$

To simplify the presentation, assume that there exists a function $\vartheta : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$[\phi_1(x), \phi_2(x)] = \vartheta(x)\phi_1(x). \quad (23)$$

As will become clear below, a similar proposition can be stated for the general case.

Proposition 5. Consider the PDE (18) with $q = 1$ and $p = 2$ and D defined by (22). Assume, the distribution spanned by the columns of ϕ is regular and involutive and that (23) holds. Furthermore, assume the monotone function is μ_2 , that is, $\mu'_2 \neq 0$.⁵ Then, the PDE (18) is solvable.

3.4 A rank condition to obviate the PDE via dynamic scaling

In this subsection result (S3) above, concerned with the dynamic scaling scheme of Proposition 2, is established. Attention is centered on the verification of Assumption 3.

Proposition 6. Suppose Φ is of the form (15), with $n \geq p$, and Assumption 4 holds. The matrix⁶

$$\mathcal{B}(x) := -D[\phi^\top(x)\phi(x)]^{-1}\phi^\top(x)$$

ensures the mapping \mathcal{Q}_B^x given in (3) is strictly monotone.

Remark 3. The assumptions that ϕ is full rank and $n \geq p \geq q$ —that relates the number of “measurements”, uncertain functions and parameters—are essential for the construction. Under these conditions, it is easy to show that the new (higher dimensional) vector $\eta := \mu(\theta)$ can be (asymptotically) reconstructed filtering (1) and left-inverting the matrix ϕ . As explained in Section 5, this approach has several serious drawbacks and practical limitations.

4. AN EXAMPLE WITH SEPARABLE NONLINEARITIES

As an illustration of the developments of Section 3, we consider the Single reaction systems with Monod’s growth laws can be described by models of the form (1) with

⁵ A similar result holds, modifying assumption (23), if μ_1 is the monotone function.

⁶ Since $n \geq p$, we have that $\text{rank } \phi = p$ and $\phi^\top \phi$ is invertible.

$$\Phi(x, \theta) = N\eta(x, \theta), \quad (24)$$

where $N \in \mathbb{R}^n$ is the stoichiometric vector and $\eta : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}_{>0}$ is the reaction kinetics—see Viel et al. [1997] for additional details on the model and the various control problem formulations. A classical model for the reaction kinetics is Monod's growth law that is given by

$$\eta(x, \theta) = \frac{\lambda_0(x)}{\lambda^\top(x)\theta} \quad (25)$$

where $\lambda_0 : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ and $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}^q$. The proposition below characterizes, in terms of a linear PDE, a set of functions $\beta(x)$ to satisfy Assumption 1.

Proposition 7. Consider the function (24) with (25). All solutions $\beta(x)$ of the linear PDE

$$\left[\frac{\partial \beta}{\partial x}(x) \right]^\top N = -\lambda(x) \quad (26)$$

satisfy Assumption 1.

Consider as an example the classical baker's yeast fed-batch fermentation process studied in Moya et al. [2002]. In this case $q = 2, n = 3$

$$N = \begin{bmatrix} 1 \\ -c_1 \\ c_2 \end{bmatrix}, \quad \eta(x, \theta) = \frac{x_2}{1 + x_3^2} \frac{1}{\theta_1 + \theta_2 x_2},$$

with $c_1, c_2 > 0$. Hence

$$\lambda_0(x) = \frac{x_2}{1 + x_3^2}, \quad \lambda(x) = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}.$$

The solutions of the PDE (26) are given by

$$\beta(x) = \begin{bmatrix} -x_1 \\ x_2^2 \\ 2c_1 \end{bmatrix} + \rho(x_2 + c_1 x_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $\rho : \mathbb{R} \rightarrow \mathbb{R}$ an arbitrary function.

5. CONCLUDING REMARKS AND FUTURE WORK

Some extensions to the new framework to design I&I adaptive controllers for nonlinearly parameterized systems of Liu et al. [2009a] have been reported. One shortcoming of the methodology proposed in Liu et al. [2009a] is that the new degree of freedom provided by the design parameter β appears in the form of a gradient—leading to the need to solve a PDE. A contribution of our paper is that this shortcoming is removed, with a suitable adaptation of the dynamic scaling of Karagiannis et al. [2009].

