Optimal Strategies in Control Problem
under Programmed Disturbances

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Abstract: The control problem under programmed disturbance with terminal type quality index is considered. An optimal strategy with full memory is designed and some properties of this strategy are provided. It is demonstrated on a class of control systems, that the strategy can guarantee the value of lower game in the case when the condition of saddle point for the small game is violated. An illustrative nonlinear example is given.

Keywords: Control problems under conflict and/or uncertainties, differential or dynamic games.

1. INTRODUCTION

The control problems under the action of a disturbance that is known to be independent both of the system’s state and of control values is considered. Such a behavior of the disturbance allows us to consider it as a programmed one. The examples of such situation are the following: a physical object under the influence of natural forces (the wind during the aircraft landing); the mass behavior relative to an individual (the economic forces relative to some enterprise), etc. Control problem with unknown parameter also should be noted here.

Every strategy of some class of strategies has its bundle of constructive motions (see Krasovskii et al. (1974, 1988)), generated by the strategy from some initial position under the influence of all possible disturbances satisfying geometrical restrictions. Call it “game bundle”. If the goal of control is to minimize some quality index, the guarantee for the strategy and the optimal guarantee for the class are represented by the supremum of values of this quality index relative to all motions from the game bundle and by the infimum of the guarantees relative to all strategies of the class, respectively.

In the case when the disturbances are programmed, the proper bundles (call it “programmed bundles”) are subsets of the game bundles above. Consequently, the value of the guarantee for any strategy from the considered class and the value of the optimal guarantee for the class are less than or equal to the corresponding values in the general case.

When the saddle point condition for the small game Krasovskii et al. (1974, 1988) is hold, the optimal guarantees in the case of programmed disturbance and in the general case coincide, but in the absence of this property, in some cases there exists a control procedure that does not utilizes the values of current disturbance and guarantees the value of the lower game Serkov (2010).

In this work, the construction of the strategy with full memory that is optimal in the case of programmed disturbances is performed. Some properties of the strategy are illustrated on one class of control systems.

2. DEFINITIONS

Consider a control system:

\[
\begin{aligned}
\dot{x}(\tau) &= f(\tau, x(\tau), u(\tau), v(\tau)), & \tau \in T, \\
x(t_0) &= z_0 \in G_0 \subset \mathbb{R}^n, \\
u(\tau) &\in \mathcal{P} \subset \mathbb{R}^p, & v(\tau) \in \mathcal{Q} \subset \mathbb{R}^q, & \tau \in T.
\end{aligned}
\]  

Here \(T \equiv [t_0, \theta], \ -\infty < t_0 < \theta < \infty; \mathcal{P}, \mathcal{Q}, \text{ and } G_0\) are compact sets; \(f(\cdot)\) is a continuous, locally Lipschitz relative to \(x\) function such that all solutions of (1) (in the sense of Carathéodory) with initial conditions from \(G_0\) can be extended on the interval \(T\). The actions of control \(u(\cdot)\) and disturbance \(v(\cdot)\) are supposed to be Borel measurable functions on \(T\). Denote by \(U_T\) the set of all such control actions, by \(V_T\) the set of all such disturbances, by \(x(\cdot, t_0, z_0, u(\cdot), v(\cdot))\) the solution of equation (1) on the interval \(T\). Let \(G \subset T \times \mathbb{R}^n\) be compact set containing all motions of control system (1) starting from \(G; G_{t_0} = G_0\). Here \(G_{t_0}\) means the intersection of \(G\) and the set \(\{(\tau, x) \in T \times \mathbb{R}^n \mid \tau = t_0\}\).

System (1) satisfies the saddle point condition for the small game (see Krasovskii et al. (1974, 1988)), if for all \((\tau, x) \in G, s \in \mathbb{R}^n\) the following equality is fulfilled:

\[
\min_{v \in \mathcal{Q}} \max_{u \in \mathcal{P}} \langle s, f(\tau, x, u, v) \rangle = \max_{v \in \mathcal{Q}} \min_{u \in \mathcal{P}} \langle s, f(\tau, x, u, v) \rangle. \tag{3}
\]

Consider the quality index of terminal type:

\[
\gamma(x(\cdot, t_0, z_0, u(\cdot), v(\cdot))) \equiv \sigma(x(\theta, t_0, z_0, u(\cdot), v(\cdot))),(\cdot))
\]
where the function $\sigma(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz. The index should be minimized by the choice of control action $u(\cdot) \in U_T$.

For $t_* \in T$ let $\Delta = \{ \tau_0 = t_* < \tau_1 < \ldots < \tau_{n\Delta} = \vartheta \}$ be a partition of an interval $[t_*, \vartheta] \subseteq T$ and $d(\Delta)$ be the diameter of $\Delta: d(\Delta) = \max_{i \in 1..n\Delta} |\tau_i - \tau_{i-1}|$. The set of all such partitions of the interval $[t_*, \vartheta]$, with $d(\Delta) < \delta$, is denoted by $\Delta_{\delta}(t_*, \vartheta)$, and $\Delta(T) \equiv \{ \Delta \in \Delta_{\delta}(t_*, \vartheta) \mid \delta \in (0, 1), t_* \in T \}$.

