Control of Distributed Stochastic Systems
– Introduction, Problems, and Approaches

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Abstract: The tutorial paper presents problems and approaches of control of distributed Gaussian systems. A distributed system consists of the interconnection of two or more subsystems. In distributed control of such systems there is no communication between controllers at all and the control synthesis is mostly via the determination of a person-by-person equilibrium. In distributed control with communication the controllers can exchange communication with other subsystems but this is done only when needed and with a restricted subset of the other subsystems. In coordination control a coordinator sets references signals for two or more subsystems. Finally control of hierarchical Gaussian systems is discussed.

Keywords: Distributed control and estimation, Stochastic control, Coordination of multiple vehicle systems (IFAC list): distributed control, decentralized control; coordination control, control with communication, hierarchical control.

1. INTRODUCTION

The purpose of this paper is to introduce to the readers the problem of control of distributed stochastic systems, to provide an overview of the existing results, and to indicate approaches for future research. The paper is an introduction to the session Stochastic control and games. The character of the paper is tutorial, it is not an exhaustive survey. The space limitations do not allow many references to be mentioned.

The motivation of the problem comes from control engineering. A distributed system is an interconnection of two or more subsystems. Most large-scale engineering systems consist of such interconnections. Examples are platoons of underwater or aerial vehicles, automated guided vehicles on a container terminal, etc.

Control of a distributed system is often structured by two or more controllers, one for each subsystem, and based on local observations. The control objectives can sometimes be met by control of each subsystem separately but often this is not the case. Then a form of coordination control is required.

The main problem of the paper is control of distributed systems, in particular coordination control. A discrete-time formulation will be used. Successively the framework, the problem, and available results will be discussed for the control architectures of distributed control, coordination control, control with communication, and hierarchical control. The main control issues are: (1) What is the state of the controller? (2) How to determine a minimal tuple of control laws? (3) Is a person-by-person equilibrium a minimal tuple of control laws? (4) How to compute a person-by-person equilibrium?

A description of the contents of the paper follows. The next section provides a problem formulation. Control theoretic concepts are provided in Section 3. Distributed control and team theory are discussed in the Sections 4 and 5 respectively. Section 6 deals with control with communication while Section 7 treats coordination control. Section 8 provides a framework of hierarchical Gaussian systems. The last section presents concluding remarks.

2. PROBLEMS

The problem discussed in this paper is: How to control a distributed stochastic system?

The notation of the paper is fairly simple. The integers are denoted by Z and the positive integers by Z+. For n ∈ Z+ denote the vector space of n tuples of real numbers by R^n. The set of positive-definite matrices with entries in the real numbers of size n × n is denoted by R^n×n^spd. A probability space (Ω, F, P) is a triple consisting of a set Ω, a σ-algebra F, and a probability measure P : F → [0, 1]. A Gaussian or normal distribution with parameters m ∈ R^n and a variance matrix Q ∈ R^n×n^spd is a probability distribution which has a characteristic function described by the formula,

\exp(iu^T m - 1/2u^T Q u), \ \forall u \in R^n.

A random variable x : Ω → R^n is said to be Gaussian if it has a Gaussian probability distribution and this will be denoted by x ∈ G(m, Q).

In this paper time is modeled as discrete and the notation for the time index set is T = {t_0, t_0 + 1, t_0 + 2, ..., t_1} ⊂ Z. A Gaussian white noise process is a stochastic process v : Ω × T → R^{nv} such that the random variables {v(t), t ∈ T} are independent and for every t ∈ T, v(t) ∈ G(0, Q_v).

