Taking into account Measurement Noise and Technological Constraints on Modelling and Controller Synthesis

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Abstract: Most of control systems are subject to physical and technological constraints, e.g. measurement noise, input and output ranges of digital devices (ADC, DAC), etc. In this paper, we first give a formalization of the practical constraints, and then show that these constraints limit the achievable bandwidth in closed loop. Consequently, it is sufficient to satisfy the specifications of modelling or controller synthesis on a limited frequency band, and not on all frequencies. By applying this approach to modelling problem, we propose a control-oriented modelling method with the guarantee of robustness margins obtained for controlled system. An experimental application, temperature control of thermal process, is presented to illustrate the applicability of the proposed approach.

Keywords: Measurement noise, technological constraint, modelling for robust control, robustness margin.

1. INTRODUCTION

Nowadays, almost all controllers are implemented using digital devices (microprocessors with input-output analog-to-digital converters and vice-versa). These devices and the experimental systems to be controlled are subject to physical and technological constraints, e.g. sensor precision, input and output ranges of actuator, etc. The digital input-output devices (analog-to-digital (ADC) and digital-to-analog (DAC) converters) are not perfect. They are technologically limited by operating range and resolution. Besides, the inputs and outputs of a control system must be a physical quantity (e.g. tension in Volt is the most used), whose value is in a pre-determined interval. For instance, the ADC, receiving sensor signal, admits only signals in a given interval, typically \( y \in [0; 5V] \). Similarly, the DAC is only able to generate a signal in a definite range, typically \( u \in [0; 5V] \). Due to practical aspects, the system designer usually normalizes input and output signals such that their operating ranges are identical. The most often encountered ranges are: \([0; 3.3V]\), \([0; 5V]\), \([-5V; 5V]\), \([0; 10V]\) and \([-10V; 10V]\). These practical features must also be considered in control-oriented modelling or controller synthesis problem.

The rest of this paper is organized as follows. In Section 2, the technological and physical constraints of a real control system are presented. The limitation of the achievable closed-loop bandwidth due to these constraints has been shown. In Section 3, by taking into account these practical constraints, we present a control-oriented modelling method with the guarantee of robustness margins obtained for controlled system. An experimental application is shown. In Section 4, finally, conclusion are drawn in Section 5.

2. TECHNOLOGICAL AND PHYSICAL CONSTRAINTS OF A REAL CONTROL SYSTEM

Consider the closed-loop system presented in Fig. 1. In this section, we will first give a formalization of the practical constraints, e.g. characteristics of acquisition chain, measurement noise.

2.1 Acquisition Chain

We first introduce some notations:
\[
\begin{align*}
    y &\in [\bar{y}; \bar{\bar{y}}] & \subset \mathbb{R}, & -\infty < \bar{y} < \bar{\bar{y}} < \infty \\
    u &\in [\bar{u}; \bar{\bar{u}}] & \subset \mathbb{R}, & -\infty < \bar{u} < \bar{\bar{u}} < \infty 
\end{align*}
\]

where \( y \) and \( u \) are the minimum and maximum values of process output and input, respectively.

From now on, we assume, without loss of generality, that \( y = -\bar{y} \) and \( u = -\bar{u} \). Besides, we denote:
\[
\alpha_{\text{IO}} = \frac{\bar{y}}{\bar{u}} \tag{1}
\]
A common practical value is \( \alpha_{\text{IO}} = 1 \).
Consider a SISO LTI physical system, represented by its transfer function \( H(s) \) such that \( Y(s) = H(s)U(s) \). We consider systems whose static gain is bounded: \( H(0) = G_0 < \infty \). Under this condition, in order that the whole operating domain of \( y \) can be reached with allowable values of \( u \), we must have:

\[
\bar{y} \leq G_0 \pi
\]  

(2)

Note that this static gain \( G_0 \) not only depends on process properties, but also on the entire acquisition chain that consists of ADC, DAC, sensor and actuator signal conditioning. From (1) and (2), we have:

\[
\alpha_{IO} \leq G_0 \quad \text{or} \quad G_0/\alpha_{IO} \geq 1
\]  