Denote by $U_{na}$ the class of non-anticipating (Krasovskii et al., 1988, Ch.11) control strategies $U \in U_{na}$ of the form

$$U : T \times C(T; \mathbb{R}^n) \times \mathbb{R}^m \times U_T \times \Delta(T) \to \mathcal{P} \times \mathbb{R}^m,$$

where $C(T; \mathbb{R}^n)$ is the space of all continuous functions from $T$ to $\mathbb{R}^n$ with the Chebyshev norm $\|x(\cdot)\|_{C(T; \mathbb{R}^n)} = \max_{t \in T} \|x(t)\|; \mathbb{R}^m$ is the state space of the guidance point (Krasovskii et al., 1988, Sec. 8.2,10.4).

Denote by $x(\cdot) \equiv x(t_*, z_*, \{U, \Delta\}, v(\cdot))$ the step-by-step motion Krasovskii et al. (1974, 1988), caused by the strategy $U \in U_{na}$ under the disturbance $v(\cdot) \in V_T$, and by $u(\cdot) \equiv u(t_*, z_*, \{U, \Delta\}, v(\cdot)) \in U(t_*, \vartheta)$ the corresponding control piecewise constant on intervals of the partitions $\Delta \in \Delta_{\delta}(t_*, \vartheta)$,

$$x(\cdot) = x(t_*, z_*, u(\cdot), v(\cdot)),$$

$$(u(\tau_i), y_i) = U(\tau_i, x(\cdot), y_{i-1}, u(\cdot), \Delta), \quad i \in 1..n\Delta;$$

here $(x, z_*) \in G$ and $y_0 \equiv Y(z_*)$. Let $(t_*, z_*) \in G$, $U \in U_{na}$ and $v(\cdot) \in V_T$. The symbol $X(t_*, z_*, U, v(\cdot))$ stands for the closure in $C([t_*, \vartheta]; \mathbb{R}^n)$ of the set of all solutions $x(\cdot), t_*, z_*, \{U, \Delta\}, v(\cdot): k \in \mathbb{N}$ such that $\lim_{k \to \infty} z_k = z_*, \Delta_k \in \Delta_{\delta}(t_*, \vartheta)$, $\lim_{k \to \infty} \delta_k = 0$. The set partition

$$X^{pr}(t_*, z_*, U) \equiv \text{cl}_{C([t_*, \vartheta]; \mathbb{R}^n)} \bigcup_{v(\cdot) \in V_T} X(t_*, z_*, U,(v(\cdot), v(\cdot)),$$

where the symbol $\text{cl}_{C}$ $A$ means “the closure of the set $A$ in topology of the space $C$”, is called the programmed bundle of the strategy $U$. The guarantee $\Gamma^{pr}(t_*, z_*, U, x(\cdot))$ for a strategy $U \in U_{na}$ and the optimal guarantee $\Gamma^{pr}(t_*, z_*, U_{na})$ for the class of strategies $U_{na}$ in a position $(t_*, z_*) \in G$ with respect to programmed disturbances are as follows

$$\Gamma^{pr}(t_*, z_*, U) \equiv \sup_{x(\cdot) \in X^{pr}(t_*, z_*, U)} \gamma(x(\cdot)),$$

$$\Gamma^{pr}(t_*, z_*, U_{na}) \equiv \inf_{U \in U_{na}} \Gamma^{pr}(t_*, z_*, U).$$

A strategy $U \in U_{na}$ is said to be optimal for initial position $(t_*, z_*) \in G$ (with respect to programmed disturbances), if $\Gamma^{pr}(t_*, z_*, U) = \Gamma^{pr}(t_*, z_*, U_{na})$.

3. RESULTS

1. Begin with an informal outline of provided optimal strategy (denote it by $U_\ast$).

In the control process, the strategy simulates the motion of auxiliary control system (called y-model). For the motion of y-model on current interval of some partition $\Delta \in \Delta(T)$ the strategy $U_\ast$ generates (reconstructs) a disturbance that is equivalent (in some special sense) to a real disturbance influencing the control system. This step-by-step reconstruction is based on the motion of the control system on the current interval of partition $\Delta$. Then the value of control that makes the motion of y-model be optimal on current interval is found. And then this control is applied to the “real” control system at the next interval of the partition. The procedure is repeated for all intervals of the partition $\Delta$.

When the lengths of the intervals tend to zero, the motions of y-model converge to some optimal constructive motion, and so do the motions of the control system, as far as they converge (in some proper sense) to corresponding motions of y-model. This closeness of the motions of y-model and the motions of control system follows from the programmed character of disturbance and from properties of step-by-step reconstructions of the disturbance.

The mentioned reconstruction of the disturbance is essential part of the strategy. The algorithm of the reconstructions assumed to be dynamical, i.e. it reconstructs the disturbance “on the fly” step-by-step. The algorithm belongs to the area of inverse problems of dynamics Kryazhimskii et al. (1988); Osipov et al. (1995).

This y-model, plays the role of the auxiliary guidance point in the procedure of guidance control Krasovskii et al. (1974, 1988). In addition to that, the provided control strategy includes time shift between control action in the y-model and in the control system: in the model the control works for one step earlier then in the system. In other words, the model is leader in reactions on the disturbance. The constructions similar to provided one were also used in Serkov (1991), Serkov (2008) in control problems, formalized on the base of Savage type criterion.

