On Using Distorted Sensors for Set-based Multi-scale Actuator Fault Diagnosis

C. Combastel ∗ Q. Zhang ∗∗ S.A. Raka ∗

* ECS-Lab (EA n°3649) - ENSEA, 6 avenue du Ponceau, 95014 Cergy-Pontoise Cedex, France (e-mail: {combastel, raka}@ensea.fr).
** INRIA, Campus de Beaulieu, 35042 Rennes Cedex, France. (e-mail: zhang@irisa.fr)

Abstract: The off-line fault detection, isolation and identification of multiplicative actuator faults using sensors providing measurements affected by unknown nonlinear distortions and bounded noises are addressed. The only assumption about the unknown functions modeling the sensor distortions is their monotonicity. Bounded disturbances are also considered as unknown inputs of the system time-varying dynamics. Polytopic faulty parameter sets resulting from the search of robust output pairs and from zonotope computations are the basis for a decision scheme which makes use of multi-scale temporal windows. A numerical example modeling part of a distillation column illustrates the off-line diagnosis of three multiplicative actuator faults from a single low-cost noisy sensor with unknown nonlinear distortion and constant bias.

Keywords: Fault diagnosis, Uncertain dynamic system, Nonlinear distortion, Sets, Bounding method, Intervals, Zonotopes, Polytopes, Fault identification, Robust estimation.

1. INTRODUCTION

Fault diagnosis is becoming an essential functionality integrated in more and more engineering systems due to their increasing complexity, from safety-critical infrastructures to household appliances. Research activities have been motivated by this trend in industry to develop theory and methods for fault diagnosis, as demonstrated by the vast literature on this topic, for instance, Basseville and Nikiforov (1993); Gertler (1998); Chen and Patton (1999); Blanke et al. (2003); Simani et al. (2003); Korbič et al. (2004); Isermann (2005); Vachtsevanos et al. (2006); Ding (2008); Stoustrup and Zhou (2008); Sobhani-Tehrani and Khorasani (2009). A common aspect of all these methods is that decisions of diagnosis are always based on the analysis of sensor signals, eventually assisted by some prior information about the monitored system. As the sensors themselves are also subject to several kinds of uncertainties (noises, distortions, failures), it is important to take into account such eventualities in order to design reliable fault diagnosis methods.

This paper focuses on off-line actuator fault diagnosis when all the sensors are subject to unknown nonlinear distortions and bounded noise; bounded disturbances are also considered. For usual methods dealing with both actuator and sensor faults, a sufficient degree of redundancy is typically assumed among the sensors, such that the valid sensors can provide the information required for fault diagnosis; see, e.g., Chen and Patton (1999). However, most analytical redundancy techniques based on decoupling and/or elimination (including those dedicated to non-linear systems such as DePersis and Isidori (2001)) do not consider unknown non-linear distortions possibly affecting all the available sensors. Moreover, a passive approach based on the choice of conservative thresholds on approximately decoupled residuals may lead to strong miss-detections. The off-line diagnosis approach proposed in this paper is mainly formulated as a set-membership estimation of some parameters modeling actuator faults under very weak assumptions about the available sensor(s). Of course, if a single sensor is considered, the isolation of a sensor fault from other faults is not possible; however, under sufficient input excitation, isolation of actuator faults remains possible in spite of bounded disturbances and unknown non-linear sensor distortions.

The assumptions made in this paper about sensors are motivated by practical considerations: for products of mass production, costs reduction both leads to reducing the number of sensors and to choosing cheaper ones. The former involves a low degree of redundancy making it difficult to apply usual decoupling techniques; the latter may result in highly non linear sensor characteristics. Unfortunately, sensor nonlinearity may vary within a series of production and/or during the normal life duration of each piece in use. For example, most oxygen sensors used in cars equipped with catalytic converters are highly nonlinear: they roughly indicate if the oxygen concentration is over or below a reference value. Nevertheless, such sensors are sufficient for the purpose of engine control. It is assumed in this paper that, upon a bounded noise, each sensor can be affected by an unknown and arbitrary, but strictly monotonic, nonlinearity. The monotonicity is a weak assumption, since any non-monotonic distortion would lead to ambiguous sensor values. Compared to the previous studies reported in Zhang (2006, 2008), bounded uncertainties, both before and after sensor distortion, are fully

Copyright by the International Federation of Automatic Control (IFAC) 2015
addressed in the present paper; whereas in the previous studies the uncertainty after sensor distortion was ignored, and the system uncertainties (before sensor distortion) were incompletely analyzed. Dealing with bounded uncertainties is achieved through the use of interval analysis, Jaulin et al. (2001), and, more specifically, zonotopes, Kühn (1998); Combastel (2005), and polytopes, Ziegler (1994); Blesa et al. (2010). Moreover, the assumption of constant fault parameters is partially relaxed by using a multi-scale approach in a logically sound decision step. Compared to existing results about robust methods for fault diagnosis Benbouzid et al. (1999); Tse et al. (2004); Ma et al. (2004); Kumamaru et al. (2004), the main novelty of the approach presented in this paper is its ability to deal with unknown nonlinear sensor distortions without requiring sensor redundancy.

This paper is organized as follows: After a description of the problem statement in section 2, an outline of the proposed FDI scheme is given in section 3. Firstly, reachable set computations based on zonotopes combined with parameter sensitivity evaluation provide a parameterized expression of bounds for the unknown output preceding the nonlinear sensor distortion (section 4). Secondly, an algorithm returning a set of robust output pairs is studied in section 5. It provides a constructive way to take into account the monotonicity of the unknown nonlinear sensor distortion. Thirdly, a FDI decision based on polytopic parameter sets and multi-scale temporal windows is proposed in section 6. A numerical example modeling part of a distillation column (section 7) finally illustrates the FDI decision results, before drawing some conclusions.

2. PROBLEM STATEMENT

Let us consider a linear system with sensor distortion in the form of

\[
\begin{align*}
x_{k+1} &= A_k x_k + B_k \text{diag} (u_k) \theta + E_k v_k \quad (1a) \\
z_k &= C_k x_k + D_k \text{diag} (v_k) \theta + F_k v_k \\ y_k &= h(z_k) + \text{diag} (g(v_k)) w_k \quad (1c)
\end{align*}
\]

where \( x_k \in \mathbb{R}^n \) is the state, \( u_k \in U \subset \mathbb{R}^r \) the known and bounded input, \( z_k \in \mathbb{R}^m \) the unknown output before sensor distortion, \( y_k \in \mathbb{R}^m \) the known output after sensor distortion. Each element of \( z_k \) refers to an unknown physical value, a distorted image of which is available through the measurement given by the related sensor output in \( y_k \). \( v_k \in [-1, 1]^p \) and \( w_k \in [-1, 1]^m \) are bounded uncertainties, \( \theta \in \mathbb{R}^q \) is a coefficient vector describing the efficiency loss of actuators (multiplicative actuator faults). The initial state \( x_0 \) is assumed to be bounded: \( x_0 \in [x_0] \), where \( [x_0] \subset \mathbb{R}^n \) is a known domain. \( A_k, B_k, C_k, D_k, E_k, F_k \) are time-varying real matrices of appropriate sizes. \( h : \mathbb{R}^m \rightarrow \mathbb{R}^q \) represents nonlinear sensor distortions and the vector \( g \) is such that \( g > 0 \) models the bounded noise magnitude of each independently distorted sensor. In this paper, the relational operators like “\( \geq \)” are vectorized using a direct element-by-element extension of the scalar case. \( \text{diag}(g) \) is a diagonal matrix filled with \( g \). For fault-free actuators, \( \theta \) takes some nominal value \( \theta_0 \). Typically \( \theta_0 = [1, \ldots, 1]^T \).

