A Sector Bound Approach to Feedback Control of Nonlinear Systems with State Quantization

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Abstract:
This paper studies the feedback control problem of nonlinear systems in strict-feedback form with state quantizers, which are static and bounded by sectors. Through a novel set-valued map based recursive control design approach, the quantized control system is transformed into an interconnection of several input-to-state stable (ISS) subsystems. The ISS property of the closed-loop system is guaranteed by the recently developed cyclic-small-gain theorem. With an appropriately designed quantized controller, the output of the quantized control system can be steered to within an arbitrarily small neighborhood of the origin.

Keywords: Sector bound, state quantization, nonlinear systems, input-to-state stability (ISS), small-gain.

1. INTRODUCTION

Recent years have seen considerable efforts devoted to quantized control of linear and nonlinear systems. In a quantized control system, the control inputs and/or the measurement outputs are processed by quantizers, which are discontinuous maps from continuous spaces to finite sets. The theoretical and practical importance of quantized control lies in the study of digital control, hybrid systems, discrete-event systems and control with information constraints (see e.g., Lunze (1994); Brockett & Liberzon (2000); Liberzon (2003a,b); Elia & Mitter (2001); Tatikonda & Mitter (2004); Fu & Xie (2005); Ceragioli & De Persis (2007), and the references therein).

The early work in Lunze (1994) introduced a discrete-event model of quantized control systems. Based on the idea of scaling quantization levels, Brockett & Liberzon (2000); Liberzon (2003b); Tatikonda & Mitter (2004) studied feedback control of linear and nonlinear systems with dynamic quantization. Elia & Mitter (2001) characterized the coarsest quantizer for single-input-single-output (SISO) linear systems to achieve quadratic stabilization, and showed that the coarsest quantizer should follow a logarithmic law. By characterizing the coarsest quantizer as a sector bounded uncertainty, Fu & Xie (2005) considered the quantized control of multi-input-multi-output (MIMO) linear systems and analyzed the robustness of the quantized control systems.

We are concerned with feedback control design of nonlinear systems with coarsest/logarithmic quantization. Earlier results in this direction appeared in Liu & Elia (2004), in which the general idea of using (robust) control Lyapunov functions to design (robust) quantized controllers was employed. In Liberzon (2003b), a general class of quantizers were implemented in nonlinear control systems which can be designed input-to-state stable (ISS) with respect to the perturbations caused by the quantization errors. Ceragioli & De Persis (2007) studied the conditions under which a logarithmic quantizer does not cancel the stabilizing effect of a continuous feedback control law, for quantized control of dissipative nonlinear systems. In Ceragioli & De Persis (2007), a set-valued map was employed to describe the sector bounded uncertainty caused by the logarithmic quantizer.

One of the widely studied nonlinear control problems is the control design of nonlinear systems in the strict-feedback form. Backstepping is a powerful tool for stabilization and robust adaptive control design (see Krstić, Kanellakopoulos & Kokotović (1995)). Based on the ISS small-gain theorem in Jiang, Teel & Praly (1994), a small-gain based backstepping method was proposed in Jiang & Mareels (1997) in order to handle input and state dynamic uncertainties. This paper makes a step forward towards backstepping-based quantized feedback control design. A novel recursive design scheme will be presented for quantized control of nonlinear systems with a triangular structure.

Over the past few years, there has been a renewed interest in generalizing the ISS small-gain theorem in Jiang, Teel & Praly (1994) to general dynamical networks composed of ISS subsystems. It is in Teel (2003) that such an extension was firstly announced for dynamical networks of discrete-time ISS systems. Shortly, Dashkovskiy, Rüffer & Wirth (2007) independently developed a matrix-small-gain criterion for dynamical networks with plus-type interconnections and mentioned the cyclic-small-gain condition. An ISS-Lyapunov based matrix-small-gain theorem was also developed in Dashkovskiy, Rüffer
& Wirth (2010). In Jiang & Wang (2008), the dynamical network with max-type interconnections in the ISS gain formulation was systematically studied and more general cyclic-small-gain criteria were developed for networks of input-to-output stable (IOS) systems. The ISS-Lyapunov based cyclic-small-gain theorem was developed in Liu, Hill & Jiang (2011a). As pointed out in Dashkovskiy, Rüffer & Wirth (2007), the matrix-small-gain and the cyclic-small-gain are mathematically equivalent. In this paper, the quantized control system will be transformed into a dynamical network with max-type interconnections, and the cyclic-small-gain theorem will be employed to guarantee the stability of the closed-loop quantized system and to construct an ISS-Lyapunov function to evaluate the influence of the quantization error.