(3)

The ratio \( \frac{G_0}{\alpha_{IO}} \) is adjusted by the gains of acquisition chain in order to satisfy the trade-off between dynamical capacity and static precision. Indeed, for \( \frac{G_0}{\alpha_{IO}} \approx 1 \), the whole operating range of \( y \) can be obtained while \( u \) fully sweeps its own range. Thus, the closed-loop dynamic is close to the one of the process. To obtain closed-loop dynamics faster than the one in open loop, the ratio \( \frac{G_0}{\alpha_{IO}} \) must be much greater than 1. But a large value may decrease control precision. For instance, with \( \alpha_{IO} = 1 \) (commonly used in practice) and \( \frac{G_0}{\alpha_{IO}} = 100 \) (the maximum output value \( \bar{y} \) is obtained with a value of only 1% of control maximum amplitude \( \pi \)), an extremely precise control and a very high noise insensitivity are required. Generally, in practice the ratio \( \frac{G_0}{\alpha_{IO}} \in [1; 10] \).

2.2 Measurement Noise

The magnitude of measurement noise is an important parameter that limits bandwidth, and hence closed-loop dynamics. Sensor noise lies in high frequencies. Let denote \( \omega_N > 0 \), the pulsation from which the effect of noise on control signal is considered. This pulsation is usually greater than the gain-crossover pulsation of open-loop system (\( L(s) = H(s)C(s) \), with \( C(s) \) the controller). Let assume that:

\[
\omega_N \geq \omega_{gc} = \sup_\omega |L(j\omega)| = 1
\]  

(4)

This means that \( \omega_N \) is higher than the greatest pulsation at which the loop transfer function is equal to one.

An important information, before controller synthesis, is the RMS (root mean square) value of measurement noise \( P_N \|_{2, \omega_N} \). We define parameter \( \beta_P \) as:

\[
\|P_N\|_{2, \omega_N} := \beta_P \bar{y}
\]  

(5)

Typically, \( \beta_P \in [0.01; 0.1] \), i.e. sensor noise varies from 1% to 10% of the maximum of measured signal.

Another significant feature for control implementation is the maximum allowed noise level on control signal \( \|U_N\|_{2, \omega_N} \). We define parameter \( \beta_u \) as:

\[
\|U_N\|_{2, \omega_N} := \beta_u \bar{y}
\]  

(6)

We can also define \( \|U_N\|_{2, \omega_N} \) as a function of the control value at which the output reaches its maximum, \( u_0 \), 

\[
\|U_N\|_{2, \omega_N} := \beta_u u_0 |y = \bar{y}|
\]  

(7)

where

\[
u_0 |y = \bar{y} := \frac{\bar{y}}{G_0}\]

(8)

From (1) and (8), it comes:

\[
\frac{\bar{y}}{G_0} = \frac{\alpha_{IO}}{G_0} \pi
\]

(9)

From (7) and (9), we have:

\[
\|U_N\|_{2, \omega_N} \leq \beta_u \frac{\alpha_{IO} \pi}{G_0}
\]

(10)

From (6) and (10), we obtain:

\[
\beta_u = \frac{\alpha_{IO} \pi}{G_0}
\]

(11)

Equation (11) shows that the parameter \( \beta_u \), given by design specifications, is obviously related to the ratio \( \frac{G_0}{\alpha_{IO}} \). Therefore, while establishing control specifications, the control designer must take into account hardware configuration. For example, if one accepts a noise level of \( 10\% \) on control signal \( (\beta_u = 0.1) \) with a ratio \( \frac{G_0}{\alpha_{IO}} = 100 \), this leads to \( \beta_u = 0.001 \) which is extremely difficult to be obtained in practice.

2.3 Controller Gain Limited by Measurement Noise

We will demonstrate that the sensor noise limits the high frequencies gain of controller.