Return to exact definitions. At the beginning consider the partitions with uniform intervals. Let $(t_*, z_k) \in G$, a sequence of partitions $\Delta_k = \{ t_{ki} = t_* + ihk, h_k = (\vartheta - t_*)/k, i \in 0..k \}, k \in \mathbb{N}$, and a disturbance $v(\cdot) \in V_T$ be given. The motions of the system satisfy the equations

$$x_{ki}(\tau) = z_k + \int_{t_*}^{\tau} f(s, x_{ki}(s), u_{ki}(s), v(s)) ds, \quad \tau \in [t_*, \vartheta],$$

where $\tau \in [t_*, \vartheta]$ and $u_{ki}(\cdot)$ stands for the actions of control, caused by the strategy $U_\ast$. Introduce the notation for a position of the control system and for values of control at moments of partition $\Delta_k$ (all step functions are supposed to be right–continuous):

$$x_{ki} \equiv x_{ki}(t_{ki}), \quad u_{ki} \equiv u_{ki}(t_{ki}), \quad v_{ki} \equiv v_{ki}(t_{ki}),$$

where $x_{ki}(\cdot)$, $u_{ki}(\cdot)$, and $v_{ki}(\cdot)$ are the actions of control and disturbance in the y-model generated in the control process. So, the motions the y-model of y-model are described by the equations

$$y_k(\tau) = z_{ki} + \int_{t_*}^{\tau} f(s, y_k(s), \bar{u}_k(s), \bar{v}_k(s)) ds, \quad \tau \in [t_*, \vartheta],$$

for $\tau \in [t_*, \vartheta]$. At the first interval of the partition $\Delta_k$ we set the value of control equal to arbitrary permitted value: $u_{k0} \in \mathcal{P}$.

Till the moment $t_{ki}$ the part $x_{ki}(\cdot)/[t_*, t_{ki})$ of the motion $x_{ki}(\cdot)$ and the part $u_{ki}(\cdot)/[t_*, t_{ki})$ of the control $u_{ki}(\cdot)$ are available. On the base of these data, the disturbance $\bar{v}_{k0} \in \mathcal{Q}$ that is constant on the interval $[t_*, t_{ki})$, is chosen.
Given some closed set $W \subset \mathbb{R}^{n+1}$ such that $W|_{\tau} \neq \emptyset$, $\tau \in [t^*, \vartheta]$, the value of control $\bar{u}_k$ on interval $[\tau, \tau_k)$ is defined as the value of counterstrategy extremal on the set $W$ (Krasovskii et al., 1974, §82), (Krasovskii et al., 1988, Ch.10):

$$\bar{u}_k \in \arg\min_{w \in P} \{ z_k - w(z_a, f(t_*, z_k, u, \bar{u}_k)): w(z_k) \in \arg\min_{w \in W_{t_*}} \| w - z_k \|, \}
$$

where $\arg\min_{a \in A} (g(a) \equiv \{ a \in A | g(a) \equiv \inf_{a \in A} g(a) \}$.

Set the value of control $u_k(\cdot)$ on the interval $[\tau_k, \tau_{k+1})$ equal to the value $\bar{u}_k(\cdot)$ on the interval $[\tau_{k(i-1)}, \tau_k)$:

$$u_k = \bar{u}_k(\cdot), \quad i \in \{0, k \in \mathbb{N}. \}$$

The value $u_k(\cdot)$, formally defined for $i = k$ on the interval $[\theta, \theta + h_k)$, does not affect the motion $x_k(\cdot)$, but is necessary for further estimates.

Till the moment $\tau_{k+1}$ the part $x_k(\cdot)[\tau, \tau_{k+1})$ of the motion $x_k(\cdot)$ and the part $u_k(\cdot)[\tau, \tau_{k+1})$ of control $u_k(\cdot)$ are available. On the base of these data, the disturbance $\bar{v}_k \in Q$ is chosen and the value $\bar{u}_k$ of the control for the $y$-model on the interval $[\tau_k, \tau_{k+1})$ is defined:

$$\bar{u}_k \in \arg\min_{w \in P} \{ y_k - w(y_k, f(\tau_k, y_k, u, \bar{v}_k)): w(y_k) \in \arg\min_{w \in W_{\tau_k}} \| w - y_k \|, \}
$$

Turn to the choice of the disturbances $\bar{v}_k(\cdot)$, Denote by $U_{[\tau_*, \vartheta]}(\Delta) \subset U_{[\tau_*, \vartheta]}$ and $V_{[\tau_*, \vartheta]}(\Delta) \subset V_{[\tau_*, \vartheta]}$ the sets of all actions of control and disturbance, respectively, piecewise constant on intervals of $\Delta \subset \Delta_{[\tau_*, \vartheta]}$.