In our very recent work Liu, Jiang & Hill (2011b), we studied the nonlinear output-feedback control problem with actuator dynamic quantization, and discovered that the problem with actuator quantization is closely related to the problem with input dynamic uncertainties; see Jiang & Mareels (1997) for a small-gain based design with input dynamic uncertainties. A modified gain assignment technique together with the cyclic-small-gain theorem was employed to develop a backstepping-based quantized control law with actuator quantization. On the one hand, the method in Liu, Jiang & Hill (2011b) was developed for quantized output-feedback control of a smaller class of nonlinear systems. It cannot be directly generalized to the present class of strict-feedback systems with state quantization, mainly because the discontinuity of quantizers prevents differentiation-based standard backstepping from being applied directly to the state quantization case. See Remark 2 for a detailed discussion. On the other hand, the gain assignment technique in Liu, Jiang & Hill (2011b) was developed for handling the actuator quantization error and cannot be directly generalized to the case of state quantization, which is addressed in the present paper.

In this paper, we make a significant modification of the standard backstepping, in which virtual control laws are not designed explicitly as some functions of the state variables but are restricted to some set-valued maps. The employment of the set-valued maps can effectively overcome the difficulties caused by the discontinuity and the sector bounded uncertainties of the quantizers. By appropriately designing the set-valued maps, the subsystems with virtual control laws belonging to the set-valued maps are rendered ISS with respect to the states of the other subsystems. Furthermore, the gain assignment technique originally introduced in Jiang, Teel & Fraly (1994) will have to be modified to adapt to the set-valued map based design and to assign the ISS gains of the subsystems. As a result, the quantized control system is transformed into an interconnection of ISS subsystems, and its stability property can be analyzed with the cyclic-small-gain theorem. Because of the discontinuity of the quantizers, we will employ the extended Filippov solution introduced in Heemels & Weiland (2007, 2008) to represent the motion of the quantized control system (see Filippov (1988); Clarke et al (1998) for the concept of Filippov solution). One of the few results using set-valued maps in backstepping can be found in Freeman & Kokotović (1996) for non-quantized nonlinear systems.

The rest of the paper is organized as follows. Section 2 presents the problem formulation. Section 3 states the main results of our paper and describes the cyclic-small-gain design for quantized feedback control. Finally, Section 4 offers some concluding remarks. The proofs of the technical lemmas are arranged in the Appendix.

Notations and Definitions: A function \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is positive definite if \( \gamma(s) > 0 \) for all \( s > 0 \) and \( \gamma(0) = 0 \). \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a \( \mathcal{K} \) function (denoted by \( \gamma \in \mathcal{K} \)) if it is continuous, strictly increasing and \( \gamma(0) = 0 \); it is a \( \mathcal{K}_\infty \) function (denoted by \( \gamma \in \mathcal{K}_\infty \)) if it is a \( \mathcal{K} \) function and also satisfies \( \gamma(s) \rightarrow \infty \) as \( s \rightarrow \infty \). Id represents the identity function. A set \( \Omega \subset \mathbb{R}^n \) is called convex if for every \( x, y \in \Omega \) and every \( \lambda \in [0,1] \), \( \lambda x + (1-\lambda)y \in \Omega \). The closed convex hull \( \overline{\mathcal{C}}(\Omega) \) of \( \Omega \subset \mathbb{R}^n \) is the intersection of all the closed convex sets containing \( \Omega \). Let \( X \) and \( Y \) be two sets. A set-valued map \( F : X \rightrightarrows Y \) is a map that associates with any \( x \in X \) a subset \( F(x) \) of \( Y \). A set-valued map \( F \) is upper semi-continuous at \( x_0 \in X \) if for any open \( \Omega_F \) containing \( F(x_0) \) there exists a neighborhood \( \Omega_{x_0} \) of \( x_0 \) such that \( F(\Omega_{x_0}) \subseteq \Omega_F \); it is upper semi-continuous if it is upper semi-continuous at all \( x_0 \in X \).