**Remark 1.** The uncertainty terms \( E_k v_k \) and \( F_k v_k \) may seem correlated. By setting some columns of \( E_k \) and the complementary columns of \( F_k \) to zeros, then \( E_k v_k \) and \( F_k v_k \) are uncorrelated, but this is not mandatory for the FDI algorithm proposed in this paper to work properly.

The nonlinear function \( h \) describing sensor distortions is component-wise defined. For \( s = 1, \ldots, m \), let \( z_{s,k}, g_s, w_s, y_s \) be respectively the \( s \)-th component of \( z, g, w, y \), then:

\[
y_{s,k} = h_s(z_{s,k}) + g_s w_{s,k}
\]

(2)

Each sensor distortion \( h_s \) is an **unknown and arbitrary** nonlinear function, but subject to the following assumption:

**Assumption 2.** Each sensor distortion \( h_s : \mathbb{R} \rightarrow \mathbb{R} \) is a strictly monotonic increasing function. In other words, for any \( \zeta, \xi \in \mathbb{R} \),

\[
\zeta < \xi \Rightarrow h_s(\zeta) < h_s(\xi)
\]

(3)

In order to ensure bounded trajectories of system (1), the following assumption on system matrices is introduced:

**Assumption 3.** The matrices \( A_k, B_k, C_k, D_k, E_k, F_k \) filled with real entries are bounded. \( A_k \) is such that the homogeneous system \( x_{k+1} = A_k x_k \) is exponentially stable.

The problem considered in this paper is the detection and isolation of multiplicative actuator faults, modeled by changes in the coefficient vector \( \theta \), from input-output data (sampled records of \( u_k \) and \( y_k \), for \( k = 1, \ldots, k_{\text{max}} \)), despite the unknown sensor distortions \( h_s \), \( s = 1, \ldots, m \), and the bounded additive uncertainties.

In the previous studies reported in Zhang (2006, 2008), the uncertainty \( \text{diag}(g) w_k \) after sensor distortion was ignored; the system uncertainties \( E_k v_k \) and \( F_k v_k \) were considered under assumptions similar to those made above, but their assumed bounds were not explicitly used in the analysis for fault diagnosis. In the present paper, all these uncertainties are fully considered in a rigorous analysis, so that the decisions for fault diagnosis are entirely determined by the assumed uncertainty bounds.

3. OUTLINE OF THE PROPOSED FDI SCHEME

The aim of the proposed Fault Detection and Isolation (FDI) scheme consists in estimating the actuator faults modeled by \( \theta \) based on the known inputs \( u_k \) and the known measurements \( y_k \). However, as \( h \) and \( z_k \) are unknown, only a partial information relying on the monotonicity of \( h_s \) can be extracted from the measurements. A fault diagnosis scheme for systems like (1) should make explicit links between \( u_k, y_k, \) and \( \theta \), while rigorously taking the assumed bounds of the uncertainties \( v_k, w_k \) and \( x_0 \) into account. To that purpose, the off-line FDI scheme proposed in this paper decomposes the task into three main steps.

Firstly, parameter dependent interval vectors enclosing \( z_k \) are computed from the system model (1) and the input excitation \( u_k \) for each sample time \( k \). \( z_k \) is bounded as:

\[
\forall k, \quad z_k \in z_k^c + z_k^g \theta + [-z_k^c, +z_k^c]
\]

(4)

\( z_k^c \in \mathbb{R}^m \) and \( z_k^g \in \mathbb{R}^{m \times q} \) define the center of the interval vector enclosing \( z_k \) as an affine function of \( \theta \), and \( z_k^g \in (\mathbb{R}^+)^m \) refers to the (positive) radius of the same interval vector. The matrix \( z_k^g \) represents the sensitivity of \( z_k \) to \( \theta \) at time \( k \). The computation of \( z_k^c, z_k^g \) and \( z_k^c \) will be made explicit in section 4.

12016
Secondly, it is necessary to characterize some information about $z$ that can be extracted from the distorted sensor outputs $y$. As $h$ is unknown (apart from its monotonicity), no absolute value of $z$ can be deducted from $h(z)$. However, the monotonicity of $h$ makes it possible to reason about relative values between some pairs of samples. As a simple example, let us consider the 5th sensor with no noise ($g_5 = 0$ so that (2) reduces to $y_{5,k} = h_k(z_{5,k})$). Then, from a pair of measurements at times $k_1$ and $k_2$ satisfying $y_{5,k_1} \leq y_{5,k_2}$, the strict monotonicity of $h_k$ (i.e. the contraposition of (3)) combined with (2) allows deducing $z_{5,k_1} \leq z_{5,k_2}$. By extending such an approach to the case of noisy sensors ($g_s > 0$), an algorithm computing a set of robust output pairs is presented in section 5.

Finally, as each robust output pair implies an inequality statement like $z_{s,k_1} \leq z_{s,k_2}$, a set of linear inequalities satisfied by $\theta$ can be derived from (4). A polytopic domain enclosing the set of possible actuator fault parameter values is so obtained and is further used to compute a FDI decision which will be detailed in section 6.

4. PARAMETER DEPENDENT BOUNDS FOR THE UNKNOWN OUTPUT

The purpose of this section is to compute parameter dependent (i.e. $\theta$ dependent) bounds for $z_k$ as indicated in (4). The proposed algorithm mainly relies on the superposition theorem and the computation of reachable outputs for linear dynamical systems like (1a)-(1b). It makes use of zonotopes, Kühn (1998), a particular class of convex and centrally symmetric polytopes that can be implicitly defined by a matrix without requiring any facets and/or vertices enumeration.

Definition 4. (Zonotope). Let $R$ be an $n \times p$ real matrix. The centered zonotope $Z(\mathbf{R}) \subseteq \mathbb{R}^n$ is defined as the linear image of the $p$-dimensional unit hypercube, $[-1, +1]^p$, by the linear application $R$:

$$Z(\mathbf{R}) = \{z = \mathbf{R}v, \|v\|_\infty \leq 1\}$$

Proposition 5. (Interval hull of a zonotope). The smallest aligned box (or interval hull) enclosing the centered zonotope $Z(\mathbf{R}) \subseteq \mathbb{R}^n$ is the interval vector $[-b(\mathbf{R}), +b(\mathbf{R})] \subseteq \mathbb{R}^n$, where $b(\mathbf{R}) \in (\mathbb{R}^+)^n$ is defined as:

$$b(\mathbf{R}) = |\mathbf{R}|\mathbf{1}$$

where $|\mathbf{R}|$ denotes the determinant of $\mathbf{R}$ and $\mathbf{1}$ denotes a column vector of ones with appropriate size.