\textit{Lemma 1.}

Consider closed-loop system presented in Fig. 1 with system transfer function \( H(s) \) such that its static gain \( |H(0)| = G_0 < \infty \). Let a controller \( C(s) \) such that the module margin \( M_m = \min_{\omega \geq 0} |1 + H(j\omega)C(j\omega)| \geq M_{md} \), with \( M_{md} \) desired module margin. The noise level on control signal \( \|U_N\|_{2, \omega_N} \) will be less than \( \|U_N\|_{2, \omega_N} \) if \( |C(s)|_{\omega_N} \leq M_{md} \frac{\beta_u}{\beta_y G_0} \), with \( \omega_N \), \( \beta_y \) and \( \beta_u \) defined by (4), (5) and (7).

\textbf{Proof.} See Appendix 6.1.

\textbf{Example:} Consider two experimental cases:

\( \beta_u = 0.1 \), \( \beta_y = 0.05 \), \( M_{md} = 0.5 \) with first \( G_0 = 1 \) and then \( G_0 = 10 \).

\( \beta_y = 0.05 \) means that the RMS value of sensor noise is 5% of the maximum output, \( \beta_u = 0.1 \) means that we tolerate a noise level of 10% on control signal. With these parameters and from Lemma 1, the controller gain at high frequencies is bounded by \( |C(s)|_{\omega_N} \leq 1 \) with \( G_0 = 1 \), and by \( |C(s)|_{\omega_N} \leq 0.1 \) with \( G_0 = 10 \).

We see the importance of the constraint on control signal due to measurement noise. Indeed, in the case of \( G_0 = 1 \), if we synthesized a controller whose high frequencies gain \( |C(s)|_{\omega_N} = 10 \), the level noise on control signal will be 100% of the static value of \( u \). This noise level will be unacceptable in practice. As a consequence, the sensor noise amplitude limits crucially the bandwidth that can be obtained in closed loop. Therefore, it is sufficient to satisfy the specifications of control-oriented modelling or controller synthesis on a narrow frequency band, but not on the whole frequency band. In Section 3, we will demonstrate this feature in the case of modelling for robust control.

2.4 Effect of Measurement Noise and Technological Constraints on Module Margin

The next proposition will show the effect of sensor noise and technological constraints on module margin.
Proposition 1.
Consider the closed-loop system of Fig. 1 with system transfer function $H(s)$ such that its static gain $|H(0)| = G_0 < \infty$. Let pulsation $\omega_0$ be such that:
$$\frac{G_0\omega_0}{\omega} \geq |H(j\omega)|, \quad \forall \omega \geq 0$$
(12)
Let the parameters $\beta_y$ and $\beta_{mr}$ defined by (5) and (7), if there exists $\omega_1 > 0$ such that:
$$\omega_1 \geq \frac{\beta_{mr}}{\beta_y} \frac{M_{md}}{1 - M_{md}} \omega_0$$
(13)
and a controller $C(s)$ such that the module margin restricted on $\omega \leq \omega_1$ is $M_{md} = \min \{ |1 + H(j\omega)C(j\omega)| \geq M_{md}, 1 > M_{md} > 0 \}$ the desired module margin, then the module margin for any $\omega \geq 0$ satisfies:
$$M_m = \min_{\omega \geq 0} \{ |1 + H(j\omega)C(j\omega)| \geq M_{md}$$
(14)

Proof. See Appendix 6.2

The Proposition 1 shows that the module margin obtained on a limited frequency band $[0; \omega_1]$ can be extended on whole frequency band. In other words, we only need to synthesize a controller guaranteeing the module margin on a limited frequency band $[0; \omega_1]$ with $\omega_1$ given by (13), and hence the global module margin over all frequencies will be also guaranteed.

Example: Consider a classical practical case:
$$\beta_{mr} = 0.1, \beta_y = 0.05, M_{md} = 0.5$$
We have $\omega_1 \geq \frac{0.05}{0.1} \frac{0.5}{1 - 0.5} \omega_0 = 2\omega_0$, with $\omega_0$ satisfying (12). It is necessary to guarantee the module margin only on $[0; 2\omega_0]$, but not on $\omega > 0$.