2. PROBLEM FORMULATION

We consider a class of nonlinear systems in disturbed strict-feedback form:

\[
\dot{x}_i = x_{i+1} + \Delta_i(x_i), \quad 1 \leq i \leq n
\]

\[
x_{i+1} \triangleq u
\]

\[
x_i^q = q_i(x_i), \quad 1 \leq i \leq n
\]

where \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R} \) is the control input, \( x_i = [x_1, \ldots, x_{i-1}, x_i]^T \). \( x_i^q \) is the quantization of \( x_i \). \( q_i \)‘s are state quantizers each of which is a map from \( \mathbb{R} \) to some set-valued map based design and to construct an ISS-Lyapunov function to evaluate the influence of the quantization error.

Assumption 1. For each \( \Delta_i \) (1 \leq i \leq n), there exists a known \( \psi_{\Delta_i} \in \mathcal{K}_\infty \) such that for all \( \hat{x}_i \in \mathbb{R}^d \),

\[
|\Delta_i(\hat{x}_i)| \leq \psi_{\Delta_i}(\hat{x}_i).
\]

Assumption 2. For each 1 \leq i \leq n, the quantizer \( q_i \) is a piecewise constant function, and there exists a known constant 0 \leq b_i < 1 such that for all \( x_i \in \mathbb{R} \),

\[
|q_i(x_i) - x_i| \leq b_i|x_i|.
\]

Remark 1. Following the idea of the sector bound approach in Fu & Xie (2005) and Ceragioli & De Persis (2007), in Assumption 2, we directly assumed the sector bound property of the quantizers, instead of discussing the nonlinearity and the discontinuity of the quantizers in detail.

The objective of this paper is to find, if possible a quantized controller in the form of

\[
u = u(x_1^q, \ldots, x_n^q)
\]

to stabilize system (1)–(3) and to steer \( x_1 \) to within an arbitrarily small neighborhood of the origin.

3. MAIN RESULT

The main result of this paper is summarized in Theorem 1.

Theorem 1. With Assumptions 1 and 2 satisfied, the quantized feedback control problem of system (1)–(3) is solvable. More specifically, under a quantized controller (6), the closed-loop signals are bounded and the state \( x_1 \) is steered to within an arbitrarily small neighborhood of the origin.
We prove Theorem 1 by designing explicitly a quantized controller of the form (6). We will transform the system into a new $[e_1, \ldots, e_n]^T$-system through a recursive control design procedure. Our design procedure is different from the standard backstepping approach. That is, we first recursively design set-valued maps for the $e_i$-subsystems such that if the virtual control laws belong to the set-valued maps, then the $e_i$-subsystems are ISS with respect to the other states $e_j$’s ($j \neq i$). Then, we use the cyclic-small-gain theorem to guarantee the ISS (more precisely, input-to-state practical stability (ISpS); see Jiang, Teel & Praly (1994)) of the $[e_1, \ldots, e_n]^T$-system. Denote $\tilde{e}_i = [e_1, \ldots, e_i]^T$.

3.1 Recursive Control Design Procedure

For convenience of notations, we define the directed distance from a point $\tilde{x} \in \mathbb{R}^n$ to a closed convex set $\Omega \subset \mathbb{R}$ as

$$d(\tilde{x}, \Omega) = \begin{cases} \frac{\tilde{x} - \max \Omega}{\max \Omega}, & \text{if } \tilde{x} > \max \Omega; \\ \frac{\tilde{x} - \min \Omega}{\min \Omega}, & \text{if } \tilde{x} < \min \Omega; \\ 0, & \text{otherwise.} \end{cases}$$

**Initial Step:** The $e_1$-subsystem

Let $e_1 = x_1$. The $e_1$-subsystem can be written as

$$\dot{e}_1 = x_2 - e_2 + (\Delta_1(e_1) + e_2)$$

with variable $e_2$ to be defined later. From Assumption 1, one can find a $\Psi_{\Phi^*_1} \in \mathcal{K}_{\omega}$ such that

$$|\Phi^*_1(\tilde{e}_2)| \leq \Psi_{\Phi^*_1}(|\tilde{e}_2|).$$

Introduce the following set-valued maps

$$\tilde{S}_1(x_1) = \left\{ \kappa_1(x_1 + d_1) : |d_1| \leq b_1|x_1| \right\}$$

and

$$S_1(x_1) = \left\{ d_{12}p_1 : p_1 \in \tilde{S}_1(x_1), \frac{1}{1+b_2} \leq d_{12} \leq \frac{1}{1-b_2} \right\}$$

where $\kappa_1$ is some continuously differentiable, odd, strictly decreasing and radially unbounded function.