The Minkowski sum of two zonotopes $Z(R_1)$ and $Z(R_2)$ is a zonotope that can be computed by a matrix concatenation: $Z(R_1) + Z(R_2) = Z([R_1 \quad R_2])$ and the image of a zonotope $Z(\mathbf{R})$ by a linear application $L$ is a zonotope that can be computed by a matrix product: $LZ(\mathbf{R}) = Z(LR)$.

Assumption 6. The set $[c_0] \subseteq \mathbb{R}^n$ enclosing the initial states of (1) is assumed to be a known zonotope centered on $c_0 \in \mathbb{R}^n$ and with a shape defined by the matrix $R_0$:

$$x_0 \in [c_0] = c_0 + Z(R_0)$$

In order to compute the parameter-dependent bounds of $z(k)$ as formulated in (4), three dynamical systems are introduced:

$$\begin{align*}
\sigma_{k+1}^c &= A_k \sigma_k^c, \\
\sigma_{k+1}^\theta &= A_k \sigma_k^\theta + B_k \text{diag}(u_k), \\
\sigma_{k+1}^v &= A_k \sigma_k^v + E_k v_k, \\
\sigma_0^c &= c_0 \\
\sigma_0^\theta &= 0_{n \times q} \\
\sigma_0^v &= Z(R_0)
\end{align*}$$

It can be noticed that the initial states and the inputs are known in (8) and (9), whereas only the bounds of the initial state and of the inputs are known in (10). More precisely, $\sigma_0^c$ is only known to belong to the zonotope $Z(R_0)$ (where $R_0$ is a known matrix) and $v_k \in [-1, +1]^p$. However, neither $\sigma_0^\theta$ nor $\sigma_0^v$ are assumed to be exactly known. Let us define $x_k$ as follows:

$$x_k = \sigma_k^c + \sigma_k^\theta + \sigma_k^v$$

By respectively substituting (8), (9) and (10) for $\sigma_{k+1}^c$, $\sigma_{k+1}^\theta$ and $\sigma_{k+1}^v$ in the expression of $x_k$ following from (11), $A_k$ can be factorized. It becomes then clear that the superposition of (8)-(10) as indicated in (11) leads to a definition of $x_k$ which is equivalent to the one arising from (1a), (7) and $v_k \in [-1, +1]^p$.

A similar decomposition distinguishing between known terms, parameter dependent terms (sensitivity to $\theta$) and unknown but bounded terms leads to the following definitions for the output terms:

$$\begin{align*}
z_k^c &= C_k \sigma_k^c \\
\theta_k^\theta &= C_k \sigma_k^\theta + D_k \text{diag}(u_k) \\
\theta_k^v &= C_k \sigma_k^v + F_k v_k
\end{align*}$$

Superposition arguments show that the definition of $z_k$ in (15) is equivalent to the one in (1b). $z_k = z_k^c + z_k^\theta + z_k^v$

From (15), it comes: $z_k \in z_k^c + z_k^\theta + [z_k^v]$ where $z_k^c$ and $z_k^\theta$ can recursively be computed using (8), (12) on the one hand, and (9), (13) on the other hand. Moreover, the domain $[z_k^v]$ refers to the reachable output set of the system (10), (14) at time $k$. This discrete time-varying linear dynamical system has bounded inputs: $v_k \in [-1, +1]^p$. The unknown inputs being bounded by a particular zonotope (here, a unit hypercube), the reachable output set $[z_k^v]$ is a zonotope. The interval hull of $[z_k^v]$ is denoted $[-z_k^v, +z_k^v]$ and can be computed as follows:

$$R_{k+1} = [A_k R_k E_k]$$

(16) describes the recursive computation of the exact reachable state set of (10), $[z_k^v] = Z(R_k)$. Indeed, replacing the variables in (10) by their membership set gives $[z_{k+1}^v] = A_k Z(R_k) + Z(E_k)$ and (16) follows from the elementary zonotope properties (Minkowski sum, linear image).

Similar considerations based on (14) first give $[z_k^v] = C_k Z(R_k) + Z(F_k)$ and (17) follows from zonotope properties (Minkowski sum, linear image and interval hull).

Remark 7. When the size $p$ of $v_k$ and the final sample time, $k_{\text{max}}$, are not too large, the number of columns of $R_k$ may remain reasonable as far as the memory and the computation time are concerned. In this case, the implementation of the off-line FDI scheme does not require a complexity reduction step preventing the number of columns of $R_k$ from exceeding a predefined value. Then, the reachable sets are exactly computed (up to numerical errors) and $[-z_k^v, +z_k^v]$ provides a tight box enclosure of the
true reachable output set. Otherwise, a reduction operator has to be used to limit the number of columns of $R_k$, as explained in [2005] for instance.

**Remark 8.** When $E_k = 0$, (16) can be simplified as $R_{k+1} = A_k R_k$. More generally, the null columns of $E_k$ need not be concatenated to build $R_{k+1}$. Notice also that the null columns are the first to be removed when a reduction operator like the one in [2005] is implemented.

Following the results and the computations presented in this section, $z^g_{s,k}, z^p_{s,k}$, and $z^p_{s,k}$ can now be considered as known values in (4). $z^p_s$ has been bounded while preserving the dependencies with respect to the parameters $\theta$ modeling actuator faults in the system (1). Such dependencies are explicitly computed through the sensitivity term $z^p_{s,k}$ in (4).

5. A SET OF ROBUST OUTPUT PAIRS

After the characterization of some explicit links between $\theta$ and $z^p_s$ in the previous section, the aim of this section consists in linking the measurements $y_{s,k}$ to $z^p_s$ so that a set enclosing the possible values of $\theta$ can be computed from the available measurements in the next section.

As previously mentioned in section 3, the only knowledge about $h$ being its monotonicity, this limited knowledge does not allow deducing information about $z_{s,k}$ (value, sign, etc. . . ) directly from $y_{s,k}$, even when the measurement is noise free ($g_s = 0$). However, it is possible to get information about the relationship between signal values taken at two different samples times $k_1$ and $k_2$. But unlike the case studied in [2008] ignoring the measurement noise, not all pairs of measurements $(y_{s,k_1}, y_{s,k_2})$ can provide here information about $(z_{s,k_1}, z_{s,k_2})$ because of the measurement noise modeled by the bounded uncertainty $w_k$ in (1e). The characterization of the link between informative measurement pairs and the resulting information about $\theta$ in the presence of bounded measurement noise is formulated in (18):

$$\forall s, \forall \{k_1, k_2\}, \quad y_{s,k_1} + 2g_s \leq y_{s,k_2} \Rightarrow z_{s,k_1} \leq z_{s,k_2} \quad (18)$$

A proof of (18) can be established by first rewriting the equation (2) modeling the $s$th sensor taken at times $k_1$ and $k_2$: $h_s(z_{s,k_1}) = y_{s,k_1} - g_s w_{s,k_1}$ and $h_s(z_{s,k_2}) = y_{s,k_2} - g_s w_{s,k_2}$. By contradiction of (3), the strict monotonicity of $h_s$ ensures that $h_s(z_{s,k_1}) \leq h_s(z_{s,k_2}) \Rightarrow z_{s,k_1} \leq z_{s,k_2}$. In addition, $h_s(z_{s,k_1}) \leq h_s(z_{s,k_2})$ can equivalently be rewritten as $y_{s,k_1} - g_s w_{s,k_1} + g_s w_{s,k_2} \leq y_{s,k_2}$ using the model of the $s$th sensor at times $k_1$ and $k_2$. As $w_{s,k_1}$ and $w_{s,k_2}$ both belong to $[-1, +1]$, then $y_{s,k_1} - g_s w_{s,k_1} + g_s w_{s,k_2} \leq y_{s,k_1} + 2g_s$. Therefore, $y_{s,k_1} + 2g_s \leq y_{s,k_2} \Rightarrow h_s(z_{s,k_1}) \leq h_s(z_{s,k_2})$ and the strict monotonicity of $h_s$ completes the proof of (18).