Assumption 1. For control system (1), there exists a non-anticipating algorithm of reconstruction of disturbance with the following property. For arbitrary $\varepsilon > 0$, $v(\cdot) \in V_T$ there exists $\delta(\varepsilon) > 0$ such that for all $(t_*, z_\varepsilon) \in G$, $\Delta \in \Delta_{[\tau_*, \vartheta]}$, $\delta < \delta(\varepsilon)$ and (6) and (11) $u_1(\cdot) \in U_{[\tau_*, \vartheta]}(\Delta)$ the algorithm provides the disturbance $\bar{v}(\cdot) \equiv \bar{v}_k(\cdot, x_1(\cdot), u_1(\cdot), \Delta) \in V_{[\tau_*, \vartheta]}(\Delta)$, based on the information on $x(\cdot) \equiv x(\cdot, t_*, z_\varepsilon, u_1(\cdot), v(\cdot))$ and $u_1(\cdot)$, that satisfies the inequality

$$\|x(\cdot, t_*, z_\varepsilon, u_2(\cdot), v(\cdot)) - x(\cdot, t_*, z_\varepsilon, u_2(\cdot), \bar{v}(\cdot))\|_{C_{[\tau_*, \vartheta]} \mathbb{R}^n} \leq \varepsilon \quad (8)
$$

for all $u_2(\cdot) \in U_T$.

It can be verified that Assumption 1 is true for systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t, u(t)) + \Delta(t)C(v(t)), \quad (9)
$$

and of the form

$$\dot{x}(t) = A(t)x(t) + B(t, u(t))C(v(t)), \quad (10)
$$

for all $u \in P$ and $t \in T$ the matrices $B(t, u)$ and $D(t)$ have the zero kernel.

In the cases, when Assumption 1 is true, the disturbance $\bar{v}_k(\cdot)$ is defined by means of the algorithm:

$$\bar{v}_k \equiv \bar{v}(\tau_k, x_k(\cdot), u_k(\cdot), \Delta_k), \quad (11)
$$

where $i \in \{0, (k-1), k \in \mathbb{N} \}$ and $\bar{v}(\cdot, x_k(\cdot), u_k(\cdot), \Delta_k)$ is reconstructed on the base of observing the motion $x_k(\cdot) \equiv x(\cdot, t_*, z_k, u_k(\cdot), v(\cdot))$ of control system (1).

This completes the definition of the strategy $U_t$ depending on the set $W$.

Remark 2. The construction of the strategy $U_t$ shows that it is non-anticipating and non-discriminative (does not utilize the values of the disturbance).

Remark 3. As far as linear systems (9) satisfy condition (3) of the saddle point in the small game, the optimal guarantee in the case of programmed disturbance (considered here) equals to the optimal guarantee for arbitrary disturbance Sverkov (2010) and, consequently, has well-known solution in the theory of differential games.

In the case of disturbances of general form, the optimal guarantee obtains the value of lower game (see Krasovskii et al. (1988)) in the class of non-discriminative strategies under condition (3) and the additional regularity conditions (see Krasovskii et al. (1988); Subbotin (1981)).

For systems, where property (3) is violated, for example, for systems of the form (10), there are cases, when the optimal guarantee reaches the value of lower game (Sverkov, 2010, Example 2) in the class of strategies with full memory.

Thus, the information about programmed character of disturbances can be effectively used for the substantial enhancement of optimal guarantee in the classes of non-discriminative strategies.

The set $W \subset \mathcal{G}$ is called u-stable, if for arbitrary $(\tau_*, x_\varepsilon) \in W$, $\tau^* \in [\tau_*, \vartheta]$ and $v_\varepsilon \in Q$ there exists an absolutely continuous solution $x(\cdot)$ of the inclusion

$$\dot{x}(\cdot) \in \mathcal{F}_u(\tau, x(\cdot), v_\varepsilon), \quad \text{for a.a. } \tau \in [\tau_*, \tau^*], \quad (12)
$$

such that $(\tau^*, x(\tau^*)) \in W$. Here

$$\mathcal{F}_u(\tau, x, v_\varepsilon) \equiv \mathcal{C}_{o_\varepsilon}[\cdot \in \mathbb{R}^n : f(\tau, x, u, v_\varepsilon), u \in P]$$

the symbol “a.a.” stands for “almost all”, and the symbol $\mathcal{C}_{o_\varepsilon}B$ stands for “closed convex hull of the set $B \subseteq A$ in the linear topological space $A$”.

Lemma 4. If control system (1) satisfies Assumption 1, $(t_*, z_\varepsilon) \in W \subset \mathcal{G}$, $W$ is closed and u-stable set, then for all motions $x(\cdot) \in X^{\tau_\varepsilon}(t_*, z_\varepsilon, U_\varepsilon)$ of the strategy $U_\varepsilon$, defined by relations (6), (7), (11), the inclusions

$$(\tau, x(\tau)) \in W, \quad \tau \in [t_*, \vartheta] \quad (13)
$$

are fulfilled.

