Infimal SNR for Output Disturbance Rejection

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Abstract: In the present paper we address the problem of plant output disturbance rejection in the context of feedback control over a signal-to-noise ratio (SNR) constrained communication channel. We study separately the case of the communication channel when located between the controller and the plant (control path) and when located between the plant and the controller (measurement path). We also study separately the case of a memoryless additive white Gaussian noise channel model and the case of an additive colored Gaussian noise channel with memory. When possible we express the resulting infimal SNR for performance in closed-form, whilst where not possible we use a numerical approach such as linear quadratic Gaussian (LQG) optimization with loop transfer recovery (LTR).

1. INTRODUCTION

Stabilizability, performance and robustness in the area of Control over Networks have been topics of increased interest in recent years. The most general results in the area that consider the problem of stabilizability call for information theoretic arguments to obtain necessary and sufficient lower bounds on the channel transmission data rate Nair and Evans [2004], Nair et al. [2004], Freudenberg et al. [2006], Nair et al. [2007], Charalambous and Farhadi [2008]. For linear plant models in [Nair and Evans, 2004, Theorem 2.1] and [Freudenberg et al., 2006, Proposition III.1] it is proved that if the plant is to be stabilized, then the transmission data rate has to satisfy a lower bound that depends on the open loop unstable eigenvalues of the plant.

Another line of research introduced a framework to study stabilizability of a feedback loop over channels that have a signal to noise ratio (SNR) constraint Braslavsky et al. [2007], with related work in Rantzer [2006], Bassam and Voulgaris [2005]. Braslavsky et al. [2007] obtained the infimal SNR required to stabilize an unstable plant over a memoryless additive white Gaussian noise (AWGN) channel. A distinctive characteristic of the SNR approach is that it is a linear formulation and that it is caused, in the discrete-time framework, by the presence of unstable plant poles, non minimum phase (NMP) zeros or time delay in the plant model. For the case of linear time invariant (LTI) controllers and minimum phase plant models with no time delay, these conditions match those derived in Nair and Evans [2004] by application of Shannon’s theorem [Cover and Thomas, 1991, §10.3].

The communication channel model in a feedback loop can either be located on the control or measurement path. In the present paper we consider both locations, first for the case of a memoryless AWGN channel and the control path location for the case of an additive colored Gaussian noise (ACGN) channel with memory.

The first contribution of this paper is then the infimal SNR required for plant output disturbance rejection when the channel is a memoryless AWGN channel in the control path. We then obtain the infimal SNR required for plant output disturbance rejection when the same type of channel is in the measurement path. As different from the plain stabilizability result, see for example Rojas et al. [2007], the performance objective results in a different infimal SNR requirement depending on the channel location.

The second contribution of the present work is the extension of the results for a memoryless AWGN channel to the case of an ACGN channel with memory. This type of channel is both more realistic, by including for example bandwidth limitation, and more flexible, by including for example coloring of the additive noise channel. As a result we obtain an infimal SNR required for plant output disturbance rejection that accounts for more factors limiting the performance of the feedback control loop.

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We then briefly discuss the results obtained in light, for example, of the plant input disturbance rejection case discussed previously in Rojas et al. [2006b] or the case of channel models simultaneously located in the control and measurement paths.

The present paper is organised as follows: In Section 2 we present the standing assumptions for this work and introduce the SNR problem. In Section 3 we solve the infimal SNR for plant output disturbance rejection for the case of a memoryless AWGN channel located either over the measurement path or the control path. In Section 4 we extend the model channel to the ACGN channel with memory case. In Section 5 we further discuss the results from the two previous sections and their interpretations, whilst in Section 6 we conclude the present work with our final remarks and highlight possible lines of future research.

2. PRELIMINARIES

In this section we present the general standing assumptions for the present work, as well as introduce a general version of the infimal SNR problem for plant output disturbance rejection.

2.1 Assumption

Plant model assumptions: We assume the plant model $G(z)$ to be a minimum phase real rational function with the following properties:
- relative degree $n_g \geq 1$.
- $m$ unstable poles, $|\rho_i| > 1$, each with multiplicity $n_i \geq 1$.
- Matrices $(A_G, B_G, C_G)$ represent a minimal realization of $G(z)$.

Channel model assumptions: The channel model will be further specified in the next two sections, nevertheless it is in general characterized by two parameters: The admissible channel input power level $P$, the channel additive noise process $n(k)$.