Given the measurements of the $s$th sensor over a temporal window $\kappa$, the following of this section is devoted to the search for a set of pairs $(k_1, k_2)$ such that the condition $C(k_1, k_2)$ defined as $y_{s,k_1} + 2g_s \leq y_{s,k_2}$ holds. According to (18), each of such pairs allows deducing $z_{s,k_1} \leq z_{s,k_2}$, what can further be used to get information about $\theta$. Some notations defining a temporal window are first introduced:

$$\kappa = \{k, \ldots, K\} \subset \mathbb{N}, \quad y_{s,\kappa} = [\ldots, y_{s,\kappa}(i), \ldots]^T \quad (19)$$

This refers to a set of consecutive sample times $k \in \mathbb{N}$ between $k$ and $K$. The number of samples in the temporal window defined by $\kappa$ is denoted $\eta$: $\eta = |\kappa| \in \mathbb{N}$ card returns the cardinality of a set. Notice that all the temporal windows considered in the paper are such that $\eta = k - k + 1$. In (19), $\kappa(i)$ denotes the $i$th element of $\kappa$, and the vertical concatenation illustrated in the definition of $y_{s,\kappa}$ can be generalized to any value (other than $y$) which is indexed by $k$. As $C(k_1, k_1)$ and $C(k_1, k_2)$ imply $C(k_1, k_2)$ because $g_s \geq 0$, the set of pairs $(k_1, k_2) \in \kappa^2$ satisfying $C(k_1, k_2)$ may have a large cardinality. However, taking all such pairs into account is not needed due to some redundancies. For instance, let us consider three pairs $(k_1, k_1), (k_1, k_2)$ and $(k_1, k_2)$ all satisfying the condition $C$. The resulting knowledge about $\theta$ is respectively expressed as: $z_{s,k_1} \leq z_{s,k_1}, z_{s,k_1} \leq z_{s,k_2}$ and $z_{s,k_1} \leq z_{s,k_2}$. It becomes then clear that the pair $(k_1, k_2)$ need not be taken into account as the information about $\theta$ it provides is redundant to the one provided by the combination of the consecutive pairs $(k_1, k_1)$ and $(k_1, k_2)$. The purpose of an algorithm returning a set of robust pairs thus consists in finding a maximal set of non redundant pairs satisfying $C$.

The algorithm proposed to compute robust pairs satisfying $C$ takes $Y = y_{s,\kappa}$ and $d = 2g_s$ as inputs and returns a two columns integer matrix $P \in \mathbb{N}^{\eta \times 2}$ defining a set of $r$ pairs in the temporal window $\kappa$. These pairs are stored in $\pi(s, \kappa)$:

$$\pi(s, \kappa) = \kappa(P) \in \mathbb{N}^{\eta \times 2} \quad (20)$$

where $P = \text{robust_pairs}(Y, d)$ (see algorithm 1). $\kappa(P)_{j,l} = \kappa(P)_{j,l} \forall (j, l) \in \{1, \ldots, r\} \times \{1, 2\}$. $\pi(s, \kappa)$ refers to the set of robust pairs obtained from the measurements of the $s$th sensor over the temporal window $\kappa$. The first (resp. second) column of $\pi(s, \kappa)$ is denoted $\pi(s, \kappa)$ (resp. $\pi(s, \kappa)$). The jth pair denoted $\pi(s, \kappa)_j$, $j = 1, \ldots, r$ can be identified with the jth line of $\pi(s, \kappa)$ as follows: $\pi(s, \kappa)_j = (\pi(s, \kappa)^j, \pi(s, \kappa)^j)$. For the jth pair to satisfy $C$, it is necessary that $y_{s,\kappa}(\pi(s, \kappa)_j) + 2g_s \leq y_{s,\kappa}(\pi(s, \kappa)_j)$. In order to find robust pairs while avoiding redundancies, the algorithm $\text{robust_pairs}$ first proceeds by sorting $Y = y_{s,\kappa}$ in ascending order (line 3 in algorithm 1): $S = \gamma$ denotes a permutation of $\{1, \ldots, n\}$ such that $\forall i \in \{1, \ldots, n\}, \ y_{s,\kappa}(\pi(s, \kappa)_i) \leq y_{s,\kappa}(\pi(s, \kappa)_i))$, where $n$ is the cardinality of $\kappa$ (and, consequently, also the cardinality of $\gamma$). Then, the search for a set of robust pairs makes use of two indices, $i_1$ and $i_2$, such that $i_1 < i_2$. Starting from $(i_1, i_2) = (n - 1, \eta)$ (line 6), the principle of the algorithm consists in decreasing $i_1$ and $i_2$ in such a way that non redundant pairs satisfying $C(\kappa(\gamma(i_1)), \kappa(\gamma(i_2)))$ can be found. It can be noticed that $C(\kappa(\gamma(i_1)), \kappa(\gamma(i_2)))$ can be equivalently rewritten as $y_{s,\kappa}(\pi(s, \kappa)_i) + 2g_s \leq y_{s,\kappa}(\pi(s, \kappa)_i)$ or $Y(S(11)) + d \leq y(Y(S(12)))$ when using the notations of the proposed Matlab implementation. The permutation $\gamma$ resulting from the sort in ascending order of $y_{s,\kappa}$ makes it possible to move incrementally between consecutive measurement values (given by the $s$th sensor over the temporal window $\kappa$) through indices like $i_1$ and $i_2$. In order to find robust pairs like $\kappa(\gamma(i_1)), \kappa(\gamma(i_2)))$ satisfying $C$, the main loop (lines 8-27 of algorithm 1) iteratively proceeds as follows:

- $P_{(1, 2)}$ represents the current pair candidate expressed as indices of $Y$ (line 9).
- While keeping $i_2$ constant, the index $i_1$ is decreased until the condition $C(\kappa(\gamma(i_1)), \kappa(\gamma(i_2)))$ is satisfied (lines 11-14).
Algorithm 1. (Robust pairs for a single sensor).