Proof. First of all, due to u-stable property, $W|_{\tau} \in \mathbb{R}^n$ are closed non-empty sets for all $\tau \in [t_*, \vartheta]$. So, definitions (7) are correct. Let $\{x_\varepsilon(\cdot) | k \in \mathbb{N}\}$ any sequence of step-by-step motions (4), generated by $U_\varepsilon$, that converges in $C_{[\tau_*, \vartheta]} \mathbb{R}^n$ to some motion $x(\cdot) \in X^{\tau_\varepsilon}(t_*, z_\varepsilon, U_\varepsilon)$; $\{y_\varepsilon(\cdot) | k \in \mathbb{N}\}$ is the sequence of corresponding step-by-step motions of y-model (5). Let $y(\cdot)$ be a partial limit in $C_{[\tau_*, \vartheta]} \mathbb{R}^n$ of the sequence $\{y_\varepsilon(\cdot) | k \in \mathbb{N}\}$. From the definition of the strategy $U_\varepsilon$ as the counterstrategy extremal to the u-stable set $W$ (see (7)), it follows (Krasovskii et al., 1974, Lemma 82.2), (Krasovskii et al., 1988, Theorem 10.2.3) that the inclusion

$$(\tau, y(\tau)) \in W, \quad \tau \in [t_*, \vartheta] \quad (14)
$$

is fulfilled. Denote

$$z_\varepsilon(\cdot) \equiv x(\cdot, t_*, z_k, \bar{u}_k(\cdot), v(\cdot)), \quad \varepsilon \in \mathbb{N}. \quad (15)$$
From definition (11) and inequality (8) of Assumption 1, it follows that
\[
\lim_{k \to \infty} \|z_k(\cdot) - y_k(\cdot)\|_{C([t_*, \vartheta]; \mathbb{R}^n)} = 0. \tag{16}
\]

Let us prove the equality
\[
\lim_{k \to \infty} \|x_k(\cdot) - z_k(\cdot)\|_{C([t_*, \vartheta]; \mathbb{R}^n)} = 0. \tag{17}
\]

The equalities
\[
z_k(\tau) = z_k + \int_{t_0}^{\tau} f(s, z_k(s), u_k(s + h_k), v(s))ds, \quad \tau \in [t_*, \vartheta].
\]

are confirmed by definitions (15), (6) of the functions \(z_k(\cdot)\) and \(u_k(\cdot)\). Being continuous, the function \(f\) possesses the modulus of continuity on compact set \(G \times P \times Q\) (Warga, 1972, sec.1.2) with respect to the first argument; denote it \(\mu_1(\cdot)\):
\[
\max_{(t',z), (t'',z) \in G} \|f(t', z, u, v) - f(t'', z, u, v)\| < \mu_1(|t' - t''|).
\]

Represent functions \(z_k(\cdot)\) in the form
\[
z_k(\tau) = z_k + \int_{t_0}^{\tau} f(s + h_k, z_k(s), u_k(s + h_k), v(s))ds
\]
\[+ \frac{1}{\kappa} \int_{t_0}^{\tau} \int f(s, z_k(s), u_k(s + h_k), v(s))ds,
\]
\[\tau \in [t_*, \vartheta].
\]

Denote the last integral by \(\varphi_1(\tau, h_k)\) and, using the modulus of continuity \(\mu_1(\cdot)\), come to the inequality
\[
z_k(\tau) = z_k + \int_{t_0}^{\tau} f(s + h_k, z_k(s), u_k(s + h_k), v(s))ds
\]
\[+ \varphi_1(\tau, h_k) \cdot |\varphi_1(\tau, h_k)| < (\vartheta - t_*) \mu_1(h_k), \quad \tau \in [t_*, \vartheta]. \tag{18}
\]

Denote \(G_1 \equiv G \cup \{ (\tau, x) \mid \tau \in [\vartheta, \vartheta + 1], x \in G|_{\{\vartheta\}} \}. \) The set \(G_1\) is closed and bounded as the set \(G\) is closed and bounded. With Lipschitz constant \(L_f(G_1)\) of function \(f\) with respect to the second variable on the set \(G_1\) and with the constant \(\kappa\) that bound the norm of the function \(f\) on the set \(G_1 \times P \times Q\) we can get
\[
z_k(\tau) = z_k + \int_{t_0}^{\tau} f(s + h_k, z_k(s + h_k), u_k(s + h_k), v(s))ds
\]
\[+ \varphi_1(\tau, h_k) + \varphi_2(\tau, h_k),
\]
\[|\varphi_1(\tau, h_k)| < (\vartheta - t_*) \mu_1(h_k),
\]
\[|\varphi_2(\tau, h_k)| < (\vartheta - t_*) L_f(G_1) \kappa h_k, \quad \tau \in [t_*, \vartheta].
\]

Define the value of the function \(v(\cdot)\) in the region \((\vartheta, \vartheta + 1)\) equal to an arbitrary constant \(\bar{v} \in Q\). Analogously to (18), we modify the above equality, utilizing modulus of continuity \(\mu_2(\cdot)\) of the function \(f\) with respect to the fourth variable that is uniform in the region \(G_1 \times P\):
\[
z_k(\tau) = z_k + \int_{t_0}^{\tau} f(s + h_k, z_k(s + h_k), u_k(s + h_k), v(s + h_k))ds
\]
\[+ \varphi_1(\tau, h_k) + \varphi_2(\tau, h_k) + \varphi_3(\tau, h_k),
\]
\[|\varphi_1(\tau, h_k)| < (\vartheta - t_*) \mu_1(h_k),
\]
\[|\varphi_2(\tau, h_k)| < (\vartheta - t_*) h_k L_f(G_1) \kappa,
\]
\[|\varphi_3(\tau, h_k)| < \int_{t_0}^{\tau} \mu_4(|v(s) - v(s + h_k)|)ds, \quad \tau \in [t_*, \vartheta].
\]