Define

$$e_2 = d(x_2, S_1(x_1)).$$

Thus, we have $x_2 - e_2 \in S_1(x_1)$.

**Remark 2.** With the method in Liu, Jiang & Hill (2011b), we should design a virtual control law $x_2 - e_2 = \kappa_1(q_1(x_1))$ for the $e_1$-subsystem. However, the discontinuity of the quantizer makes it impossible to take the derivative of $e_2 = x_2 - \kappa_1(q_1(x_1))$, and the differentiation-based standard backstepping cannot proceed. In this paper, by the definitions of $\tilde{S}_1$ and $S_1$, max $S_1(x_1)$ and min $S_1(x_1)$ are continuously differentiable almost everywhere with respect to $x_1$. Lemma 1 below shows that the motion of $e_2$ defined in (12) can be described with a differential inclusion. Furthermore, the new $e_i$-subsystems ($2 \leq i \leq n$) will be constructed in a recursive manner in the following procedure and will also be represented with differential inclusions.

**Recursive Step:** The $e_i$-subsystem ($2 \leq i \leq n$)

By default, $\tilde{S}_i(x_i) := \{0\}$. For each $1 \leq k \leq i - 1$, the set-valued maps $\tilde{S}_k(x_k)$ and $S_k(x_k)$ are defined as

$$\tilde{S}_k(x_k) = \left\{ \kappa_k(x_k - p_{k-1} + d_{k1}) : p_{k-1} \in \tilde{S}_{k-1}(x_{k-1}), |d_{k1}| \leq b_k|x_k| \right\}$$

$$S_k(x_k) = \left\{ d_{k2}p_k : p_k \in S_{k}(x_k), \frac{1}{1+b_{k+1}} \leq d_{k2} \leq \frac{1}{1-b_{k+1}} \right\}$$

where $\kappa_k$ is continuously differentiable, odd, strictly decreasing and radially unbounded; $e_{i+1}$ is defined as

$$\tilde{e}_{i+1} = d(x_{i+1}, \tilde{S}_{i+1}(x_{i+1})).$$

**Lemma 1.** Consider system (1)–(3) with Assumptions 1 and 2 satisfied. With $\tilde{S}_k, S_k, e_{k+1}$ defined in (13), (14) and (15) for each $1 \leq k \leq i - 1$, for any variable $e_{i+1}$, when $e_1 \neq 0$, the $e_i$-subsystem can be represented as

$$\dot{e}_i \in \{x_i - e_{i+1} + \Phi^*_i : \Phi^*_i \in \Phi^*_i(\tilde{e}_{i+1}, x_i)\}$$

where $\Phi^*_i(\tilde{e}_{i+1}, x_i)$ is a convex, compact and upper semi-continuous set-valued map, and there exists a $\Psi_{\Phi^*_i} \in \mathcal{K}_{\omega}$ such that for any $\Phi^*_i \in \Phi^*_i(\tilde{e}_{i+1}, x_i)$,

$$|\Phi^*_i| \leq \Psi_{\Phi^*_i}(|\tilde{e}_{i+1}|).$$

The proof of Lemma 1 is in Appendix A.

**Remark 3.** Due to the “flattened” type definition of $e_i$, the dynamics of the $e_i$-subsystem might be discontinuous at the origin. However, this does not influence the stability property of the $e_i$-subsystem. In Clarke et al (1998), the asymptotic stability of systems described with differential inclusions was discussed on $\mathbb{R}^n \setminus \{0\}$. The ISS results for discontinuous systems in Heemels & Weiland (2007, 2008) can also be easily extended to systems with dynamics discontinuous at the origin.

Define set-valued maps $\tilde{S}_i$ and $S_i$ as in (13) and (14) with $k = i$. Define $e_{i+1}$ as in (15) with $k = i$. Then, it always holds that $x_{i+1} - e_{i+1} \in S_i(x_i)$.

Thus, when $e_1 \neq 0$, we can represent the $e_i$-subsystem with a differential inclusion as:

$$\dot{e}_i \in S_i(x_i) + \Phi^*_i(\tilde{e}_{i+1}, x_i).$$

The extended Filippov solution of the $e_i$-subsystem can be defined with differential inclusion (18) because the set-valued map $S_i(x_i) + \Phi^*_i(\tilde{e}_{i+1}, x_i)$ is convex, compact and upper semi-continuous. See Heemels & Weiland (2007, 2008) for details.