3. MODELLING OF SYSTEM UNDER TECHNOLOGICAL CONSTRAINTS

Due to accuracy requirements of complex engineering, high-order models are often required to represent real systems. The controller implementation for high-order systems may be a very expensive computation task. Indeed, the advanced control design methods, such as, LQG (Doyle [1978]), $H_\infty$/$H_2$ (Hindi et al. [1998]), or convex synthesis (Boyd and Barrat [1991]) usually yield controllers with order comparable to system order. In practice, low-order controllers are often required for real-time implementation. To deal with this problem, several order reduction techniques may be used (Obinata and Anderson [2001]). However, these approaches deal only with the properties of system model without taking into account the controller synthesis. Within the context of controller design, it is desirable to guarantee the closed-loop properties for real system. Frequency-weighted reduction techniques can be used to preserve some closed-loop properties (Anderson and Y. [1989]). But with uncertain systems, for example with parameters in a given domain, the choice of the model used to design the controller, on which the model reduction is done, is not an easy task.

In this section, we present a modelling method, for robust control design, with the guarantee of closed-loop robustness margins (see our previous work (Le and Mendes [2009])). The proposed method is expanded by eliminating some conservative assumptions such that it can deal with more general cases. Moreover, by taking into account the practicals constraints mentioned in Section 2, we demonstrate that it is sufficient to consider the real system on a finite frequency range, but not on all frequencies.

3.1 Problem Statement

Consider a stable SISO LTI process characterized by a transfer function $H(s)$, which may be uncertain, or by a Bode diagram, that satisfies the property:
$$\forall \omega \geq 0 : \frac{\partial \arg H(j\omega)}{\partial \omega} < 0$$
(15)

The modelling problem, for robust control design, is to determine a model $H_r(s)$ such that, with a controller $C(s)$ which is designed based on $H_r(s)$, we have at least the desired robustness margins with $H(s)$.

3.2 Some Results of Modelling for Robust Control

By using the results of Section 2, we present some results that are the basis of the modelling approach proposed in this paper. This method can guarantee both of phase margin and module margin.

Modelling with Phase Margin Guarantee

Proposition 2.
Consider the closed-loop system presented in Fig. 1 with system transfer function $H(s)$ satisfying (15), (12), and with static gain $|H(0)| = G_0 < \infty$. Let another transfer function $H_r(s)$ satisfying (15) and the following properties:
$$\forall \omega \in [0, \omega_1] : \begin{cases} P1: |H_r(j\omega)| \geq |H(j\omega)| \\ P2: \arg H_r(j\omega) \leq \arg H(j\omega) \end{cases}$$
(16)
with $\omega_1$ defined in (13).

Let a controller $C(s)$, designed based on $H_r(s)$, satisfying:
$$\forall \omega \in [0, \omega_1] : \begin{cases} C1: \frac{\partial |L_r(j\omega)|}{\partial \omega} < 0 \\ C2: \frac{\partial \arg L_r(j\omega)}{\partial \omega} < 0 \end{cases}$$
(17)

and $P_{mr} \geq \varphi_d$ where $P_{mr}$ is the phase margin obtained with $L_r(s) = H_r(s)C(s)$ and $\varphi_d$ is the desired phase margin.

Then the phase margin $P_m$ obtained with $L(s) = H(s)C(s)$ will satisfy:
$$P_m \geq P_{mr} \geq \varphi_d$$
(18)

Proof. See Appendix 6.3

Modelling with Module Margin Guarantee

Theorem 1.
Consider the closed-loop system presented in Fig. 1 with system transfer function $H(s)$ satisfying (15), (12), and with static gain $|H(0)| = G_0 < \infty$. Let another transfer function $H_r(s)$ satisfying (15) and (16).