Definition 2.1. A Gaussian system is a stochastic dynamic system with the state-space representation,
\[ x(t+1) = Ax(t) + Bu(t) + Mv(t), \quad x(t_0) = x_0, \quad (1) \]
\[ y(t) = Cx(t) + Du(t) + Nv(t), \quad (2) \]
\[
n, m, p \in \mathbb{Z}^+, \quad v : \Omega \times T \to \mathbb{R}^{m_v}, \quad \]
\[
a \text{ Gaussian white noise process,} \quad u : \Omega \times T \to \mathbb{R}^n, \quad \]
\[
denotes the input process, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad M \in \mathbb{R}^{n \times m_v}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m_v}, \quad \]
\[ x : \Omega \times T \to \mathbb{R}^n, \quad y : \Omega \times T \to \mathbb{R}^p. \]

Here \( x \) is called the state process and \( y \) the output process. To save space in this paper, the initial conditions will mostly be omitted from the state recursions. The system representation contains the special case in which the system noise and the output noise are independent.

A distributed stochastic system consists of the interconnection of two or more subsystems. In this paper attention is often restricted to two subsystems. Each subsystem has an input from the environment and an output to the environment in addition to links with the other subsystems.

**Definition 2.2.** A distributed Gaussian system is a Gaussian system with the state-space representation,

\[
x_1(t+1) = A_{11}x_1(t) + B_{11}u_1(t) + B_{12}u_2(t) + M_1v(t), \quad (3) \]
\[
y_1(t) = C_1x_1(t) + D_{11}u_1(t) + N_1v(t), \quad (4) \]
\[
u_{21}(t) = C_{21}x_1(t) + D_{21}u_1(t) + N_{21}v(t), \quad (5) \]
\[
x_2(t+1) = A_{22}x_2(t) + B_{22}u_2(t) + B_{21}u_1(t) + M_2v(t), \quad (6) \]
\[
y_2(t) = C_2x_2(t) + D_{22}u_2(t) + N_2v(t), \quad (7) \]
\[
u_{12}(t) = C_{12}x_2(t) + D_{12}u_2(t) + N_{12}v(t). \quad (8) \]

The state of Subsystem 1 is represented by \( x_1 \), the input from the outside to that system by \( u_1 \), the input of Subsystem 1 provided by Subsystem 2 is \( u_{21} \), the output to the outside by \( y_1 \), the output from that system to the other subsystem by \( u_{21} \), etc. The system representation covers the case in which the different noise components are independent.

The interconnected system is then represented by

\[
x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T, \]
\[
x(t+1) = Ax(t) + Bu(t) + Mv(t), \quad (9) \]
\[
y_1(t) = C_1x_1(t) + D_{11}u_1(t) + N_1v(t), \quad (10) \]
\[
y_2(t) = C_2x_2(t) + D_{22}u_2(t) + N_2v(t), \quad (11) \]
\[
= \begin{pmatrix} A_{11} & B_{12}C_{12} \\ B_{21}C_{12} & A_{22} \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_{11} \\ B_{12} \end{pmatrix}, \quad B_2 = \begin{pmatrix} B_{21} \\ B_{22} \end{pmatrix}, \quad M = \begin{pmatrix} M_1 + B_{12}N_{12} \\ M_2 + B_{21}N_{12} \end{pmatrix}. \quad (12) \]

The above listed system representation is also called a decentralized Gaussian system. Conversely, a monolithic system with two or more inputs and two or more outputs may be decomposed into the interconnection of two or more subsystems. The interconnection inputs \( u_{i,j} \) and \( u_{j,i} \) of a distributed system are not available for control.

The control laws are defined formally in Section 3. The closed-loop system is the combination of the control system and the control law. A control objective is a property a control engineer strives to obtain in the closed-loop system. General control objectives of control of distributed systems concern stability, noise reduction, performance optimization, and robustness. For stochastic systems noise reduction is a primary control objective.

**Problem 2.3.** The problem of control of distributed Gaussian systems is to construct a tuple of control laws, one for each controller, such that the closed-loop system meets the control objectives.

The following control architectures for Problem 2.3 will be discussed in the remainder of the paper.