1: function P=robust_pairs(Y,d)
2: % Sort Y in ascending order
3: [Y cs, S]=sort(Y);% Initialization
4: 5: P=[]; j=1; exitloop=0;
6: n=length(Y); i1=n-1; i2=n;
7: % Main loop
8: while (~exitloop)
9:  Pj2=S(i2); P(j,1)=Pj1; P(j,2)=Pj2; j=j+1;
10:  if (i1>0), Pj1=S(i1); else exitloop=1; end
11:  i1=i1-1;
12:  while ((Y(Pj1)+d)>=Y(Pj2)) && (~exitloop),
13:  % Decrease i1 until the gap is satisfied
14:  end
15:  if (i1>0), Pj1=S(i1); P(j,1)=Pj1; P(j,2)=Pj2; j=j+1; end
16:  % Store current pair (except if exitloop is true)
17:  end
18:  % Store pairs by decr. i1 until i2 can be decr.
19:  while ((Y(Pj1)+d)>=Y(nextPj2)) && (~exitloop),
20:  % Except if exitloop is true
21:  i2=i2-1;
22:  if (~exitloop), P(j,1)=Pj1; P(j,2)=Pj2; j=j+1; end
23:  % Next value of Pj2
24:  end
25:  % Decr current pair (except if exitloop is true)
26:  i1=i1-1;
27: end
28: return

A careful study of the algorithm which is not detailed here for place reasons shows that an upper bound for the number of returned pairs is \(2(\eta -1)\). Roughly speaking, the algorithm does not return a number of pairs which exceeds twice the number \(\eta\) of samples in the temporal window under study. To complete this section, an example illustrates a simple running of the algorithm 1.

Example 9. Let \(Y=[9 5 7 4 3 1]\) and \(d=2.5\). Then, the function robust_pairs returns \(P=[2 1; 4 1; 4 3; 5 3; 6 3; 6 2; 6 4]\). The interpretation of the result is more explicit when computing the transpose of \(YP\):

\[
YP^T = \begin{bmatrix} 5 & 4 & 4 & 3 & 3 & 1 & 1 & 1 & 1 \\ 9 & 9 & 9 & 7 & 7 & 7 & 5 & 5 & 4 \end{bmatrix}
\]

The difference between the second line and the first line does not contain any element inferior to \(d=2.5\).

This example illustrates a fundamental property of the proposed algorithm: all the pairs returned in \(P(s, \kappa)\) (20) satisfy the robustness condition \(C\) i.e. \(v_j \in \{1, \ldots, r\}, y_{s}(\pi(s, \kappa, j)) + \eta \leq y_{s}(\pi(s, \kappa, j)).\) The related knowledge about \(z\) given by the \(s\)th sensor over the temporal window \(\kappa\) can then be expressed in vector form as (see (18)):

\[
z_{s, \pi}(s, \kappa) \leq z_{s, \pi}(s, \kappa)
\]

6. MULTI-SCALE FDI DECISION BASED ON POLYTOPIC PARAMETER SETS

A FDI decision based on the previous results is the subject of this section. It first consists in characterizing the knowledge about \(\theta\) by a polytopic set. Under the single fault assumption, this set is then used to define the multi-scale decision procedure itself.

6.1 Polytopic parameter set over a temporal window

Section 4 has explained how \(z_{s, \pi}^k, z_{s, \pi}^d\) and \(z_{s, \pi}^c\) can be computed so that (4) holds. From (4) and (22), it comes:

\[
H(s, \kappa) = z_{s, \pi}^k(s, \kappa) - z_{s, \pi}^d(s, \kappa) \leq \theta_i(\kappa) \leq K(\kappa)
\]

Proposition 10. (Parameter set, \(s\)th sensor). The knowledge about \(\theta\) given by the \(s\)th sensor, \(s \in \{1, \ldots, m\}\), over the temporal window \(\kappa\) is consistent with the linear inequality \(H(s, \kappa) \theta \leq K(\kappa)\), where:

\[
H(s, \kappa) = z_{s, \pi}^k(s, \kappa) - z_{s, \pi}^d(s, \kappa) + z_{s, \pi}^c(s, \kappa)
\]

Proof. (4) can equivalently be rewritten as: \(\forall s, \forall k, z_{s, k} = z_{s, k}^k + z_{s, k}^d \theta + z_{s, k}^c\) and \(z_{s, k} \in [-z_{s, k}^c, z_{s, k}^c]\) where \(z_{s, k}^c \geq 0\). Replacing this expression of \(z_{s, k}\) in (22), it comes:

\[
z_{s, \pi}^k(s, \kappa) + z_{s, \pi}^d(s, \kappa) \theta + z_{s, \pi}^c(s, \kappa) \leq z_{s, \pi}^k(s, \kappa) + z_{s, \pi}^d(s, \kappa) \theta + z_{s, \pi}^c(s, \kappa).
\]

Moreover, \(z_{s, \pi}^k(s, \kappa) \in [-z_{s, \pi}^c(s, \kappa), z_{s, \pi}^c(s, \kappa)]\) and \(z_{s, \pi}^c(s, \kappa) \in [-z_{s, \pi}^c(s, \kappa), z_{s, \pi}^c(s, \kappa)]\), where the upper bounds are vectors with positive elements. Then, by factorizing \(\theta\) in the previous inequality, it comes \(H(s, \kappa) \theta \leq K(\kappa)\) where \(H(s, \kappa) \) and \(K(\kappa)\) are defined as in (23) and (24), what completes the proof.

Proposition 11. (Parameter set, all sensors). The knowledge about \(\theta\) given by all the sensors (indexed by \(s \in \{1, \ldots, m\}\)) over the temporal window \(\kappa\) is consistent with (25), where \(H(\kappa)\) (resp. \(K(\kappa)\)) is the vertical concatenation of \(H(s, \kappa)\) (resp. \(K(s, \kappa)\)), for \(s = 1, \ldots, m\).

\[
H(\kappa) \theta \leq K(\kappa)
\]

Proof. The definition of \(H(\kappa)\) and \(K(\kappa)\) ensures the equivalence between \(H(\kappa) \theta \leq K(\kappa)\) and \(\forall s, H(s, \kappa) \theta \leq K(s, \kappa)\), where the latter comes from the proposition 10.

Remark 12. An upper bound for the number of lines of \(H(\kappa)\) is \(2(\eta - 1) m\) where \(\eta\) is the size of the temporal window \(\kappa\) and \(m\) is the number of elementary (scalar) sensors. This is a direct consequence of the upper bound for the number of pairs returned by the function robust_pairs in section 5.

From a membership set point of view, the Feasible Parameter Set (FPS) for \(\theta\) over the temporal window \(\kappa\) is included in the polytope \(P(\kappa) = \{\theta, \ H(s, \kappa) \theta \leq K(s, \kappa)\}\) which is the intersection of the polytopes \(P(s, \kappa) = \{\theta, \ H(s, \kappa) \theta \leq K(s, \kappa)\}\) enclosing the FPS related to each single sensor \(s \in \{1, \ldots, m\}\). Moreover, the computation of \(H(s, \kappa)\) and \(K(s, \kappa)\) has been made explicit for any temporal window \(\kappa \subset \{1, \ldots, \kappa_{max}\}\).