Define $\Phi^*_i(\tilde{e}_i) = \{\Phi^*_i(\tilde{e}_i)\}$ and $\Psi_{\Phi^*_i} = \Psi_{\Phi^*_i}$. Then, the $e_i$-subsystem is also in the form of (18). With Lemma 1, through the recursive design procedure, we can transform the $[x_1, \ldots, x_n]^T$-system into the $[e_1, \ldots, e_n]^T$-system with each $e_i$-subsystem ($1 \leq i \leq n$) in the form of (18).

3.2 ISS of the $e_i$-subsystems ($1 \leq i \leq n$)

Define $\alpha_{s} = \frac{1}{2}s^2$ for $s \in \mathbb{R}_+$. In this subsection, we show that each $e_i$-subsystem ($1 \leq i \leq n$) can be rendered ISS (or, more precisely, ISpS in the sense of Jiang, Teel & Praly (1994)) with the ISS-Lyapunov function

$$V_i(e_i) = \alpha_{s}(|e_i|)$$

by appropriately choosing the $\kappa_i$'s for the $\tilde{S}_i$'s and the $S_i$'s. For convenience of notations, define $V_{n+1} = \alpha_{s}(|e_{n+1}|)$.

**Lemma 2.** Consider the $e_i$-subsystem ($1 \leq i \leq n$) in (18) with the set-valued map $S_i$ defined in (11) or (14). For any specified
\( \varepsilon_i > 0, \upsilon_i > 0, \gamma_{e1}^{i-1}, \ldots, \gamma_{ei}^{i-1} \in K_{\infty}, \) one can find a continuously differentiable, odd, strictly decreasing and radially unbounded \( \kappa_i \) for \( S_i(\xi_i) \) such that \( V_i(e_i) \) satisfies

\[
V_i(e_i) \geq \max_{k=1,\ldots,i-1,i+1} \{ \psi_{ei}^i (V_i(e_i)), \varepsilon_i \} \Rightarrow \max V_i(e_i)(S_i(\xi_i) + \Phi_i^i(\xi_i, \xi_i)) \leq -t_i V_i(e_i). \tag{20}
\]

The proof of Lemma 2 is in Appendix B.

With the Lyapunov formulation of discontinuous ISS systems in Heemels & Weiland (2008), \( V_i(e_i) \) is an ISS-Lyapunov function of the \( e_i \)-subsystem and the \( e_i \)-subsystem is ISS. With Lemma 2, the \( e_i \)-subsystems can be rendered ISS one-by-one in the recursive design procedure. Furthermore, the ISS gains \( \beta^{i}(\cdot) \)'s and \( \chi^{i}(\cdot) \)'s can be designed arbitrarily small or small enough to verify certain cyclic-small-gain conditions.

### 3.3 Quantized Controller

At Step \( i = n \), the true control input \( u \) occurs, and thus we can set \( \bar{e}_{n+1} = 0 \) and \( v_{n+1} = 0 \) in (20). Indeed, our desired quantized controller \( u \) can be chosen as follows:

\[
p_i^1 = \kappa_i(q_i(x_1)) \tag{21}
\]

\[
p_i^i = \kappa_i(q_i(x_i) - p_i^{i-1}, \quad 2 \leq i \leq n - 1 \tag{22}
\]

\[
u = \kappa_n(q_n(x_n) - p_{n-1}^i). \tag{23}
\]

It is directly checked that

\[
p_i^1 \in S_1(x_1) \Rightarrow \cdots \Rightarrow p_i^i \in S_i(\xi_i) \Rightarrow \cdots \Rightarrow u \in S_n(x_n). \tag{24}
\]

### 3.4 Small-gain based Synthesis

Based on the study of interconnected discontinuous dynamical systems in Heemels & Weiland (2007, 2008), the cyclic-small-gain theorem for continuous systems in Jiang & Wang (2008); Liu, Hill & Jiang (2011a) can be generalized to discontinuous systems. We can also construct ISS-Lyapunov functions for discontinuous systems in the same way as for continuous systems. With the help of the cyclic-small-gain theorem, the purpose of this subsection is to fine tune the quantized controller designed above, to yield the ISS property of the closed-loop system.