1. **Distributed control**, where the expression distributed refers to the property that the two controllers do not communicate with each other online.
2. **Distributed control with communication**, where any controller may communicate part of its online observations or state estimate to other controllers.
3. **Coordination control**, where the subsystems are distinguished into a coordinator and the remaining subsystems.
4. **Hierarchical control**, in which the subsystems are distinguished into layers and in which a subsystem at any one layer is related to one or more subsystems at the next lower layer.

### 3. CONTROL THEORETIC CONCEPTS

**Definition 3.1.** Consider a distributed Gaussian system. A tuple of control laws, one for each of the controller, is a tuple of functions defined by

\[ G_1 \times G_2 = \{ (g_1, g_2) \text{ satisfying conditions below} \}, \]
\[
g = (g_1, g_2) \in G_1 \times G_2, \quad Y_1 = \mathbb{R}^{p_1}, \quad U_1 = \mathbb{R}^{m_1}, \]
\[
g_i(t, \cdot) : Y_{i,\{t_0, t-1\}} \times U_{i,\{t_0, t-1\}} \to U_i, \]
\[
y_i(t, y_i, y_{i,\{t_0, t-1\}}, u_i, u_{i,\{t_0, t-1\}}), \]
\[
y_{i,\{t_0, t-1\}} = (y(t_0), y(t_0 + 1), \ldots, y(t)), \]
\[
i \in Y_{i,\{t_0, t\}}, \quad \forall \ t \in T, \ i = 1, 2, \]
\[
g_i(t, \ldots), \text{ is a measurable function.} \]

Note that the tuple of control laws defined above is such that its memory grows with time. For practical implementation attention has to be restricted to a tuple of control laws with a finite or finite-dimensional memory.

The closed-loop system consisting of the distributed Gaussian system and the tuple of control laws then has the representation,

\[ x^g(t+1) = A^g(t) + B_1g_1(t, y_1, \{t_0, t-1\}, u_1, \{t_0, t-1\}) + \]
\[ + B_2g_2(t, y_2, \{t_0, t-1\}, u_2, \{t_0, t-1\}) + M^g(t), \quad (14) \]
\[
y_1^g(t) = C_1x^g(t) + D_1g_1(t, y_1, \{t_0, t-1\}, u_1, \{t_0, t-1\}) + \]
\[ + N_1v(t), \quad (15) \]
\[
y_2^g(t) = C_2x^g(t) + D_2g_2(t, y_2, \{t_0, t-1\}, u_2, \{t_0, t-1\}) + \]
\[ + N_2v(t), \quad (16) \]
\[
u_1^g(t) = g_1(t, y_1, \{t_0, t-1\}, u_1, \{t_0, t-1\}), \quad (17) \]
\[
u_2^g(t) = g_2(t, y_2, \{t_0, t-1\}, u_2, \{t_0, t-1\}). \quad (18) \]
The superindex $g$ of the control laws is used to remind the reader of the fact that the processes and hence the probability distribution of those processes depend on the tuple $g$ of control laws used.

Denote the cost function by

$$J : G_1 \times G_2 \rightarrow \mathbb{R},$$

$$J(g_1, g_2) = E \left[ \sum_{s,t=0}^{t-1} b(s, x_0(s)) g_1(s, y_1(t, s, x_1(s)), u_1(t, s, x_1(s))) + g_2(s, y_2(t, s, x_2(s)), u_2(t, s, x_2(s))) + b_1(x^{g_1}(t_1)) \right].$$

(19)

The cost function may impose interaction between the subsystems.

The following control research issue is relevant. What is the state of a controller of distributed control system? A naive thought is to think that the state of a controller consists of the estimation of the state of the plant only. But note that the state of the plant has as input also the input of the other controller. Thus any controller also has to estimate the state of the other controller.