Channel additive noise process: The channel additive noise process is labelled $n(k)$ and it is a zero-mean i.i.d. Gaussian white noise process with variance $\sigma^2$.

Output disturbance process: The output disturbance process is labelled $d(k)$ and it is a zero-mean i.i.d. Gaussian white noise process with variance $\sigma_d^2$.

Notice that if we lift the assumption of $G(z)$ minimum phase, then we would be required to invoke an inner factorization argument similar to the one presented in Zhang [1996], Zhang and Freudenberg [1993].

2.2 Infimal SNR Problem

We assume that $C(z)$ in Figure 1 is such that the closed-loop system is stable in the sense that for any distribution of initial conditions, the distribution of all signals in the loop will converge exponentially fast to a stationary distribution. The channel input power, defined by $\|s\|_P^2 \triangleq \lim_{k \to \infty} \mathcal{E}\{s^2(k)\}$, is required to satisfy an imposed power constraint $P > \mathcal{E}\{s^2\}$, for some predetermined power level $P$, where $\mathcal{E}\{s^2\}$ stands for $\lim_{k \to \infty} \mathcal{E}\{s^2(k)\}$ and it is introduced to easy the notation. Under reasonable stationarity assumptions [Åström, 1970, §4.4], the power in the channel input may be computed as $\mathcal{E}\{s^2\} = \|T_{sd}(z)\|^2_2 \sigma_d^2 + \|T_{sn}(z)\|^2_2 \sigma^2$. The closed-loop transfer function $T_{sd}(z)$ is structurally different depending on the channel location, but not on the channel model, see Figure 1. On the other hand, $T_{sn}(z)$ is structurally different depending on the channel model location, but not on the channel location. The power constraint at the input of the channel translates to a SNR lower bound defined by the squared $H_2$ norm of $T_{sd}(z)$ and $T_{sn}(z)$

$$P^2 > \|T_{sd}(z)\|^2_2 \sigma_d^2 + \|T_{sn}(z)\|^2_2. \tag{1}$$

From (1) we observe that a fundamental limitation in the SNR of the fading channel will be given by the simultaneous infimum of $\|T_{sd}(z)\|^2_2$ and $\|T_{sn}(z)\|^2_2$, which is at the core of the infimal SNR problem definition that follows.

Problem 1. (Infimal LTI SNR for Stabilizability). Find a proper rational stabilizing controller $C(z)$ such that the feedback control loop, subject to the assumptions presented in Subsection 2.1, is stable and the transfer functions in (1) achieve the infimum admissible channel SNR.

3. MEMORABLE ADDITIVE WHITE GAUSSIAN NOISE CHANNEL

In this section we assume the communication channel model to be a memoryless AWGN channel, thus we do not need to add any other assumption to the one already presented in Subsection 2.1.

3.1 Measurement Path

We first consider the memoryless AWGN channel to be located on the measurement path. For this choice of communication channel model and location, we obtain the following expressions for $T_{sd}(z)$ and $T_{sn}(z)$

$$T_{sd}(z) = \frac{1}{1 + G(z)C(z)}, \quad T_{sn}(z) = -\frac{G(z)C(z)}{1 + G(z)C(z)} \tag{2}$$

Next, we express the infimal SNR for plant output disturbance rejection in closed-form.

Theorem 2. (Infimal SNR for Plant Output Disturbance Rejection) Consider the discrete-time LTI output feedback represented in Figure 1 and that the disturbance process and the channel model, located on the measurement path, satisfy the assumptions listed in Subsection 2.1. The channel SNR is then lower bounded by

![Fig. 2. Discrete-time output feedback loop subject to plant output disturbance over a memoryless AWGN channel located on the measurement path.](image-url)
Fig. 3. Discrete-time output feedback loop subject to plant output disturbance over a memoryless AWGN channel located on the control path.