Denote \( e = [e_1, \ldots, e_n]^T \). With the recursive control design, the \( e \)-system is an interconnection of ISS subsystems with \( e_1, \ldots, e_n \) as the inputs. The cyclic-small-gain condition in Jiang & Wang (2008) can be described as follows: the composition of the gain functions along every simple loop is less than the identity function. According to the recursive control design, given the \( \bar{e}_{i-1} \)-subsystem, by designing the set-valued maps \( S_i \) and \( S_i \), for the \( e_i \)-subsystem, we can assign the ISS gains \( \chi_{ei} \) for \( 1 \leq k \leq i - 1 \) such that

\[
\chi_{e1}^2 \circ \chi_{e1}^1 \circ \cdots \circ \chi_{ei}^1 \circ \chi_{ei}^i < \text{Id}
\]

\[
\chi_{e2}^2 \circ \chi_{e1}^1 \circ \cdots \circ \chi_{ei}^i < \text{Id}
\]

\[
\vdots
\]

\[
\chi_{e1}^i \circ \chi_{ei}^{i-1} < \text{Id}
\]

Applying this reasoning repeatedly, we can guarantee (25) for all \( 2 \leq i \leq n \). In this way, the \( e \)-system satisfies the cyclic-small-gain condition.

Motivated by the Lyapunov-ISS function construction in Jiang, Mareels & Wang (1996), we construct an ISS-Lyapunov function for the \( e \)-system as the maximum influence from

\[
V_i(e_1), \ldots, V_n(e_n) \rightarrow V_i(e_1):
\]

\[
V(e) = \max_{1 \leq i \leq n} \{ \sigma_i(V_i(e_i)) \} \tag{26}
\]

with \( \sigma_i(s) = s \) and \( \sigma_i(s) = \chi_{ei}^i \circ \cdots \circ \chi_{ei}^1(s) \) (\( 2 \leq i \leq n \)). With the \( \chi_{ei}^i \)'s are \( K_{\infty} \) functions continuously differentiable on \((0, \infty)\), slightly larger than the corresponding \( \beta^{i}(\cdot) \)'s and still satisfy the cyclic-small-gain condition. Clearly, \( V(e) \) is differentiable almost everywhere. This type of ISS-Lyapunov functions (26) was used by Dashkovskiy, Rührer & Wirth (2010) for the construction of ISS-Lyapunov function for their matrix-small-gain theorem. Also notice that (26) was utilized in Liu, Hill & Jiang (2011a) in the construction of ISS-Lyapunov function for the ISS cyclic-small-gain theorem.

We can represent the maximum influence from the \( \varepsilon_i \)'s to \( V_i(e_1) \) as:

\[
\theta = \max_{1 \leq i \leq n} \{ \sigma_i(e_i) \}. \tag{27}
\]

With the cyclic-small-gain theorem in Liu, Hill & Jiang (2011a) and the discussions of the discontinuous ISS system and its small-gain theorem in Heemels & Weiland (2008), we have

\[
V(e) \geq \theta \Rightarrow \nabla V(e) \leq -\alpha V(e) \tag{28}
\]

with \( \alpha \) positive definite, wherever \( VV(e) \) exists.

The Lyapunov formulation of ISS in (28) means, \( V(e) \) ultimately converges to within the region \( V(e) \leq \theta \). Recall the definitions of \( V_i \) and \( V \) (see (19) and (26)). One can see \( V(e) \geq \sigma_i(V_i(e_i)) \rightarrow V_i(e_i) = \alpha_i(e_i) \). Thus, \( x_1 := e_1 \) ultimately converges to within the region \( |x_1| \leq \alpha_i^{-1}(\theta) \).

In the recursive design procedure, we can design the \( \chi_{ei}^i \)'s (and thus the \( \beta^{i}(\cdot) \)'s arbitrarily small to get arbitrarily small \( \sigma_i \)'s (\( 2 \leq i \leq n \)). We can also design the \( \varepsilon_i \)'s (\( 1 \leq i \leq n \)) arbitrarily small. In this way, we can design the \( \theta \) arbitrarily small. Thus, with the quantized controller appropriately designed in the recursive design procedure, \( x_1 := e_1 \) ultimately converges to within an arbitrarily small neighborhood of origin.

### 4. CONCLUSIONS

This paper has proposed a novel recursive control design method for quantized nonlinear feedback control by using set-valued maps, gain assignment technique and the recently developed cyclic-small-gain theorem. A new class of quantized feedback controllers is obtained for the popular class of nonlinear strict-feedback systems. Moreover, the output \( x_1 \) of the closed-loop quantized system can be steered to within an arbitrarily small neighborhood of the origin.