4. DISTRIBUTED CONTROL

Consider the following distributed Gaussian system with two inputs and two outputs,

$x(t + 1) = Ax(t) + B_1u_1(t) + B_2u_2(t) + Mv(t), \quad (20)$

$y_1(t) = C_1x(t) + D_1u_1(t) + N_1v(t), \quad (21)$

$y_2(t) = C_2x(t) + D_2u_2(t) + N_2v(t). \quad (22)$

All processes and matrices are defined similarly to those of Definition 2.2. There are two controllers, referred to as Controller 1 and Controller 2. Controller 1 receives the observations $y_1$ and correspondingly Controller 2 receives the observations of $y_2$. Controller 1 produces the input $u_1$ and Controller 2 correspondingly produces $u_2$.

Problem 4.1. The problem of distributed control of a distributed Gaussian system as described above with the above defined cost function is to determine a tuple of control laws which achieves the lowest cost, to be called the minimal tuple if it exists. Thus, find a tuple of optimal control laws $(g_1^*, g_2^*) \in G_1 \times G_2$ such that

$$J(g_1^*, g_2^*) = \inf_{(g_1, g_2) \in G_1 \times G_2} J(g_1, g_2). \quad (23)$$

If there does not exist a minimal tuple of control laws then the problem is to determine a tuple of $\epsilon$-optimal control laws for an $\epsilon \in (0, \infty) \subset \mathbb{R}$.

The author of this paper is not aware of any publication which solves Problem 4.1 directly. The difficulty is both in the optimization over the product space $G_1 \times G_2$ and in the fact that the two controllers have different partial observations of the system.

The approach to handle the problem with a tuple of two or more control laws is to use the approach of game theory and to restrict the search for a minimal tuple to a person-by-person equilibrium, Ho [1980].

Definition 4.2. Consider the distributed control problem 4.1 and the cost function defined above. The tuple of control laws $(g_1^*, g_2^*) \in G_1 \times G_2$ is called a person-by-person equilibrium for this problem if,

$$J(g_1^*, g_2^*) \leq J(g_1, g_2^*) \quad \forall g_1 \in G_1, \quad (24)$$

$$J(g_1^*, g_2^*) \leq J(g_1^*, g_2) \quad \forall g_2 \in G_2. \quad (25)$$

A person-by-person equilibrium is such that none of the two controllers can deviate unilaterally from the equilibrium and gain from doing so. The justification for the restriction to an equilibrium is that if the two controllers cannot communicate with each other during the control synthesis or the operation of the system, then they cannot do better than behave as if they strive for this equilibrium.

A person-by-person equilibrium of distributed control is identical to the concept of a Nash equilibrium of game theory when the game setting is restricted to distributed control.

The control research issues that appear with the equilibrium are: (1) Does such an equilibrium exist? (2) If an equilibrium exists, are there two or more? (3) How to compute an equilibrium? (4) Is such an equilibrium a minimal tuple? A minimal tuple is by definition of minimality a person-by-person equilibrium. In general, a person-by-person equilibrium is not a minimal tuple. Sufficient conditions for the latter implication are of interest to distributed control. A case where this implication holds is that of distributed control of discrete-event systems, see Overkamp and van Schuppen [2000].

The determination of a person-by-person equilibrium for a distributed stochastic system is still not straightforward because the concept of state of the controller has to be defined. Therefore a further restriction is imposed next.

Definition 4.3. Consider the following set of finite-dimensional control laws consisting of the following two stochastic systems and their output equations (producing the inputs to the subsystems),

$$r_1(t + 1) = f_1(r_1(t), g_1(t, r_1(t)), g_2(t, r_2(t)), \quad (26)$$

$$r_2(t + 1) = f_2(r_2(t), g_1(t, r_1(t)), g_2(t, r_2(t)), \quad (27)$$

$$u_1(t) = g_1(t, r_1(t)), \quad u_2(t) = g_2(t, r_2(t)), \quad (28)$$

$$r_1 : \Omega \times T \rightarrow \mathbb{R}^{n_1}, r_2 : \Omega \times T \rightarrow \mathbb{R}^{n_2}. \quad (29)$$

The above definition presumes that Controller 1 formulates an assumption for a class of dynamic control laws for Controller 2 with a finite-dimensional state space. The motivation for this definition is that because of this assumption, the state set of Controller 1 now becomes finite dimensional. This makes a solution via stochastic control with partial observations feasible if attention is restricted to a person-by-person equilibrium with this class of control laws. It is assumed that either controller knows the class from which the other controller has chosen its control law.