Let a controller $C(s)$, designed based on $H_r(s)$, satisfying (17) and
$$M_{mr} = \min_{\omega \in [0, \omega_1]} \{ |1 + L_r(j\omega)| \geq M_{md} \}$$
where $M_{mr}$ is the module margin obtained with $L_r(s) = H_r(s)C(s)$ and $M_{md}$ is the desired module margin.
Then the global module margin obtained with \( L(s) = H(s)C(s) \) will satisfy:

\[ M_m \geq M_{nd}, \quad \forall \omega \geq 0 \]  

(19)

**Proof.** See Appendix 6.4

In summary, the Theorem 1 and the Proposition 2 demonstrate that under certain conditions, a controller, synthesized based on a suited model, guarantees the module margin and the phase margin for controlled system. Moreover, by taking into account the practical constraints, the model, used for controller design, needs only to consider the controlled system on a finite frequency band \([0; \omega_1]\). Therefore, this model may be of low order to deal with design problems where low-order controllers are required. Whatever controller, \( \omega_1 \) is determined by four factors: \( \beta_y, \beta_u \) (depending on hardware configuration), \( M_{nd} \) desired module margin, and \( \omega_{0r} \) (depending on system).

### 3.3 Modelling Method

The method consists in determining a model \( H_c \) such that the proposition 2 and the Theorem 1 are satisfied and the following criterion is minimized:

\[
J = \alpha_1 \Delta G_{\infty} + \alpha_2 \Delta G_{2} + \alpha_3 \Delta \varphi_{\infty}
\]  

(20)

where \( \theta \) is the parameter vector to be determined of \( H_c \); \( \omega_1 \) is defined in (13); \( \alpha_i = 1, 2, 3 \geq 0 \) are the weighting coefficients, and:

\[
\Delta G_{\infty} = \| H - H_c \|_{\infty}, \quad \Delta G_{2} = \int_{\omega_{min}}^{\omega_1} (|H| - |H_c|)^2 \, d\omega, \quad \Delta \varphi_{\infty} = |\arg H - \arg H_c|_{\infty}.
\]

The first term \( \Delta G_{\infty} \) represents the maximum gain distance between \( H \) and \( H_c \). The second term \( \Delta G_{2} \), which is the RMS value of gain distance, represents the average gain distance in the operating frequency band \([\omega_{min}; \omega_1]\). The third term \( \Delta \varphi_{\infty} \) represents the maximum phase distance between \( H \) and \( H_c \). Fig. 2 represents the magnitude Bode diagrams of system model (dashed curve) and of two possible \( H_c \) models: \( H_{1r} \) (solid curve) and \( H_{2r} \) (solid bold curve). \( H_{1r} \) presents bigger average gain distance \( \Delta G_{2} \), while \( H_{2r} \) presents bigger maximum gain distance \( \Delta G_{\infty} \). It is therefore desirable to consider these two criteria for choosing model \( H_c \).

The weighting coefficients can be chosen as: \( \alpha_1 = 1/G \), \( \alpha_2 = 1/\int_{\omega_{min}}^{\omega_1} |H|^2 \, d\omega \), \( \alpha_3 = 1/|\arg H(\omega_1)| \); with \( G \) the static gain of \( H \) if \( H \) does not have zero pole, otherwise \( \alpha_1 = 1 \). This choice of weighting coefficients permits normalizing the different distances in the criterion (20).

### 3.4 Determination of model \( H_c \)

Many models are possible candidate for model \( H_c \). For example, we can choose the following one:

\[
H_c(s) = \frac{G_c}{1 + \frac{s}{\omega_0} + (s + \omega_d)}
\]  

(21)

It is easy to verify that model (21) satisfies the property (15). The three parameters \( \theta = [G_c, \omega_{0r}, \omega_d]^T \) are determined by solving the optimization problem (20). The structure of (21) presents some advantages. Indeed, the first part is a first-order system helping to satisfy P1 in (16); whereas, the second part, with unity gain, is used to satisfy P2 in (16). The main advantage of the choice of this form is that the parameters of \( H_c \) could be determined independently one after another, and hence the optimization problem (20) becomes simpler:

\[
\begin{align*}
(G_c)_{\text{opt}} &= \max_{\omega \in [0, \omega_1]} |H(j\omega)| \\
(\omega_{0r})_{\text{opt}} &= \arg \min_{H_c(\omega) \geq |H(\omega)|, \forall \omega \in [0, \omega_1]} |\omega_{0r}| \\
(\omega_d)_{\text{opt}} &= \arg \max_{H_c(\omega) \leq |H(\omega)|, \forall \omega \in [0, \omega_1]} |\omega_d|
\end{align*}
\]  