### REFERENCES


A.R. Teel. Private communications.

Appendix A. PROOF OF LEMMA 1

We simply use $S_1$ and $S_2$ to denote $S_k(x_k)$ and $S_k(x_k)$ for $1 \leq k \leq i−1$. We only consider the case of $e_i > 0$. The proof for the case of $e_i < 0$ is similar.

Consider the recursive definitions of $S_k$ in (14). Note that $0 < b_k < 1$ for $1 \leq k \leq n$. For $1 \leq k \leq i−1$, the strictly decreasing properties of the $k_o^i$’s imply

$$\max S_{k−1} = \max \left\{ d_{k2} \max S_k : \frac{1}{1+b_{k+1}} \leq d_{k2} \leq \frac{1}{1-b_{k+1}} \right\}$$ (A.1)

$$\max S_k = k_o^i (x_k - \max S_{k−1} - b_k |x_k|)$$ (A.2)

From (A.1), $\max S_{k−1}$ is continuously differentiable almost everywhere with respect to $\max S_{k−1}$. By iteratively using (A.2), we can see $\max S_{k−1}$ is continuously differentiable almost everywhere with respect to $x_{k−1}$. Thus, $\max S_{k−1}$ is continuously differentiable almost everywhere with respect to $x_{k−1}$. Define

$$\partial \max S_{k−1} = \bigcap_{\epsilon > 0} \bigcap_{\mu, \tilde{\theta} \in [0,1]} \{ \mathcal{V} \max S_{k−1} (\mathcal{B}_\epsilon (x_{k−1}), \mu, \tilde{\theta}) \}$$ (A.3)

where $\mathcal{B}_\epsilon (x_{k−1})$ is a ball of radius $\epsilon$ around $x_{k−1}$. Then, $\partial \max S_{k−1}$ is convex, compact and upper semi-continuous. Refer to Heemels & Weiland (2007) for a recent result on this kind of definitions for discontinuous systems.

In the case of $e_i > 0$, the $e_i$-subsystem can be represented with a differential inclusion as

$$e_i \in \left\{ x_i+1 + \Delta_i (x_i) - \phi_i : \phi_i \in \partial \max S_{k−1} \right\}$$

$$\Phi_i^*(\bar{e}_{i+1}, x_i) = \left\{ e_i+1 + \Delta_i (x_i) - \phi_i : \phi_i \in \partial \max S_{k−1} \right\}$$ (A.5)

Because $\Delta_i (x_i)$ and $x_{k−1}$ are locally Lipschitz, and $\partial \max S_{k−1}$ is convex, compact and upper semi-continuous, $\Phi_i^*(\bar{e}_{i+1}, x_i)$ is convex, compact and upper semi-continuous.

From (1)–(2) and Assumption 1, $|\Delta_i (x_i)|$ is bounded by a $\mathcal{K}_\infty$ function of $|x_i|$ and $|x_{k−1}|$ is bounded by a $\mathcal{K}_\infty$ function of $|x_i|$. Hence, there exists a $\Psi_{\Phi_i^*} \in \mathcal{K}_\infty$ such that for any $\phi_i^* \in \Phi_i^*(\bar{e}_{i+1}, x_i)$, it holds that

$$|\phi_i^*| \leq \Psi_{\Phi_i^*} (|\bar{e}_{i+1}|)$$ (A.6)

We show that $|x_i|$ is bounded by a $\mathcal{K}_\infty$ function of $|\bar{e}_{i+1}|$. The definitions of $e_{k+1}$ ($1 \leq k \leq i−1$) in (15) implies

$$|x_{k+1}| \leq \max \{ |\max S_k|, |\min S_k| \} + |e_{k+1}|$$ (A.7)

Define $\kappa^s_k (s) = |\kappa_k (s)|$ for $s \in \mathbb{R}^+$. Then, $\kappa^s_k \in \mathcal{K}_\infty$. From (A.1) and (A.2), for $1 \leq k \leq i−1$. 