The control research issues based on the above definition are then: (1) How to choose the dimensions of the states of the control laws, thus $n_1, n_2 \in \mathbb{Z}^+$? (2) What is the state of Controller 1? (3) How to determine a person-by-person equilibrium within the class of tuples of control laws $G_1 \times G_2$? (4) Is a person-by-person equilibrium also a globally minimal tuple of control laws?
How to check whether a candidate tuple of control laws \((g_1^*, g_2^*)\) satisfies the conditions of a person-by-person equilibrium? If say \(g_2^* \in G_2\) is fixed, then one obtains an optimal stochastic control problem with partial observations for the control law \(g_1 \in G_1\) of which the optimal solution then has to equal \(g_1^*\). The solution procedure for such a control problem is known from optimal control with partial observations, see for example the books Bertsekas [1976], Kumar and Varaiya [1986].

**Determination of an equilibrium.** To determine a tuple of control laws which is an equilibrium one has to solve a coupled set of equations as is well known from the theory of dynamics games. Control research issues are then the existence of a tuple, uniqueness, and the construction of a solution. There are few examples where the control laws can be obtained in analytic form.

It is well known that the solution of a control problem for a distributed stochastic system may exhibit a signaling effect. Signaling takes place if a controller signals part of its private partial observations of the plant to the other controller via its input to the plant and from there to the other controller. This works well in deterministic systems. It also works in case of stochastic systems. The problem of how to signal information is identical to the main problem of information and communication theory, how to communicate information via a noisy channel, where in control of distributed systems the channel is the plant. A book about signaling in an economic context is Spence [1974]. D. Tenekeetzis and co-workers have analyzed signalling in distributed systems using concepts of control, information, and of communication theory, see Tenekeetzis [1997, 2006].

Control research issues for signaling are: (1) To exhibit the signaling effect on actual distributed control problems. (2) The synthesis of communication protocols for the signaling.

### 5. TEAM THEORY

In this section a brief summary is provided of team theory of optimization problems.

J. Marschak and R. Radner in a series of papers introduced a framework that was later referred to as team theory, see Marschak [1955], Radner [1962], Marschak and Radner [1972]. It is a special case of a game problem with two or more players. The time axis has only one point, the decision is made only once. Therefore the problem is referred to as a static problem in distinction with a control problem which refers to a system developing over time. The team problem considers an optimization problem with two or more decision makers. The decision makers have different information about the state of the world, but have the same optimization criterion. Thus they strive to minimize the same criterion but based on different information. This makes the team problem a static case of the problem of distributed control.

The main result of Marschak and Radner is that a tuple of decision laws satisfying a person-by-person equilibrium is also a minimal tuple of decision laws. The conditions for this result are that the team problem is with continuous input spaces and a criterion which is continuous, differentiable, and convex in the two inputs. Thus a minimal tuple is obtained if the decision makers individually optimize their criteria in a parallel fashion. This is a relatively important result for economics.

The main result of Marschak and Radner was extended in papers, see Krainak et al. [1982a,b,c]. In those papers the team problem is considered with an expected cost function of the exponential of a quadratic cost. The papers provide formulations of conditions of integrability and of first-order conditions for optimality. The main result is then that with respect to particular conditions, a person-by-person equilibrium is a minimal element of the team problem. The equilibrium has to be determined by solving a set of coupled equations.

The result on team theory has been partly generalized to the case of finite and discrete input spaces, see de Waal and van Schuppen [2000]. This is of interest to control of queueing systems and to other computer science models.

Control research issue: What form does the signaling in team problems take if this is allowed in a team problem? An exposition of the problems and issues of team theory may be found in the paper Ho [1980].