\[
\frac{P}{\sigma^2} > \frac{\sigma^2}{\sigma^2} + \inf_{C(z) \text{ stab.}} \|T(z)\|^2_2 \left( 1 + \frac{\sigma^2}{\sigma^2} \right).
\]

where we have introduced the infimum notation as to make explicit that we want to solve Problem 1. We are left now with the task of finding the infimum of \( \|T(z)\|_2^2 \) over all stabilizing controllers \( C(z) \) by means of a Youla parameterization approach. Such a result is readily available in the form of

\[
\inf_{C(z) \text{ stab.}} \|T(z)\|^2_2 = \delta + \sum_{i=1}^{m} \sum_{l=1}^{n_i} \frac{r_{i,l}}{(l-1)!} + \sum_{j=1}^{m} \sum_{p=1}^{n_j} \frac{d^{p-1}}{2^{p-1}} \left( \frac{\rho_j(z)}{z^p} \right) |_{z=p_l}.
\]

with \( r_{i,l} \) as in (4), \( \delta \) and \( \mu_k \) as in (5). The above expression is a special case of [Rojas and Yuz, 2008, Theorem 9] with \( F(z) = 1 \) and \( H(z) = 1 \).

**3.2 Control Path**

We now consider the memoryless AWGN channel to be located on the control path. For this choice of communication channel model and location, we obtain the following expressions for \( T_{sd}(z) \) and \( T_{sn}(z) \)

\[
T_{sd}(z) = -\frac{C(z)}{1 + G(z)C(z)}, \quad T_{sn}(z) = -\frac{G(z)C(z)}{1 + G(z)C(z)}.
\]

In this occasion, since we cannot pose the minimization of the channel SNR only on terms of the squared norm of the complementary sensitivity, we introduce a numerical argument through a linear quadratic Gaussian (LQG) optimization with loop transfer recovery (LTR). We start by noticing that the augmented system with input \( s(k) \) and output \( m(k) \), see Figure 3, is given by

\[
x(k+1) = A_o x(k) + B_o u(k) + B_{G_o} n(k)
\]

\[
m(k) = C_o x(k) + d(k)
\]

Since we wish to minimize the channel input power, this calls for an observer equation governed by the noise covariance from the above system, that is

\[
\Sigma_o = A_o \Sigma_o A_o^T - (A_o \Sigma_o C_o^T + S) \left( C_o \Sigma_o C_o^T + V \right)^{-1} \left( C_o \Sigma_o A_o^T + S^T \right) + W^T,
\]

\[
K_{po} = A_o \Sigma_o C_o^T \left( C_o \Sigma_o C_o^T + V \right)^{-1},
\]

with \( W = B_{G_o} B_{G_o}^T \sigma^2, S = 0 \) and \( V = \sigma_d^2 \). The regulator gain, on the other hand, is obtained by posing the following dual Riccati equation

\[
P_o = A_o^T P_o A_o - A_o^T P_o B_o \left( B_o^T P_o B_o + R \right)^{-1} B_o^T P_o A_o + Q.
\]

**Theorem 3.** Consider a system with feedback over a memoryless AWGN channel as shown in Figure 3. Let \( K_{po} \) be the optimal observer gain obtained from (8) with \( A_o, B_o, C_o, W, V, S \) as defined for the augmented plant/channel structure given by (7) and \( K_o \) the optimal regulator gain obtained from (9) with weights \( Q = 0 \) and \( R = 1 \).

Then the infimal LTI SNR for plant output disturbance rejection problem is solved by the following controller

\[
C_f(z) = z K_o (zI - (I - K_o C_o) (A_o - B_o K_o))^{-1} K_{fo}.
\]

with \( K_{fo} = A_o^{-1} K_{po} \).

**Proof.** The proof is standard in that for an LQG optimization the quadratic cost index is defined as

\[
J = \mathbb{E} \left\{ \sum_{k=0}^{\infty} (x(k)^T Q x(k) + u(k)^T R u(k)) \right\}
\]

The proposed weight choice for \( Q = 0 \) and \( R = 1 \) is such that the “recovered” functional is \( \mathbb{E} \{ s^2 \} \), the channel input power (since, from Figure 3, we have that \( u(k) = s(k) \)).
or $T_{sd}(z)$ does not achieve the lowest SNR for plant output disturbance rejection. Indeed only a simultaneous minimization, achieved through the LQG optimization with LTR approach, is capable to reduce the SNR to its lowest value in an LTI context. Notice that, as $\sigma_d^2 \to 0$, the minimum for $T_{sn}(z)$ and the infimal SNR match, whilst the minimization for $T_{sd}(z)$ converges to a non-zero value. On the other hand, as $\sigma_d^2$ grows the minimal solution derived from $T_{sd}(z)$ and the minimal solution derived from minimizing (1), upon replacing (6), converge due to the increased relevance of the plant output disturbance process $d(k)$ relative to the channel additive noise process $n(k)$.