(22)

### 4. EXPERIMENTAL APPLICATION - TEMPERATURE PROCESS

#### 4.1 System Configuration

The process control trainer is presented in Fig. 3. The operating principle is the following: a heating resistor, controlled by voltage (0-10 V), generates heat which is transferred in the tube than to a fan ventilator. A thermistor probe sends a signal (0-10 V) which gives the temperature in certain point of the tube. A signal presents a noise of \( \|P_N\|_{\omega_N} \approx 0.3 \text{V} \). Using the information provided by the sensor, a voltage control of heating resistor can be made such that the temperature in the tube is raised to a desired value. Besides, the controller must also be capable of rejecting disturbances for examples: speed variation of fan ventilator or partial obstruction at the tube output. By identifying the elements of system (heating resistor, tube and thermistor), the system model at one operating point is:

\[
H(s) = \frac{e^{-0.2s}}{(s/\omega_0 + 1)^2}
\]  

(23)

where \( \omega_0 \in [2; 3] \) is uncertain pole. With this configuration, we have \( G_0 = 1 \), \( \alpha_{IO} = 1 \), \( \beta_y = 0.03 \). Assume that we tolerate a noise level of 10\% at control signal, it means \( \beta_u = 0.1 \).
4.2 Determination of \( H_r \)

We first determine the finite frequency range where the system models will be considered by \( H_r \). By using the system models (23) we can find the pulsation \( \omega_0 \) as defined in (12), \( \omega_0 = 5\text{rad/s} \). Fig. 4 illustrates the determination of \( \omega_0 \) by equation (12). Now we can determine \( \omega_1 \) by equation (13):

\[
\omega_1 = \frac{0.1}{0.03} \frac{0.5}{1 - 0.5} = 5\text{rad/s}
\]

By solving the optimization problem (20), we obtained:

\[
H_r(s) = \frac{1}{1 + \frac{s + 3.46}{1.43}} \left( -s + 3.46 \right)
\]

with the optimal criterion \( J = 0.24 \). An illustration of the properties (16) is presented in Fig. 5.

4.3 Controller Synthesis based on \( H_r \)

Consider the Fig. 1, to validate the model (25) in the context of closed-loop behaviours, we synthesize a PID controller:

\[
C(s) = K\left(1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{a_d} s}\right)
\]

that satisfies the specifications: optimal rejection of step load disturbance \( d_l \); desired module margin \( M_{rd} = 0.5 \) and desired phase margin \( \varphi_d = 45^\circ \). This controller is synthesized by using the method proposed in (Le and Mendes [2009]). The calculation results are shown in Table 1. As we can see the measurement noise limits controller gain in high frequency as demonstrated in Lemma 1:

\[
|C(s)|_{\omega \geq \omega_N} \leq M_{rd} \frac{\beta_{u0}}{\beta_y} \frac{1}{C_0} \left( \frac{0.1}{0.03} \frac{1}{1.7} \right)
\]

Here, the gain of PID controller in high frequency is \( K(1 + a_d) = 1.6 \leq 1.7 \).