### 6. DISTRIBUTED CONTROL WITH COMMUNICATION

Communication between controllers is often used in many engineering systems. This communication will allow to meet the control objectives better than without it. A well-known example of control with communication in the discrete domain is the alternating bit protocol for the transmission of packets from a sender to a receiver via a communication channel. The communication between the controllers is then the acknowledgement sent by the receiver to the sender. Examples of distributed control with communication for stochastic systems include the control of two or more underwater vehicles and the backpressure algorithms of communication networks.

A theoretical framework for control of distributed control with communication was proposed and analyzed by H. Witsenhausen, see Witsenhausen [1971, 1975, 1988]. Periodic information sharing of Yoshikawa is also of interest, see Yoshikawa [1975].

**Problem 6.1.** The distributed control with communication problem is then to synthesize communication protocols and distributed control laws such that the closed-loop system meets the control objectives.

The problem of control of distributed systems with communication was also formulated by the author in van Schuppen [2004].

Control research issues of distributed control with communication include: (1) What should be communicated? When should information be communicated? To or from whom should information be delivered or requested respectively? (2) How to integrate the information streams of different sources into one state estimate? (3) What is the effect of the extra observations on the control performance?
The following forms of communication between controllers may be useful, depending on the engineering system considered. The first form is *subset communication*, a controller sends a linear combination of its observations to the other controller. The second form is *state-based observation* in which a communication is sent only if the state of the observer takes values in a particular subset of the state set. A third form of communication is *requesting information*, a controller requests the information if it seems useful for its control task.

An overview of research progress of distributed control with communication follows. What information is to be communicated? In general a subset of the events in control of discrete-event systems or a linear combination of the partial observations of a Gaussian system is sent. A problem of this type is formulated below.

*When should the information be sent?* For a Gaussian system a communication law has been derived in Boel and van Schuppen [2010]. The communication law prescribes to request information when the uncertainty has become too large for the purpose of state estimation. In a particular case, the communication law with periodic communication is the best for the considered cost function.

Control research issues are: (1) When is periodic communication the optimal communication law for the problem? (2) A conjecture is that if the graph of the interconnection of the distributed system is a tree and if the cost function is compatible with this then the optimal control law with respect to an average cost criterion is of nearest neighbor type. The concept of conditional independence will be useful for this. (3) A further conjecture is that, in case the underlying graph has a cycle, the control laws for all subsystems of the cycle have to be determined jointly.

A particular distributed control with communication is formulated below. The problem is to find that linear combination of the output of a Gaussian system of a controller to be sent to the other controller, so that the latter controller can do its control task as best as possible.

**Definition 6.2.** Consider a distributed Gaussian control system with the space-state representation,

\[
x(t + 1) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) + M v(t),
\]

\[
y_1(t) = C_1 x(t) + D_1 u_1(t) + N_1 v(t),
\]

\[
y_2(t) = C_2 x(t) + D_2 u_2(t) + N_2 v(t).
\]

A linear combination \( y_{12} = L_{12} y_2 \) of the output components observed by Controller 2, \( y_2 \), is to be sent to Controller 1 with \( L_{12} \in \mathbb{R}^{p_2 \times p_2} \),

\[
y_{12}(t) = L_{12} y_2(t)
\]

\[
= L_{12} C_2 x(t) + L_{12} D_2 u_2(t) + L_{12} N_2 v(t).
\]

Controller 1 then faces the following partially-observed Gaussian system where \( y_{1s} \) denotes its new observation vector,

\[
y_{1s}(t) = (y_1(t) y_{12}(t))^T,
\]

\[
x(t + 1) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) + M v(t),
\]

\[
y_{1s}(t) = \begin{pmatrix} C_1 \\ L_{12} C_2 \end{pmatrix} x(t) + \begin{pmatrix} D_1 & 0 \\ L_{12} D_2 & L_{12} N_2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + N_1 v(t).
\]

**Problem 6.3.** Consider the above defined distributed Gaussian system with communication from Controller 2 to Controller 1. Consider the problem of distributed control of a distributed Gaussian system, see Problem 2.3. Which matrix \( L_{12} \in \mathbb{R}^{p_2 \times p_2} \) achieves the lowest cost for Controller 1? Research on this problem is in progress.