6.2 FDI decision over a single temporal window

Given a polytopic approximation of the feasible parameter set over a temporal window \(\kappa\), \(H(s, \kappa) \theta \leq K(s, \kappa)\), a constructive way to estimate intervals \([\theta_i(\kappa)\]) bounding
each fault parameters, $\theta_i$, $i = 1, \ldots, q$ is the subject of this paragraph. For the sake of faster computations, the single fault assumption is made. Therefore, assuming the $i$th actuator is faulty, the others can be considered as fault-free, what can be expressed as: $\theta_j = 1$, $\forall j \neq i$ (remember that $\theta = \theta_0 = [1, \ldots, 1]^T$ when the system (1) is fault-free). Then, rewriting $H(\kappa)\theta \leq K(\kappa)$ under each of the $q$ single fault assumptions like “Any fault comes from the $i$th actuator” leads to:

$$H_i(\kappa)\theta_i \leq L_i(\kappa), \quad i = 1, \ldots, q$$

(26)

where $H_i(\kappa)$ is the $i$th column of $H(\kappa)$ and $L_i(\kappa) = K(\kappa) - \sum_{j \in \{1, \ldots, q\} \setminus \{i\}} H_j(\kappa)$. The superscripts $+, -$, and $0$ respectively referring to the indices of the strictly positive, strictly negative and null elements of $H_i(\kappa)$, then (26) can be decomposed into three subsets of inequalities:

$$\begin{align*}
H_i(\kappa)\theta_i & \leq L_i^+(\kappa) \\
H_i(\kappa)\theta_i & \leq L_i^-(\kappa) \\
H_i(\kappa)\theta_i & \leq L_i^0(\kappa)
\end{align*}$$

(27)

As $H_i^0(\kappa) = 0$, if any element of $L_i^0(\kappa)$ is strictly negative, then $[\theta_i(\kappa)] = \emptyset$ else $[\theta_i(\kappa)] = \overline{\theta}_i(\kappa)$, $\underline{\theta}_i(\kappa)$, where:

$$\begin{align*}
\overline{\theta}_i(\kappa) & = \max(L_i^+(\kappa) \div H_i^-(\kappa)) \\
\underline{\theta}_i(\kappa) & = \min(L_i^-(\kappa) \div H_i^+(\kappa))
\end{align*}$$

(28)

(29)

The operator $\cap$ refers to the element-by-element division. If $\overline{\theta}_i(\kappa) > \underline{\theta}_i(\kappa)$, then $[\theta_i(\kappa)] = \emptyset$. The FDI decision over the temporal window $\kappa$ given by the (possibly empty) intervals $[\theta_i(\kappa)]$, $i = 1, \ldots, q$ can be interpreted as follows: If $[\theta_i(\kappa)] = \emptyset$, then the assumption of a (constant) fault on the $i$th actuator leads to an inconsistency and can thus be rejected, else the interpretation depends on whether the fault-free value of $\theta_i$ (i.e. in the problem statement used in this paper) belongs or not to $[\theta_i(\kappa)]$: if $1 \in [\theta_i(\kappa)]$, then the fault-free case is consistent with the model and the observed behavior, else (if $1 \not\in [\theta_i(\kappa)]$) a (single) fault on the $i$th actuator is detected and the interval $[\theta_i(\kappa)]$ gives a bounded estimate of the fault parameter $\theta_i$.

6.3 FDI decision over multi-scale temporal windows

The motivation for a FDI decision over multi-scale temporal windows is to relax the assumption of a constant fault parameter $\theta$ in the problem statement (section 2). In fact, it may seem contradictory to assume a constant $\theta$ whereas the purpose of the fault diagnosis scheme is to detect faults (characterized by changes in $\theta$) from a set of measurements acquired between $k = 1$ and $k = k_{\text{max}}$. Indeed, if an actuator fault occurs before $k_{\text{max}}$, the real system behavior is likely to be described by the system model (1), but with time-varying $\theta$ during the fault occurrence. A consequence of such variations of $\theta$ is the possible inconsistency of (1) during the fault occurrence and, more generally, over the temporal window $\{1, \ldots, k_{\text{max}}\}$. However, the assumption of constant $\theta$ is likely to be valid before and after the fault occurrence (Notice that the bounded state and output noises ($v_k$, $w_k$) can be used to enlarge the validity domain of this assumption, if necessary). As the time (and duration) of fault occurrence is obviously unknown, the notion of before and after the fault occurrence can be hardly made explicit in a direct way. This motivates the implementation of a FDI decision over multi-scale temporal windows, so that the fault detection, isolation and identification relies on the assumption of constant parameter $\theta$ over time intervals with multiple lengths. The larger the temporal window is, the better the precision of $\theta$ estimates is expected to be, but the more sensitive to violations of the constant $\theta$ assumption the FDI scheme is also expected to be. The consequence of such a violation is the detection of an inconsistency between the model and the observed measurements and the impossibility to identify the parameters within the related temporal window: concretely, $[\theta_i(\kappa)]$ becomes empty. As the best trade-off between the precision of the estimates and the validity of the constant parameter assumption depends on the unknown variations of $\theta$ modeling possibly very different kinds of fault occurrences, a set of temporal windows with arbitrary fixed lengths parameterized by a scale index $l$ is considered:

$$\kappa^l_j = \{k \in \mathbb{N}, (j-1)k_{\text{max}}/2^l < k \leq jk_{\text{max}}/2^l\}, \quad l \in \Lambda \subset \mathbb{N}, \quad j \in \{1, \ldots, 2^l\}$$

(30)

Given a scale level $l \in \Lambda$ (where $\Lambda \subset \mathbb{N}$ is a set of scale levels fixed by the designer), $2^l$ temporal windows indexed by $j$ and denoted $\kappa^l_j$ are defined as in (30). Then, the overall FDI decision scheme consists in computing the FDI decision described in subsection 6.2 for each of the $\sum_{l \in \Lambda} 2^l$ temporal windows $\kappa^l_j$ defined in (30).

Regarding the computational complexity of the proposed scheme, it must be noticed that the computation of the parameter sensitivities and the reachable sets (i.e. the computation of $z^l_k$, $z^l_0$, and $z^l_j$) as described in section 4 need only be done once for $k$ between 1 and $k_{\text{max}}$. It does not depend on the number $\lambda = \text{card}(\Lambda)$ of scale levels used to compute the FDI decision(s). Moreover, (30) shows that the $2^l$ temporal windows $\kappa^l_j$ related to a scale level $l$ are non-overlapping. A consequence of this is that the complexity of an overall FDI decision scheme involving $\lambda$ scale levels is roughly $\lambda$ times the complexity of the FDI decision over the single window $\kappa^l_j = \{1, \ldots, k_{\text{max}}\}$. Even if this rough evaluation does not take the increased number of internal function calls into account as $l$ increases, it however underlines an interesting complexity property of the algorithm robust_pairs to compute the sets of robust pairs: the complexity of computing $\pi(s, \kappa)$ (20) is almost linear w.r.t. to the number $\eta$ of samples in the window $\kappa$, as well as the upper bound on the number of returned pairs, $2(\eta - 1)$, as previously mentioned in section 5.