Due to the property of measurable functions (see Natanson (1961)), for any \(v(\cdot) \in V\) the last integral converges to zero as \(h_k \to 0\). From the last equation and (4) we derive
\[
x_k(\tau) = z_k + \int_{t_0}^{\tau} f(s + h_k, z_k(s + h_k), u_k(s + h_k), v(s))ds
\]
\[+ \int_{t_0}^{t_0 + h_k} f(s, z_k(s), u_k(s + h_k), v(s))ds
\]
\[+ \int_{t_0}^{t_0 + h_k} f(s, y_k(s), u_k(s), v(s))ds
\]
\[+ \int_{t_0}^{t_0 + h_k} f(s, y_k(s), u_k(s), v(s))ds
\]
\[+ \int_{t_0}^{t_0 + h_k} \mu_4(|v(s) - v(s + h_k)|)ds, \quad \tau \in [t_*, \vartheta].
\]

With Lipschitz constant \(L_f(G_1)\) and the upper bound \(\kappa\), the previous relations can be modified to
\[
|x_k(\tau) - z(\tau)| \leq \int_{t_0}^{t_0 + h_k} L_f(G_1) |x_k(s) - z(s)|ds
\]
\[+ h_k(2\kappa + L_f(G_1)) \int_{t_0}^{t_0 + h_k} (\vartheta - t_*) \mu_1(h_k) + (\vartheta - t_*) h_k L_f(G_1) \kappa +
\]
\[\int_{t_0}^{t_0 + h_k} \mu_4(|v(s) - v(s + h_k)|)ds, \quad \tau \in [t_*, \vartheta].
\]

Using notation \(\Psi(h_k)\) for the sum of all right-hand side items excluding the first one, from Gronwall’s inequality (Warga, 1972, Theorem II.4.4), we obtain
\[
|x_k(\tau) - z(\tau)| \leq \Psi(h_k)(1 + L_f(G_1)(\vartheta - t_*) \exp(L_f(G_1)(\vartheta - t_*) \kappa + \int_{t_0}^{t_0 + h_k} \mu_4(|v(s) - v(s + h_k)|)ds, \quad \tau \in [t_*, \vartheta], \lim_{\vartheta \to 0} \Psi(\varepsilon) = 0.
\]

The last relations give (17). Required inclusion (13) follows from (14), (16) and (17).

Remark 5. The form of \(\Psi(\cdot)\) shows that convergence (17) is not uniform with respect to \(v(\cdot)\). In addition to that, the restriction of the set \(V\) to some subset that is compact in the strong topology of the space \(L_2([t_*, \vartheta]; \mathbb{R}^n)\) makes convergence (17) uniform.

Now for arbitrary \((t_*, z_*) \in G\) define the u-stable bridge Krasovskii et al. (1974, 1988):
\[
Z^{pr}_{\cdot}(t_*, z_*) \equiv \bigcup_{\varepsilon > 0} \bigcup_{\Gamma_\varepsilon(t_*, z_*, U) \leq t_0 + \varepsilon, \Gamma_\varepsilon(t_*, z_*, U_n) \leq t_0} Z^{pr}(t_*, z_*, U), \tag{19}
\]
\[
Z^{pr}(t_*, z_*, U) \equiv \{(\tau, x(\tau)) \mid \tau \in [t_*, \vartheta], x(\cdot) \in X^{pr}(t_*, z_*, U)\}.
\]

Lemma 6. For all \((t_*, z_*) \in G\) the set \(Z^{pr}_{\cdot}(t_*, z_*)\) is closed, u-stable and
\[
\max_{z \in Z^{pr}_{\cdot}(t_*, z_*) \cap \Gamma_\varepsilon(t_*, z_*, U_n)} \sigma(x) = \Gamma^{pr}(t_*, z_*, U_n). \tag{20}
\]
Proof. The closeness property of the set $Z_{pr}^{*}(t_*, z_*)$ and equality (20) follow from the definition. The set $Z_{pr}^{*}(t_*, z_*)$ is $u$-stable because of $u$-stability of sets under the intersection sign in (19) and of monotonicity of their sequence.

Theorem 7. If control system (1) satisfies Assumption 1, then for all $(t_*, z_*) \in G$ the strategy $U_i$ defined by relations (6), (7), (11), where $W \equiv Z_{pr}^{*}(t_*, z_*)$ is optimal for the initial position $(t_*, z_*)$

The proof of Theorem 7 follows from Lemmas 4, 6 with the use of the reasonings standard for the theory of differential games. It can be demonstrated, that under the conditions of Theorem 7 the value $\Gamma^r(t_*, z_*, U_{na})$ equals to the value of lower differential game at the initial position $(t_*, z_*)$ in $G$ (see Krassovskii et al., 1988, Ch.10).

2. Another variant of the optimal strategy can be based on the conditions of regularity of programmed maximin Krassovskii et al. (1974, 1988); Subbotin et al. (1981).