![Fig. 4. Inidmal SNR as a function of $\sigma_d^2$](image)

The result from Theorem 3 can be further developed with the use of a fairly recent result on closed-form solution of Riccati such as the one in (9) with $Q = 0$ and $R = 1$, see Rojas [2009a]. As a consequence the regulator gain $K_o$ for the case treated in this subsection can be rewritten as

$$K_o = \left[ (-1)^m \prod_{i=1}^m r_i \right]^{1/m} \left[ r_1 r_2 \cdots r_m \right], \quad \forall i = 1, \cdots, m,$$

with $r_i$ equal to $r_i = (1 - |\rho_i|^2) \prod_{j \neq i}^{m} \frac{1 - \rho_i \rho_j}{1 - \rho_i^2}$. For a plant model of the type $G(z) = \gamma/(z - \rho)$, with $\gamma \in \mathbb{R}$ and $\rho \in \mathbb{R}$, $|\rho| > 1$ such as the one discussed in the previous example we then obtain an optimal controller of the form

$$C_f(z) = \frac{(\rho^2 - 1) K_{fo}}{\rho^2 - (1 - K_{fo}) (\rho^2 (1 - \gamma) + \gamma)}.$$

4. ADDITIVE COLORED GAUSSIAN NOISE CHANNEL WITH MEMORY

In this section we extend our discussion to include the choice of an ACGN channel with memory. Present assumptions on the channel model side include the following:

**Channel model**: The channel model $F(z)$ is a stable, minimum phase, biproper transfer function. Matrices $(A_F, B_F, C_F, D_F)$ represent a minimal realization of $F(z)$.

![Fig. 5. Discrete-time output feedback loop subject to plant output disturbance over an ACGN channel with memory located on the control path.](image)

**Noise model**: The system $H(z)$ coloring the channel additive white noise $n(k)$ is assumed to be a stable, biproper and minimum phase transfer function. Matrices $(A_H, B_H, C_H, D_H)$ represent a minimal realization of $H(z)$.

4.1 Control Path

We consider here the case of an ACGN channel with memory located on the control path, see Figure 5. The expressions for $T_{sd}(z)$ and $T_{sn}(z)$ in this occasion are given by

$$T_{sd}(z) = \frac{C(z)O(z)}{1 + G(z)F(z)C(z)}, \quad T_{sn}(z) = \frac{-G(z)C(z)H(z)}{1 + G(z)F(z)C(z)}.$$  

In this occasion we assume the system $O(z)$, filtering the plant output disturbance $d(k)$, to be a stable, proper or strictly proper, and minimum phase transfer function. Matrices $(A_D, B_D, C_D, D_D)$ represent a minimal realization of $O(z)$. We use the LQG optimization with LTR approach and notice that the augmented system with input $u(k)$ and output $m(k)$, see Figure 5, is given by

$$x(k+1) = \begin{bmatrix} A_F & 0 & 0 & 0 \\ 0 & A_H & 0 & 0 \\ B_G C_F & B_G C_H & A_G & 0 \\ 0 & 0 & 0 & A_D \end{bmatrix} x(k) + \begin{bmatrix} B_F \\ 0 \\ B_G D_F \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 & 0 & B_H & 0 \\ 0 & B_G D_H & 0 & 0 \end{bmatrix} \begin{bmatrix} n(k) \\ d(k) \end{bmatrix}$$

$$m(k) = \begin{bmatrix} 0 & 0 & C_G & C_D \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 & D_D \end{bmatrix} \begin{bmatrix} n(k) \\ d(k) \end{bmatrix}$$

The above system results in noise covariance matrices

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_B B_B^T & 0 & B_B B_B^T \\ 0 & B_C D_B H_B^T & 0 & B_C D_B H_B^T \\ 0 & 0 & 0 & B_D D_D^T \end{bmatrix}, \quad V = D_D D_D^T \sigma_d^2.$$
Theorem 5. Consider a system with feedback over an ACGN channel with memory as shown in Figure 5. Let $K_{po}$ be the optimal observer gain obtained from (8) with $A_o, B_o, C_o, W, V, S$ as defined for the augmented plant/channel structure given by (12) and $K_o$ the optimal regulator gain obtained from (9) with weights $Q = 0$ and $R = 1$.