For systems with factorizable nonlinearities a rather complete understanding of the problem is available. It is shown that monotonicity is ensured if the parameterized LMI (17) is feasible, and a standard linear PDE is solved. Sufficient conditions for solvability of the PDE, in terms of the plant characteristics, are presented. As pointed out in Subsection 3.3 the solution of the LMI problem raises several challenging open questions. For instance, for which kind of functions μ_i is the following inclusion true

$$\{\nabla \mu_i(\theta), \theta \in \Theta\} \subset \text{conv}\{\nabla \mu_i(v_1), \dots, \nabla \mu_i(v_s)\}, \quad s \in \mathbb{Z}_+? \quad (27)$$

where the vectors v_i are computable from the data of the problem, *i.e.*, the knowledge of μ and the vertices t_i . In that case, the parameterized LMI (17) becomes a standard LMI that can be easily verified. In the case considered in Proposition 4, μ_i is a convexity preserving map Boyd et al.

[2004]—meaning that the image of μ_i is also convex—and the inclusion (27) holds true with identity. In spite of the huge literature on LMIs, see the recent survey Apkarian et al. [2000], it does not seem to be clear if this property is sufficient for our aim. Current research is under way to address these questions.

As discussed in Remark 3, for factorizable models, defining the vector $\eta = \mu(\theta)$, yields a linear parametrization. Classical parameter adaptation algorithms can then be used to estimate these new parameters and, under an *assumption of invertibility* of the mappings μ , generate $\hat{\theta}$ from $\hat{\eta}$. However, over-parametrization suffers from the following well-known shortcomings Ioannou et al. [1996], Sastry et al. [1989]:

- (i) Performance degradation, *e.g.*, slower convergence, due to the need of a search in a bigger parameter space.
- (ii) The more stringent conditions imposed on the reference signals to ensure the persistency of excitation needed for convergence of the parameters η .
- (iii) Inability to recover the true parameter θ —except for injecting mappings. This stymies the application of this approach in direct adaptive schemes, where the actual parameters θ are needed.
- (iv) Conservativeness introduced when incorporating prior knowledge in restricted parameter estimation.
- (v) Reduction of the domain of validity of the estimates stemming from the, in general only local, invertibility of the mappings μ .

In the recent paper Grip et al. [2010] a Lyapunov perspective is adopted to study systems with matched uncertainties—that, as indicated in Liu et al. [2009a], is the most often studied case in the literature. That is, the authors consider (possibly non-autonomous) systems of the form

$$\dot{x} = f(x) + g(x)[\Phi(x, \theta) + u].$$

Interestingly, an assumption that ensures the existence for a Lyapunov function used later in the analysis—equation (6) of Proposition 3—implies, for this particular class of systems, our monotonicity Assumption 3. Although the analysis in the paper does not invoke monotonicity properties it is our belief that there exists a connection with our developments. It should be mentioned that the examples considered in Grip et al. [2010] can be solved with the theory of Liu et al. [2009a] and the corresponding extensions given here. For instance, in Example 6 of Grip et al. [2010] the system (1) is the scalar, non-autonomous, equation $\dot{x} = -x + u + e^{\theta \sin(t)}$, which clearly verifies Assumption 1 with the choice $\beta(x, t) = x \sin(t)$. Current research is under way to further establish the connections between both approaches.

In the paper we have compromised generality of the results for simplicity of its presentation. Indeed, it has been assumed that the convexity property is strict (or even strong) and that it holds globally, that is, for $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^q$. In more practical situations only plain monotonicity, which holds locally will be attained. As discussed in Liu et al. [2009a,b], see also Remark 2, the analysis in this case is far more intricate, invoking properties of cascaded systems and, essentially, requiring some persistency of excitation conditions. It is our belief that trying to develop a com-

plete theory, at the level of generality considered here, is futile and will only lead to extremely technical—and often unverifiable—assumptions. Instead, we are pursuing this research for particular classes of systems, with emphasis in practical applications.

Acknowledgment

The authors express their gratitude to Daizhan Cheng for his help in the proof of Proposition 5. They also thanks to Franco Blanchini, Patrizio Colaneri and Dimitri Peaucelle for some useful discussions on LMIs and convexity preserving mappings.