Table 1. Closed-loop properties for \( H_r \) and system models

<table>
<thead>
<tr>
<th>Process</th>
<th>( K )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( a_d )</th>
<th>( M_m )</th>
<th>( P_m )</th>
<th>( \beta_{u0} )</th>
<th>( \beta_{u0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_r ) ((\omega_0 = 2\text{rad/s}))</td>
<td>1.43</td>
<td>0.811</td>
<td>0.147</td>
<td>0.1</td>
<td>0.55</td>
<td>52</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>( H_r ) ((\omega_0 = 3\text{rad/s}))</td>
<td>0.51</td>
<td>37</td>
<td>0.395</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 4. Determination of the pulsation \( \omega_0 \)

Fig. 5. Bode diagrams for \( H_r \) (solid) and system models with \( \omega_0 \) = 2rad/s (dash-dotted) and \( \omega_0 \) = 3rad/s (dash)

Fig. 6. Step responses to reference and input disturbance

Experimental Validation

Now the controller based on \( H_r \) is experimentally implemented. Fig. 6 presents the step response to reference and load disturbance, obtained for real system. With design specification \( \beta_{u0} = 0.1 \) we ensure that the noise level on control signal is smaller than: \( \|U_N\|_{\omega_{max} > \omega_{\text{N}}} := \beta_{u0} = \frac{2\pi}{T_{FN}} \leq 0.1410 = 1V \). In practice, we obtained the noise level on control signal 0.5V.

To validate our approach in practice, it is necessary to estimate the real phase and module margins. Hence, we use two experiments based on relay method (Garcia et al. [2007]) to estimate real robustness margins of closed-loop system. The estimated phase and module margin are 58° and 0.67, respectively, that are close to those of controlled system.

5. CONCLUSION

We presented a formalization of the practical constraints of a real control system, e.g. sensor noise, different elements of acquisition chain. We have shown that these constraints limit the achievable bandwidth in closed loop. By taking into account the measurement noise and the technological constraints, we presented a modelling method that consists in determining a model for controller synthesis. This approach has the following properties: first, a controller designed based on this model, ensures desired robustness margins for real system, second, the frequency interval on which we consider the real system is finite. The proposed method is illustrated via an experimental system, thermal process.

6. APPENDIX

6.1 Proof of Lemma 1

Consider the Fig. 1, the transfer function from sensor noise to control variable is:

\[
\frac{U}{F_N} = -\frac{C(s)}{1 + H(s)C(s)}
\]

(28)

\( C(s) \) is determined such that the desired module margin is respected:

\[
\frac{1}{\|1 + H(s)C(s)\|_{\infty}} = \frac{1}{M_m} \leq \frac{1}{M_{rd}}
\]

(29)
From (28) and (29), it comes:
\[ \| U_N \|_{2,\omega \geq \omega_N} \leq \frac{\| C(s) \|_{2,\omega \geq \omega_N}}{M_{md}} \tag{30} \]

Besides, from (1), (5), (6) and (11) we have:
\[ \| U_N \|_{2,\omega \geq \omega_N} \leq \frac{\beta_{ua} 1}{\beta_y G_0} \tag{31} \]

Hence, if \( |C(s)|_{\omega \geq \omega_N} \leq M_{md} \frac{\beta_{ua}}{\beta_y G_0} \), from (30), we obtain:
\[ \| U_N \|_{2,\omega \geq \omega_N} \leq \frac{\beta_{ua} 1}{\beta_y G_0} \tag{32} \]

Finally, from (31) and (32), we obtain:
\[ \| U_N \|_{2,\omega \geq \omega_N} \leq \| U_N \|_{2,\omega \geq \omega_N} \leq \| U_N \|_{2,\omega \geq \omega_N}. \]

6.2 Proof of Proposition 1

If \( |H(j\omega)C(j\omega)|_{\omega \geq \omega_1} \leq 1 - M_{md} \) then \( M_{md} |\omega_1| \geq M_{md} \). To demonstrate this fact, trace in Nyquist plan, robustness circle with radius \( M_{md} \) and center at \((-1,0)\). Trace another circle with radius \( 1 - M_{md} \) and center at origin. Hence, if \( |H(j\omega)C(j\omega)|_{\omega \geq \omega_1} \leq 1 - M_{md} \), it means that the image curve of loop transfer function \( H(j\omega)C(j\omega) \) rests inside the second circle and not cuts the robustness circle, then \( M_{md} |\omega_1| \geq M_{md} \).