The infimal LTI SNR for plant output disturbance rejection problem is then solved by the following controller

$$C_f(z) = zK_o(zI - (1 - K_{fo}C_o) (A_o - B_oK_o))^{-1}K_{fo}.$$  

with $K_{fo} = A_o^{-1}K_{po}$.

Proof. The proof, adapted to the case under consideration here, follows the same steps as proposed for Theorem 3.

Example 6. In the present example we consider a minimum phase plant model, with relative degree one, and one variable unstable pole $\rho$ in $[1, 5]$. The channel model is given by

$$F(z) = \frac{5}{8} \frac{z - 0.2}{z - 0.5}, \quad H(z) = \frac{7}{9} \frac{z - 0.1}{z - 0.3}$$

Both the channel additive noise variance $\sigma^2$ and the plant output disturbance variance $\sigma^2_p$ are equal to one. The disturbance filtering process is given by

$$O(z) = \frac{0.89323(z + 1)^2}{(z - 0.01)(z^2 + 1.775z + 0.7979)}$$

![Fig. 6. Infimal SNR for plant output disturbance rejection as a function of $\rho$ for the proposed ACGN channel with memory over the control path (solid line) and over the measurement path (dashed line).](image)

A similar analysis to the one proposed in Theorem 5 can be adapted to the situation when the ACGN channel with memory is located over the measurement path. We use this fact in Figure 6 where we first observe that the SNR for the measurement path tends to 1 when $\rho \to 1$ due to $\sigma^2 = 1$. On the other hand, even subject to plant output disturbance, the infimal SNR of the channel when located on the control path tends to zero as $\rho \to 1$. This might suggests to prefer, when possible by the system design, a control path location over a measurement path location for the channel model. However, we also observe that this preference is soon overturned in this example for values of $\rho$ approximately greater than 1.5. This observation then clarifies that, subject to plant output disturbance, the infimal SNR of the channel will be location dependent and that no location is apparently preferable in terms of lower channel SNR.

5. DISCUSSION

In this section we offer a discussion on the present results. Observe, for example, that besides the initial case of a memoryless AWGN channel over the measurement path, all the other infimal SNR solutions are numerical in nature. However, the loss in insight is repaid through the flexibility of studying such different scenarios. In particular we observe that the resulting infimal SNR is now different depending on the channel location, different from the stabilization only result Rojas et al. [2006a] where the channel location does not play a major role in the SNR limitation. Also on the channel location, we have that none of the two positions is preferable in terms of the resulting channel SNR limitation. However, it is also true that in any given situation, subject to plant output disturbance rejection, one location will be preferable to the other. Depending on the configuration addressed, the present study of plant output disturbance rejection is closely related to the case previously studied of plant input disturbance rejection, Rojas [2009b]. In that work and here the presence of a disturbance process suggests the possibility to consider the case of simultaneous communication through the control and measurement path. For the memoryless AWGN channel model, see Figure 7, we have that the extended model with input $s_e(k)$ and output $r_m(k)$ would be given by

$$x(k + 1) = \frac{A_G}{C_o} x(k) + \frac{B_G}{C_o} u(k) + \frac{B_G}{C_o} n_e(k)$$

$$r_m(k) = \frac{C_G}{C_o} x(k) + n_m(k)$$

The matrices $W, S$ and $V$ can be inferred from the above model, whilst we have to consider for the LQG optimization approach with LTR a weight selection of $Q = C_G T C_G$ and $R = 1$. The regulator weights selection allow us to account for the power limitation at the input of both channels simultaneously. In Figure 6 we have the resulting infimal SNR for plant model as in Example 6 with $\rho \in [-3, -1] \cup [1, 3]$, $\sigma^2 = 1$ and $\sigma^2_p = 1$. Observe that the choice of weights seems to suggest that a greedy approach can eventually benefit the channel over the measurement path with a lower SNR, against an increased infimal SNR.
6. CONCLUSION

In the present paper we have studied the infimal SNR problem for plant output disturbance rejection, when the plant is a linear, minimum phase, with arbitrary relative degree and unstable model. We presented the analysis for a communication channel model located over the control path or over the measurement path. When possible we provided the resulting infimal SNR in closed-form and when preferable we introduced a numerical approach in the control path. Finally the solution accounting for both limitations seems to grant a better overall infimal SNR for both channels over the complete range of values of $\rho$ studied here. Needless to say a more in depth study of what proposed here is necessary.

REFERENCES