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Thus, in the case of $e_i \neq 0$, for $p_{i-1} \in \bar{S}_{i-1}$ and $d_{i1} \leq b_i|x_i|$, 

$$|x_i - p_{i-1} + d_{i1}| \geq (1 - b_i)|e_i|, \quad \text{sign}(x_i - p_{i-1} + d_{i1}) = \text{sign}(e_i).$$

With (17) satisfied, one can find $\psi^*_{\phi^1_i}, \ldots, \psi^*_{\phi^1_{i+1}} \in \mathcal{K}_e$ such that for any $\phi^*_i \in \Phi_i^*(\hat{e}_i+1, \bar{x}_i)$, 

$$|\phi^*_i| \leq \sum_{k=1}^{i+1} \psi^*_{\phi^1_k}(|e_k|).$$

Note that $0 \leq b_i, b_{i+1} < 1$. From Lemma 1 in Jiang & Mareels (1997), for any $e_i > 0$, one can find a $V_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is positive, nondecreasing and continuously differentiable on $(0, \infty)$, and satisfies 

$$\frac{1 - b_i}{1 + b_{i+1}} V_i((1 - b_i)s) s \geq \frac{1}{2} s + \sum_{k=1}^{i+1} \psi^*_{\phi^1_k}(s) + \sum_{k=1}^{i+1} \psi^*_{\phi^1_k}(|e_k|).$$

for $s \geq \sqrt{2e_i}$. 

With this kind of $V_i$, define $\kappa_i(r) = -V_i(|r|)$ for $r \in \mathbb{R}$. Then, $\kappa_i$ is odd, strictly decreasing, radially unbounded and continuously differentiable on $(-\infty, 0) \cup (0, \infty)$. With direct calculation, we have $\lim_{r \to -0^+} \frac{d^2\kappa_i(r)}{dr^2} = \lim_{r \to -0^-} \frac{d^2\kappa_i(r)}{dr^2}$ and $\kappa_i$ is continuously differentiable.

Recall $V_i(e_i) = \alpha_0(|e_i|) = \frac{1}{4}|e_i|^2$. Consider the case of 

$$V_i(e_i) \geq \max_{k=1, \ldots, i-1, i+1} V_i^2(V_i(e_k), e_i).$$

In this case, we have 

$$|e_i| \leq \alpha_i^{-1} \circ (\gamma_i^1)^{-1} \circ \alpha_0(|e_i|), \quad k = 1, \ldots, i-1, i+1 \quad (B.6)$$

$$|e_i| \geq \sqrt{2e_i}. \quad (B.7)$$

For any $|d_{i1}| \leq b_i|x_i|$, $p_{i-1} \in \bar{S}_{i-1}$, and $\frac{1}{1+b_{i+1}} \leq d_{i2} \leq \frac{1}{1-b_i}$ and $\phi^*_i \in \Phi_i^*(\hat{e}_{i+1}, \bar{x}_i)$, using (B.3)–(B.7), we successfully achieve 

$$\nabla V_i(e_i)(d_{i1}\kappa_i(x_i - p_{i-1} + d_{i1}) + \phi^*_i)$$

$$= e_i(-d_{i2} V_i(|x_i - p_{i-1} + d_{i1}|)(x_i - p_{i-1} + d_{i1}) + \phi^*_i)$$

$$\leq -d_{i2} V_i(|x_i - p_{i-1} + d_{i1}|) |x_i - p_{i-1} + d_{i1}| |e_i| + |e_i||\phi^*_i|$$

$$\leq |e_i| \left( -\sum_{k=1}^{i+1} \sum_{k=1}^{i+1} \psi^*_{\phi^1_k}(|e_k|) \left( 1 - \frac{b_i}{1 + b_{i+1}} V_i((1 - b_i)|e_i|)(1 - b_i)|e_i| + \sum_{k=1}^{i+1} \psi^*_{\phi^1_k}(|e_k|) \right) \right)$$

$$\leq |e_i| \left( -\sum_{k=1}^{i+1} \sum_{k=1}^{i+1} \psi^*_{\phi^1_k}(|e_k|) \left( 1 - \frac{b_i}{1 + b_{i+1}} V_i((1 - b_i)|e_i|)(1 - b_i)|e_i| + \sum_{k=1}^{i+1} \psi^*_{\phi^1_k}(|e_k|) \right) \right)$$

$$\leq \frac{\bar{t}_i}{2}|e_i|^2 = -t_i V_i(e_i).$$

Thus, in the case of (B.5), we obtain 

$$\max \nabla V_i(e_i)(S_i(\bar{x}_i) + \Phi_i(\hat{e}_{i+1}, \bar{x}_i)) \leq -t_i V_i(e_i).$$

This ends the proof.