For the sake of FDI decision visualization, two signals, $\overline{\theta}_i(k)$ and $\underline{\theta}_i(k)$ are defined for each fault assumption, $i \in \{1, \ldots, q\}$, for each scale level $l \in \Lambda$ and over all the available samples, $k \in \{1, \ldots, k_{\text{max}}\}$ as follows:

$$\begin{align*}
k \in \kappa^l_j & \Rightarrow \overline{\theta}_i(k) = \overline{\theta}_i(\kappa^l_j), \quad \forall(i, j, l) \\
k \in \kappa^l_j & \Rightarrow \underline{\theta}_i(k) = \underline{\theta}_i(\kappa^l_j), \quad \forall(i, j, l)
\end{align*}$$

(31)

(32)

By convention, it is considered in (31) and (32) that $\overline{\theta}_i(\kappa^l_j) = 0$ and $\underline{\theta}_i(\kappa^l_j) = 0$ when $\theta_i(\kappa^l_j) = 0$. The inconsistency related to the empty interval is then reflected by the empty value of the decision signals $\overline{\theta}_i(k)$ and $\underline{\theta}_i(k)$ for $k \in \kappa^l_j$ (Remark: the empty value of decision signals has been coded using NaN, the IEEE arithmetic representation for Not-a-Number, what makes the empty value clearly distinct from zero in our implementation).

When $\theta_i(\kappa^l_j) \neq \emptyset$, $\overline{\theta}_i(k)$ and $\underline{\theta}_i(k)$ are constant for all $k \in \kappa^l_j$. These two constants are the bounds of the estimate of $\theta_i$ over the temporal window $\kappa^l_j$ under the assumption
of a single fault on the ith actuator. Finally, the whole FDI decision over multi-scale temporal windows is entirely represented by the 2qλ signals \( \tilde{\theta}^j(k) \) and \( \tilde{\theta}^i(k) \).

7. NUMERICAL EXAMPLE

To illustrate the fault diagnosis method presented in this paper, this section presents its application to a simulated distillation column. In (Glad and Ljung, 2000, page 223), a distillation column is modeled with a transfer function matrix with 3 control inputs: top draw flow rate, side draw flow rate, bottom temperature control; and 3 outputs. In order to make a simple illustrative example, let us consider this distillation column model with one of its outputs, the bottom reflux temperature. Though our state extension removing a one sample time delay on the state space matrices are (up to display rounding errors):

\[
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}
= \begin{bmatrix}
0.9412 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0.9556 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0.9001 & 0 & 0 & 1 \\
0.9151 & 0.3928 & 0.7194 & 0 & 0 & 0
\end{bmatrix}
\]  

(33a)

where the state space matrices are (up to display rounding errors):

\[
\begin{bmatrix}
\tilde{x}_k \\
\tilde{z}_k
\end{bmatrix} = \begin{bmatrix}
\tilde{A} \tilde{x}_k + \tilde{B} \tilde{u}_k \\
\tilde{C} \tilde{x}_k + \tilde{D} \tilde{u}_k
\end{bmatrix}
\]  

(33b)

The input delays are then taken into account through a state extension taking past inputs as part of the new state. To that purpose, the operator \( D_l \) defines an elementary state extension removing a one sample time delay on the ith input:

\[
D_l : \begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}
\rightarrow \begin{bmatrix}
A & B_j & B_1 \ldots B_{i-1} & 0 & B_{i+1} \ldots B_q \\
0 & 0 & 0 & 0 & \ldots & 0 \\
C & D_j & D_1 \ldots D_{i-1} & 0 & D_{i+1} \ldots D_q
\end{bmatrix}
\]  

(34)

A state-space representation \( (A, B, C, D) \) modeling a 10Ts delay on \( \tilde{u}_1 \) and a 11Ts delay on \( \tilde{u}_2 \) can be obtained from \( (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \) and multiple calls to \( D_l \) and \( D_2 \):

\[
(A, B, C, D) = D_l^{10}(D_2^{11}((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})))
\]  

(35)

Using the numerical values in (34) as inputs of (35), it comes: \( A \in \mathbb{R}^{24 \times 24}, B \in \mathbb{R}^{24 \times 1}, C \in \mathbb{R}^{1 \times 24} \) and \( D \in \mathbb{R}^{1 \times 1} \). The system model like (1) used for the numerical example is based on the corresponding numerical values for the (time-invariant) matrices \( A, B, C, D \). Moreover, the bound of the bounded uncertainties is specified by \( E = 0_{24 \times 1}, F = 1 \) and \( g = 0.5 \). By assuming time invariant uncertainty bounds in the proposed numerical example, the subscript \( k \) in \( E_k \) and \( F_k \) can be dropped. The extended initial state is \( x_0 = [x_0, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{i-11}, u_{i-11}, \ldots, u_{2}, u_{2}]^T \) where \( u_{i-1} \) denotes the (past) value of the ith input \( u_i \) at time \( k = \delta (\leq 0) \). \( x_0 \) is not exactly known but it is assumed to belong to the interval vector \( [x_0, t_0] \subset \mathbb{R}^{24} \) where \( t_0 = 0_{24 \times 1} \) and \( t_0 = [300, 300, 200, 40_1 \times 10, 301_1 \times 1]^T \). Informally, this means that, at the initial time \( k = 0 \), the past values of the delayed inputs \( u_1 \) and \( u_2 \) are respectively assumed to belong to \([0, 40]\) and \([0, 30]\). The ability to deal with unknown but bounded past inputs for the time-delay system taken at \( k = 0 \) is the main motivation for the state extension resulting from (35). Finally, the center and the shape matrix of the zonotope \( c_0 + Z(R_0) \) enclosing the initial state \( x_0(7) \) can be computed as \( c_0 = (\overline{x}_0 + \underline{x}_0)/2 \) and \( R_0 = diag((\overline{x}_0 - \underline{x}_0)/2) \). The (single) sensor distortion function is:

\[
h(z) = 100 \left( 1 + e^{-0.2(z-320)} \right) + 10
\]  

(36)

The expression \( h(z) \) is only used to simulate the "true" system. The additive constant 10 has been added to illustrate the robustness of the proposed method to any constant sensor bias (shifting error). \( h(z) \) is unknown as far as fault diagnosis is concerned, except from the fact that it is a monotonic increasing function. The \( q = 3 \) scalar input variables in \( \overline{u} \) are randomly drawn with uniform distributions within the intervals \([20, 40]\) (mol/min), \([10, 30]\) (mol/min) and \([10, 20]\) (°C). The values of \( \overline{v} \) and \( \overline{u} \) used to simulate the measurements \( y \) are computed using uniformly distributed random numbers normalized within \([-1, +1]\). The simulated data is recorded during 3000 minutes (i.e. \( k_{max} = 1500 \) because \( T_s = 2 \) minutes) and the FDI decision signals are computed for \( \lambda = 3 \) scale levels: \( l \in \Lambda = \{2, 4, 6\} \). The computation of the proposed actuator fault diagnosis procedure implemented in the Matlab environment (m-file) requires less than 2s on a usual desktop PC (Intel® Core2TM Duo CPU E4400 running at 2GHz, 2Go RAM, 32 bits OS).