7. COORDINATION CONTROL

Coordination control of distributed stochastic systems is motivated by the experience that often a distributed control cannot meet the control objectives. Therefore a form of coordination between the subsystems is necessary.

Examples of control tasks of control engineering problems include: (1) The assignment of two or more subsystems to a particular task. For example, underwater vehicles for pluming tracking. Other examples include the assignment of pumps on an oil platform and generators in a power plant or in a power network. (2) Avoiding direct conflicts. For examples, aerial vehicles should not collide. (3) Control of power networks with many local controllers and actuators.

The reader may find various forms of coordination control in the literature. At the intuitive level, the forms of coordination are related but in the details they are different. In this paper another form of coordination control is used. It consists of a coordinator and two or more subsystems which interact such that the coordinator issues signals to the subsystems but the subsystems do not directly send signals to the coordinator. Coordination control of linear systems is described in Ran and van Schuppen [2008], Kempker et al. [2009] and of discrete-event systems in Komenda et al. [2010].

**Example 7.1.** The concept of a coordinated Gaussian system is introduced by this example. Consider then a set

\[
x_{c}(t+1) = A_{cc} x_{c}(t) + B_{cc} u_{c}(t) + M_{cc} v_{c}(t),
\]

\[
r_{1c}(t) = C_{1c} x_{c}(t) + D_{1c} u_{c}(t) + M_{1c} v_{c}(t),
\]

\[
r_{2c}(t) = C_{2c} x_{c}(t) + D_{2c} u_{c}(t) + M_{2c} v_{c}(t),
\]

\[
x_{1}(t+1) = A_{11} x_{1}(t) + B_{11} u_{1}(t) + B_{1c} r_{1c}(t) + M_{11} v_{1}(t),
\]

\[
x_{2}(t+1) = A_{22} x_{2}(t) + B_{22} u_{2}(t) + B_{2c} r_{2c}(t) + M_{22} v_{2}(t).
\]

The interconnection of these systems then has the coordinated Gaussian system representation,
The set of coordination-structured matrices occurring in the definition of a coordinated Gaussian system is a ring. In control synthesis of coordinated Gaussian systems attention is restricted to a linear control law, $g(x) = Fx$, in which the feedback matrix $F$ belongs to the set of coordination-structured matrices.

Control research issues for coordination control which are currently under investigation include:

1. The geometric characterization of a coordinated Gaussian system.
2. The construction of a coordinated Gaussian system representation from the interconnection of subsystems.
3. The characterization of the minimality of the coordinator of a coordinated Gaussian system or of a mammillary Gaussian system. Minimality can be defined as the dimension of the state set of the coordinator.
4. How to formulate an approximate coordinated Gaussian system in which one or more of the zeros in the last row of the above structured matrices are nonzero though small?
5. How to carry out optimal control synthesis for coordination control of distributed Gaussian systems?

8. CONTROL OF HIERARCHICAL SYSTEMS

Hierarchical systems are a very useful modeling formalism of engineering. The concept goes back to immemorial times, the Romans used it for the organization of their army and the Aztecs used it for the organization of their society. Yet, control theory for hierarchical systems is far from satisfactory. References on optimization and control of hierarchical control systems include Arrow and Hurwicz [1963], Findeisen et al. [1980], Mesarović et al. [1970].

In a hierarchical control system one distinguishes two or more layers often based on geographical extent or on the time scale of the dynamics. Each layer has several subsystems. The relation between the subsystems of adjacent layers is then an undirected tree of graph theory with the root of the tree at the highest level; this is the main characteristic of a hierarchical system. There is a close analogy between hierarchical systems and coordinated control systems in that the coordinator is the subsystem directing other subsystems at the next-lower level.