Consider a control system of the form

$$\dot{x}(\tau) = A(t)x(\tau) + f(\tau, u(\tau), v(\tau)), \quad \tau \in T \equiv [t_0, \tau] \subset \mathbb{R}, \quad x(t_0) = z_0 \in G_0 \subset \mathbb{R}^n \quad (21)$$

under restrictions (2) and a terminal quality index defined by

$$\sigma(x) \equiv \min_{u \in \Pi} ||x - w||, \quad (22)$$

where $M \subset \mathbb{R}^n$ is a given non-empty closed convex set, cylindrical with respect to last $n - m$ coordinates. Define the function of programmed maximin:

$$c(t_*, z_*) \equiv \sup_{u \in \Pi} \inf_{v \in \Pi_T} \sigma(x(t_*, z_*, u(\cdot), v(\cdot))).$$

Following Krassovskii et al. (1974, 1988) one can show, that

$$c(t_*, z_*) = \max_{l \in \Pi} [\langle l, \Phi_{m}(\omega, t_*, z_*) \rangle + \rho(l, t_*)], \quad (23)$$

where

$$\rho(t_*, l) \equiv \int_{t_*}^{\theta} \min_{s \in \Pi} \left( l, \Phi_{m}(s, u(s), v) \right) ds, \quad \rho_M(l) \equiv \min_{x \in \Pi_M} ||l, x||,$$

$$\Phi(\omega, t)$$ is an $n \times n$-matrix function satisfying the boundary problem

$$\frac{d\Phi(\omega, t)}{dt} = -\Phi(\omega, t)A(t), \quad \Phi(\omega, 0) = E, \quad (24)$$

and the signs $\Phi_{m}(\omega, t)$ stands for the matrix of the first $m$ rows of matrix $\Phi(\omega, t)$.

In the cases when the function

$$\sigma(t, \omega, x, v) \equiv -\rho(t, l) - \rho_M(l)$$

is convex with respect to the variable $l$, value (23) equals to the value of the lower differential game (Krassovskii et al., 1988, Ch.10), and the value $u(\tau, x, v)$ of the optimal counterstrategy is defined by the conditions

$$u(\tau, x, v) \equiv \arg\min_{u \in \Pi} \langle l_0, \Phi_{m}(\omega, \tau)x \rangle, \quad (26)$$

for arbitrary $u \in \Pi$, $\tau, \tau' \in T$, $x(\cdot) \in C(T; \mathbb{R}^n)$ denote by $Q_{\tau, u}$ the factor set of $Q$ induced by the equivalence relation $f(\tau, u, v_1) = f(\tau, u, v_2)$ and by $V(u, x(\cdot), \tau, \tau')$ the following subsets of $Q$:

$$V(u, x(\cdot), \tau, \tau') \equiv \arg\min_{v \in \Pi} \left\{ \Phi(\omega, \tau') \langle x(\cdot) - f(\tau, u, v) \rangle \right\}.$$ 

Lemma 8. Suppose that the control system has the form (21) and for every $\tau \in T$ the sets $Q_{\tau, u}$ coincides for all $u \in \Pi: Q_{\tau, u} = Q_{\tau}$. Then the reconstruction defined by

$$\bar{v}(\tau, x(\cdot), u(\cdot), \Delta) \equiv V(u(\tau), x(\cdot), \bar{\tau}, \bar{\tau} + 1), \quad (27)$$

where $\tau \in [\bar{\tau}, \bar{\tau} + 1], \bar{\tau} \in \Delta$, satisfies the Assumption 1.

Combining the counterstrategy with the reconstructed values $\bar{v}(\cdot)$ into the $y$-model and using the resulting control actions in the “real” control system with one step delay (as it was done above) one get another variant of strategy $U_i$.

Theorem 9. If control system (1) satisfies conditions of Lemma 8 and function $\mathcal{R}(\cdot)$ (25) is convex with respect to the variable $l$ for all $\tau \in T$, then the strategy $U_i$, described by (6) and by relations

$$\bar{v}_{ki} = u(t_{ki}, y_{ki}), \quad \bar{v}_{ki} \equiv V(t_{ki}, x(\cdot), t_{ki}, \bar{\tau} + 1), \quad (28)$$

with $u(\cdot) \equiv t \times \mathbb{R}^n \times Q \rightarrow \Pi$ defined in (26), is optimal for the initial position $(t_*, z_*)$.

The proof of Theorem 9 is based on the asymptotic optimality of step-by-step motions of the $y$-model and on the closeness of the motions to corresponding motions of the control system (assured by Assumption 1).

4. EXAMPLE

The availability of provided strategy and some differences in optimal guarantee comparing with the case of general type disturbance are demonstrated on the next illustrative control problem (Krassovskii et al., 1974, Example 21.5).