In order to illustrate the FDI decision results, a faulty scenario is considered: \( \theta \) firstly equals \( \theta_0 = [1, 1, 1]^T \) between \( k = 1 \) and \( k = 999 \), (i.e. the system is fault-free for \( k < 1000 \)); then, \( \theta \) equals \([0.8, 1, 1]^T \) between \( k = 1000 \) and \( k = k_{max} = 1500 \). The system thus suffers from a 20% loss on the first actuator for \( k \geq 1000 \). The simulated outputs are shown in Fig. 1: \( z_k \) is the distorted unknown output and \( y_k \) is the measurement given by the distorted and noisy sensor. The FDI decision related to the above mentioned faulty scenario is illustrated by the \( q = 3 \) figures Fig. 2-4. Each figure shows the \( \lambda = 3 \) interval estimates \( [\tilde{\theta}^j(k)] = [\tilde{\theta}^j(k), \tilde{\theta}^j(k)] \), \( l \in \Lambda = \{2, 4, 6\} \), of one scalar actuator fault parameter \( \theta_i \) (for a given \( i \in \{1, \ldots, q\} \)). In Fig. 2-4, the bold (resp. thin, resp. dotted) lines stand for the inf/sup bounds of the estimates related to the scale level \( l = 2 \) (resp. \( l = 4 \), resp. \( l = 6 \)). For instance, the bold lines in Fig. 2 correspond to the estimated bounds computed for the 4 temporal windows \( \varsigma^j_k, j = 1, \ldots, 2^2 \), related to the scale level \( l = 2 \). It can be observed that: \( 1 \in [\theta_1(\varsigma^1_k)], 1 \notin [\theta_1(\varsigma^2_k)], [\theta_1(\varsigma^3_k)] = \varnothing \neq [\theta_1(\varsigma^4_k)] \). This can be interpreted as follows: assuming a single fault on the first actuator, the measurements are consistent with
the fault-free behavior for all \( k \in \kappa_2 \) and \( k \in \kappa_3 \); the measurements are inconsistent with the system model for some \( k \in \kappa_2 \); the measurements are inconsistent with the fault-free behavior for all \( k \in \kappa_3 \) but a fault on the first actuator is likely to explain them provided \( \theta_1 \in [\theta_1(\kappa_2)] \) which is not an empty interval. This interpretation is fully consistent with the simulated faulty scenario: the system is fault-free over \( \kappa_2 \) and \( \kappa_3 \), subject to non-modeled parameter variations related to the fault occurrence over \( \kappa_3 \), and affected by a fault on the first actuator over \( \kappa_2 \).

Moreover, the “true” faulty value of \( \theta_1 \), 0.8, belongs to the interval \([\theta_1(\kappa_2)]\), showing so the effectiveness of the fault identification. Looking again at the bold lines in Fig. 3 and Fig. 4, a reasoning similar to the one followed for Fig. 2 allows drawing the following conclusions: as \([\theta_2(\kappa_2)] = \emptyset\) and \([\theta_2(\kappa_3)] = \emptyset\) (Fig. 3), the system behavior is neither consistent with the fault-free behavior \((\theta_2 = 1)\) nor with the assumption of a faulty second actuator \((\theta_2 \neq 1)\) over \( \kappa_2 \) and \( \kappa_3 \); the assumption of a faulty second actuator to explain the detected inconsistencies is thus rejected. However, as \([\theta_3(\kappa_2)] = \emptyset\) and \( l \in [\theta_3(\kappa_2)] \neq \emptyset\) (Fig. 4), it can be concluded that after a fault occurrence (somewhere in \( \kappa_2 \)), the measurements can be explained by a fault on the third actuator characterized by \( \theta_3 \in [1.1, 1.2] \). This illustrates the ability of the proposed scheme to deal with decision ambiguity arising from a lack of information related to a poor instrumentation (here, only a single distorted sensor with constant bias...) and/or a lack of input excitation. In the proposed scenario, the fault isolation is partial because only the second actuator fault can be rejected. However, simulations based on other faulty scenarios (20% loss on \( \theta_2 \), on the one hand, 20% loss on \( \theta_1 \), on the other hand) has shown that the situation of ambiguous decision occurs less frequently than for the 20% loss on \( \theta_1 \) corresponding to the faulty scenario reported in this paper. This confirms the effectiveness of the proposed scheme to perform fault detection, isolation and identification from a rather weak instrumentation of the system under study. After having focused on the scale level \( l = 2 \) (i.e. decomposition of \([1, ..., k_{max}]\) into \( 2^2 = 4 \) temporal windows), the influence of the scale level \( l \) on the FDI decision is the subject of the following comments and interpretations. First, it can be noticed that the scale level \( l = 4 \) (resp. \( l = 6 \)) represented by the thin (resp. dotted) lines in Fig. 2-4 rely on a decomposition of \([1, ..., k_{max}]\) into \( 2^4 = 16 \) (resp. \( 2^6 = 64 \)) contiguous temporal windows. A closer look at Fig. 2-4 shows that the precision of the fault estimates (i.e. the width between the interval bounds, when defined) increases for lower scale levels. This is normal because lower scale levels correspond to larger temporal windows, and larger windows provide more information to estimate the fault parameters due to an increased number of available robust (and non redundant) output pairs. However, lower scale levels do not allow a precise estimate of the time of fault occurrence. For instance, the interruption of the bold lines in Fig. 2 (i.e. the inconsistency detected by \([\theta_2(\kappa_2)] = \emptyset\)) indicates that a fault may have occurred between \( k = 750 \) and \( k = 1125 \), what corresponds to 12.5 hours. Compared to \( l = 2 \) (bold), the estimates show a poor precision for \( l = 4 \) (thin) and even poorer for \( l = 6 \) (dotted). However, the interruption of thin (resp. dotted) lines indicate that a fault may have occurred within a time of 6.25 hours (resp. 2.3 hours). A similar reasoning applies to Fig. 4 and even to Fig. 3. In the latter case, even if no inconsistency is detected for \( l = 6 \), it can be observed that the three consecutive intervals described by the dotted lines just after \( k = 1000 \) have an empty intersection. This suggests that rapid and significant variations of \( \theta_2 \) are necessary to preserve the consistency between the model and the measurements, so rejecting the possibility of a fault-free behavior. As far as temporal window lengths are considered, it can be concluded that each scale level \( l \) gives a particular insight on the FDI decision by fixing the terms of the tradeoff between the precision of the fault amplitude estimates and the precision of the estimated time of fault occurrence.

8. CONCLUSION

The off-line diagnosis of multiplicative actuator faults using sensors providing measurements affected by unknown (but monotonic) nonlinear distortions, bounded noises and constant bias has been studied. The proposed scheme relies on the tight integration between an original algorithm searching for a set of robust output pairs within a given temporal window, and a reachable set computation based on zonotopes combined with parameter sensitivity evaluation. A generic FDI decision based on polytopic parameter sets has been applied over multi-scale temporal windows, which are useful to accurately estimate both the fault amplitude and the time of fault occurrence. This has been illustrated by a numerical example showing the ability
of the proposed scheme to detect, isolate and identify (with bounds) three multiplicative actuator faults from a single noisy sensor with unknown nonlinear distortion and constant bias. Based on the reasonable computational complexity properties of the underlying algorithms, an online implementation is among the improvements planned for future work, as well as a study of the requirements on the input excitation for the faults to be isolable.

REFERENCES