Examples of hierarchical control systems in engineering are the telephone network developed in the early twentieth century, the transportation networks of airline companies, the computer and communication networks developed in the latter part of the twentieth century, and road networks consisting of motorways, provincial roads, and urban roads.

The problem of control of a hierarchical system has as control objective that the combined hierarchical closed-loop system should satisfy prespecified control objectives. The controllers of the subsystems are based on state or output feedback of the same subsystem. The subsystem at a particular level different from the top level follows a reference signal or a control objective of the coordinator at the next-higher level. In addition, each subsystem at a particular level except for the lowest level, issues reference signals or control objectives to the subsystems to which

\[ x(t) = (x_1(t) x_2(y) x_3(t))^T, \]

\[ u(t) = (u_1(t) u_2(y) u_3(t))^T, \]

\[ v(t) = (v_1(t) v_2(y) v_3(t))^T, \]

\[ x(t + 1) = \begin{pmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{pmatrix} u(t) + \begin{pmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{pmatrix} v(t). \]

Define the class of mamillary Gaussian systems by the representation,

\[ y(t) = \begin{pmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{pmatrix} x(t) + \begin{pmatrix} D_{11} & 0 & D_{1c} \\ 0 & D_{22} & D_{2c} \\ 0 & 0 & D_{cc} \end{pmatrix} u(t) + \begin{pmatrix} N_{11} & 0 & N_{1c} \\ 0 & N_{22} & N_{2c} \\ 0 & 0 & N_{cc} \end{pmatrix} v(t). \]

Define the class of coordinated Gaussian systems by the representation,

\[ x(t + 1) = \begin{pmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ A_{1} & A_{2} & A_{cc} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{pmatrix} u(t) + \begin{pmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{pmatrix} v(t). \]

It is possible to provide a geometric characterization of a coordinated Gaussian system in terms of a conditional independence property of linear subspaces, an invariance condition on the linear dynamics, and input injection conditions on the input and on the noise process. This is done for coordinated deterministic linear systems in Kempker et al. [2009]. Due to space limitations, the characterization will not be provided here. The concept of a coordinated Gaussian system is inspired by the concept of conditionally independent σ-algebras.

Readers who first learn of coordinated systems are concerned about the property that there is no feedback from the subsystems to the coordinator. There are examples where the structure is this way. In case there is feedback from the subsystems to the coordinator then one obtains a mammillary system of the research area of compartmental systems. This class is defined also above and it is directly related to the class of hierarchical systems.

The usefulness of coordination control with respect to distributed control of distributed systems is in: (1) The enforcement of interaction on the subsystems by the coordinator. (2) The lower complexity of control synthesis. A disadvantage is that more information has to be communicated between the subsystems at a cost.
it is connected at the next-lower level. A closed-loop hierarchical system should satisfy the control objectives if and only if controllability and observability of the system, connecting these two levels, hold. The controllability and the observability relations between the layers have not been investigated as deeply as is needed. Progress has been made on hierarchical control of discrete-event systems by K. Schmidt and T. Moor, see Schmidt et al. [2008].

Control research issues for hierarchical control of hierarchical systems include: (1) How to abstract stochastic systems? (2) The control theoretic concepts of hierarchical control. (3) Theory for meeting the control objectives. (4) How to distributed the control objectives of a hierarchical system over the different layers?

9. CONCLUDING REMARKS

Control of distributed stochastic systems is the topic of the paper. Several control architectures were formulated and discussed. Coordination control treats a coordinator which directs two or more subsystems. Control with communication occurs of one or more controllers send or request information of other controllers. Control of hierarchical systems deals with two or more layers where any subsystem is related to one or more subsystems below and only one subsystem in the higher layer. Control issues discussed included the construction of the state of the controller, the search for a person-by-person equilibrium, the minimality of a coordinator, the communication controller, and equivalent conditions for control of hierarchical systems.

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