Apply Lemma 1 with \( \omega_N = \omega_1 \), we obtain:
\[ |C(j\omega)|_{\omega \geq \omega_1} \leq M_{md} \frac{\beta_{ua} 1}{\beta_y G_0} \tag{33} \]

From (12) and (33), we have:
\[ |H(j\omega)|_{\omega \geq \omega_1}|C(j\omega)|_{\omega \geq \omega_1} \leq M_{md} \frac{\beta_{ua} \omega}{\beta_y \omega_1} \tag{34} \]

Recall Cauchy-Schwarz inequality:
\[ |H(j\omega)C(j\omega)|_{\omega \geq \omega_1} \leq |H(j\omega)|_{\omega \geq \omega_1}|C(j\omega)|_{\omega \geq \omega_1} \tag{35} \]

With (34) and (35), we obtain:
\[ |H(j\omega)C(j\omega)|_{\omega \geq \omega_1} \leq M_{md} \frac{\beta_{ua} \omega}{\beta_y \omega_1} \tag{36} \]

If \( \omega_1 \geq \frac{\beta_{ua} M_{md} \omega}{\beta_y} \), \( \omega_1 \leq \frac{1}{1 - \frac{\beta_{ua} M_{md} \omega}{\beta_y}} \leq 1 - M_{md} \)
\[ \omega_1 \leq \frac{\beta_{ua} M_{md} \omega}{\beta_y} \tag{37} \]

From (36) and (37) we obtain:
\[ |H(j\omega)C(j\omega)|_{\omega \geq \omega_1} \leq 1 - M_{md} \]

As demonstrated above, the Nyquist images of \( I(\omega) \) and \( L_{cr}(\omega) \) images inside the robustness circle, \( M_{md} \). Finally, we have:
\[ M_{md} = \min \{ 1 + |H(j\omega)C(j\omega)|_{\omega \geq \omega_1} \} \geq M_{md}. \]

6.3 Proof of Proposition 2

The phase margin of \( L_{cr}(\omega) = H_{cr}(\omega)C(\omega) \) is:
\[ P_{mar} = \pi + \arg L_{cr}(j\omega_{ger}), \quad |L_{cr}(j\omega_{ger})| = 1 \]

The phase margin of \( L(\omega) = H(\omega)C(\omega) \) is:
\[ P_{m} = \pi + \arg L(j\omega_{ger}), \quad |L(j\omega_{ger})| = 1 \]

We will first demonstrate that \( \omega_{ger} \leq \omega_{ger} \). For that, suppose that \( \omega_{ger} > \omega_{ger} \), because both \( \omega_{ger} \) and \( \omega_{ger} \) are less than \( \omega_1 \), we can use C1 of (17) to deduce:
\[ |L_{cr}(j\omega_{ger})| < |L_{cr}(j\omega_{ger})| \tag{38} \]

With P1 of (16), we have:
\[ |L_{cr}(j\omega)| \geq |L(j\omega)|, \forall \omega \in [0; \omega_1] \tag{39} \]

With (38) and (39), we obtain: \( 1 = |L(j\omega_{ger})| < |L_{cr}(j\omega_{ger})| = 1 \). Hence, there is a contradiction. Then, we find:
\[ \omega_{ger} \leq \omega_{ger} \tag{40} \]

With P2 of (16), we have:
\[ \arg L_{cr}(j\omega) \leq \arg L(j\omega), \forall \omega \in [0; \omega_1] \tag{41} \]

From C2 of (17) and (40), we have \( \arg L_{cr}(j\omega_{ger}) \leq \arg L_{cr}(j\omega_{ger}) \), besides with (41) we have \( \arg L_{cr}(j\omega_{ger}) \leq \arg L(j\omega_{ger}) \). Finally, we find \( \arg L_{cr}(j\omega_{ger}) \leq \arg L(j\omega_{ger}) \), then \( P_{m} \geq P_{mar} \geq \frac{\pi}{2} \).

REFERENCES