REFERENCES

- A. Annaswamy, F. P. Skantze, and A.P. Loh, “Adaptive control of continuous-time systems with convex/concave parametrizations”, *Automatica*, vol. 34, pp. 33–49, 1998.
- P. Apkarian and H. D. Tuan, “Parameterized LMIS in control theory”, *SIAM Journal on Control and Optimization*, vol. 38, no. 4, pp.1241–1264, 2000.
- A. Astolfi and A. Ortega, “Immersion and invariance: A new tool for stabilization and adaptive control of nonlinear systems”, *IEEE Trans Automatic Control*, vol. 48(4), pp. 590–606, 2003.
- A. Astolfi, D. Karagiannis and R. Ortega, *Nonlinear and Adaptive Control Design with Applications*, Springer-Verlag, London, 2007.
- H. Grip, T. Johansen, L. Inslund and G. Kaasa, “Parameter estimation and compensation in systems with nonlinearly parameterized perturbation”, *Automatica*, Vol. 46, No. 1, pp. 19–28, 2010.
- A. Astolfi, R. Ortega and A. Venkatraman, “A globally exponentially convergent immersion and invariance speed observer for n degrees of freedom mechanical systems”, *48th IEEE Conference on Decision and Control*, Shanghai, P.R. China, pp. 6508–6513, 2009. (Also to appear in *Automatica*.)
- S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, NewYork, 2004.
- D. Cheng, A. Astolfi and R. Ortega, “On feedback equivalence to port controlled hamiltonian systems”, *Systems and Control Letters*, vol. 4, No. 9, pp. 911–918, 2005.
- V. Fomin, A. Fradkov and V. Yakubovich, *Adaptive Control of Dynamical Systems*, Eds. Nauka, Moskow, 1981 (in Russian).
- A. Fradkov, “Integro-differentiating velocity gradient algorithms”, *Sov. Phys. Dokl.* vol. 31, No. 2, pp. 97–98, 1986.
- A. Fradkov, R. Ortega and G. Bastin, “Semi-adaptive control of convexly parametrized systems with application to temperature regulation of chemical reactors”, *Int. J. of Adaptive Control and Signal Processing*, vol.15, pp. 415–426, 2001.
- P. A. Ioannou and J. Sun, *Robust Adaptive Control*, Printice Hall, 1996
- A. Isidori, *Nonlinear Control Systems II*, Springer-Verlag, London, 1999.
- D. Karagiannis, M. Sassano and A. Astolfi, Dynamic scaling and observer design with application to adaptive control, *Automatica*, Vol. 45, pp. 2883–2889, 2009.
- H. K. Khalil, *Nonlinear Systems*, Third Edition, Prentice Hall, 1996.
- I. Landau, *Adaptive Control: The Model Reference Approach*, Marcel–Dekker, New York, 1979.
- X. Liu, R. Ortega, H. Su and J. Chu, “A framework for design of I&I adaptive controllers for nonlinearly parameterized nonlinear systems”, *ACC’09*, St. Louis, Missouri, USA, pp. 1268–1273, 2009a. (Also to appear in *IEEE Trans. Automatic Control*.)
- X. Liu, R. Ortega, H. Su and J. Chu, “Identification of nonlinearly parameterized nonlinear models: application to mass balance systems”, *48th IEEE Conference on Decision and Control*, Shanghai, P.R. China, pp. 4682–4685, 2009b.
- P. Moya, R. Ortega, M. S. Netto, L. Praly and J. Picó, “Application of nonlinear time-scaling for robust controller design of reaction systems”, *Int. J. of Robust and Nonlinear Control*, vol. 12(1), pp. 57–69, 2002.
- M. Netto, A. Annaswamy, R. Ortega and P. Moya, “Adaptive control of a class of nonlinearly parametrized systems using convexification”, *Int. J. of Control*, vol. 73, No. 14, pp. 1312–1321, 2000.
- M. Netto, A. Annaswamy, S. Mammar and N. Minoiu, “A new adaptive control algorithm for systems with multilinear parametrization”, in *Taming Heterogeneity and Complexity of Embedded Control*, Eds F. Lamnabhi et al., ISTE Ltd, London, pp. 505–522, 2006.
- R. Ortega, “Some remarks on adaptive neuro-fuzzy systems”, *Intern. J. Adaptive Control and Signal Processing*, vol. 10, pp.79–83, 1996.
- S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*, Prentice Hall, Englewood Cliffs, N.J, 1989.
- E. Sontag and A. Teel, “Changing supply functions in input/state stable systems”, *IEEE Trans. Automatic Control*, vol. 40(8), pp. 1476–1478, 1995.
- I. Y. Tyukin, D. V. Prokhorov and V. A. Terekhov, “Adaptive control with nonconvex parameterization”, *IEEE Trans. Automatic Control*, vol. 48(4), pp. 554–567, 2003.
- I. Y. Tyukin, D. V. Prokhorov and C. V. Leeuwen, “Adaptation and parameter estimation in systems with unstable target dynamics and nonlinear parametrization”, *IEEE Trans. Automatic Control*, vol. 52(9), pp. 1543–1559, 2007.
- B. Yao and M. Tomizuka, “Adaptive robust control of SISO nonlinear a semi-strict feedback form”, *Automatica*, vol. 33(5), pp. 983–900, 1997.
- F. Viel, F. Jadot and G. Bastin, “Robust feedback stabilization of chemical reactors”, *IEEE Trans. Automatic Control*, vol. 42(4), pp. 473–481, 1997. 893–900, 1997.