Consider two mass points (with masses $m_1$ and $m_2$, respectively) in the horizontal plane, where some coordinate system is chosen. These points are moving under the forces $F_1$ and $F_2$. Vector $F_1$ coincides with a vector $(u_1, u_2)$, rotated counter-clockwise on the angle $v_3$, if $v_3 > 0$, and in opposite direction, otherwise. Vector $F_2$, in its turn, coincides with vector $(v_1, v_2)$, rotated counter-clockwise on the angle $v_3$, if $v_3 > 0$, and in opposite direction, otherwise. The vector $u \equiv (u_1, u_2, u_3)$ can be chosen within set $\Pi$ to control the system. The values $v \equiv (v_1, v_2, v_3)$ are unknown in advance and programmed to vary in the set $Q$. The sets $\Pi$ and $Q$ are given by the relations:

$$\Pi \equiv \{(u_1, u_2, u_3) | (u_1^2 + u_2^2) \leq \lambda_1, |u_3| \leq \beta_1\},$$

$$Q \equiv \{(v_1, v_2, v_3) | (v_1^2 + v_2^2) \leq \lambda_2, |v_3| \leq \beta_2\}.$$ 

Let the vector $x^{(1)} \in \mathbb{R}^4$ denotes coordinates of the first point and of its velocity, and $x^{(2)} \in \mathbb{R}^4$, the same values for the second point. Then these coordinates satisfy the equations:
According to the aim of control and to the construction of the $y$-model define the set $M$ from (22) as $M \equiv \{(0, 0)\}$. Calculate maximin (23):

$$
\varepsilon_{0}(\tau, y) = \max_{\|l\| = 1} \left[ (l, y) - \frac{(\vartheta - \tau)^2}{2} \left( \frac{\lambda_1}{m_1} - \frac{\lambda_2 \cos \beta_1}{m_2} \right) \|l\| \right] = \|y\| - \frac{(\vartheta - \tau)^2}{2} \left( \frac{\lambda_1}{m_1} - \frac{\lambda_2 \cos \beta_1}{m_2} \right) \|l\| .
$$

Function $\varkappa(\cdot)$ (25) takes the form

$$
\varkappa(\tau, l) = -\frac{(\vartheta - \tau)^2}{2} \left( \frac{\lambda_1}{m_1} - \frac{\lambda_2 \cos \beta_1}{m_2} \right) \|l\|
$$

and under the condition

$$
\frac{\lambda_1}{m_1} > \frac{\lambda_2 \cos \beta_1}{m_2}
$$

becomes convex relative to $l$ for all $\tau \in [t_0, \vartheta]$. In this case, the vector providing maximum in (30) can be chosen as

$$
l_{0} \equiv \begin{cases} y/\|y\|, & \|y\| > 0, \\ (0, 1), & \|y\| = 0. \end{cases}
$$

and, the values $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ of counterstrategy (26) can be written as follows

$$
\begin{align*}
\bar{u}_1 &= -\lambda_1 \frac{\|T(\bar{v})\Phi_2(\vartheta, \tau) y\|_2}{\|\Phi_2(\vartheta, \tau) y\|_2}, \\
\bar{u}_2 &= -\lambda_2 \frac{\|T(\bar{v})\Phi_2(\vartheta, \tau) y\|_2}{\|\Phi_2(\vartheta, \tau) y\|_2}, \\
\bar{u}_3 &= \arg\max_{|\bar{v}| \leq \beta_1} \langle y, \Phi_2(\vartheta, \tau) T(u) \bar{v} \rangle,
\end{align*}
$$

where prime means the transposition operation and $\{A\}_\tau$ stands for the first 2 rows of the matrix $A$. Turning to the original variables, one get

$$
\begin{align*}
\bar{u}_1 &= -\lambda_1 \frac{\|T(\bar{v})\Phi_2(\vartheta, \tau) y\|_2}{\|\Phi_2(\vartheta, \tau) y\|_2}, \\
\bar{u}_2 &= -\lambda_2 \frac{\|T(\bar{v})\Phi_2(\vartheta, \tau) y\|_2}{\|\Phi_2(\vartheta, \tau) y\|_2}, \\
\bar{u}_3 &= \arg\max_{|\bar{v}| \leq \beta_1} \langle y, \Phi_2(\vartheta, \tau) T(u) \bar{v} \rangle.
\end{align*}
$$

Let us calculate values $\bar{v}_k(\cdot)$ following Lemma 8. Denote the coordinates of vector $\bar{v}_k$ by $\bar{v}_k \equiv (\bar{v}_1, \bar{v}_2, \bar{v}_3)$, then they can be rewritten as follows:

$$
\begin{align*}
\bar{v}_1 &= \frac{m_2}{h_k} \langle T(\bar{v}) \Phi(\tau_{k+1}, \vartheta) (x_{k+1}^{(1)} - x_{k+1}^{(2)}) \rangle, \\
\bar{v}_3 &= \frac{m_3}{h_k} \langle \Phi(\tau_{k+1}, \vartheta) (x_{k+1}^{(1)} - x_{k+1}^{(2)}) \rangle,
\end{align*}
$$

Estimate (8) of Assumption 1 is fulfilled for the values $x_{k+1}^{(1)} - x_{k+1}^{(2)}$ defined by the motions of control system (29). And due to the goal of control, this estimate is enough for Theorem 9 to be fulfilled.

The computer simulations of the control process with various programmed disturbances and various steps $h_k$ of partition demonstrated, that the difference $\|x_{k+1}^{(1)}(\vartheta) - x_{k+1}^{(2)}(\vartheta)\|_2$ ranks over the lower differential game value (30) no more than some quantity vanishing with $h_k \to 0$.

REFERENCES


