On performance and optimality tradeoffs in guidance and control law design

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Abstract: A number of control applications require that the system outputs track a specified trajectory, usually designed by solving an optimal control problem. Then, a tracking controller guarantees that the error between the outputs and their desired time evolutions is sufficiently small. The design of the tracking controller will depend on the linear time-varying (LTV) linearised model, the properties of which might render the design itself "easier" or "harder" in some sense. The goal of this paper is to define an index that measures the limitations on the dynamic performance associated with each possible trajectory. Such an index will be defined by specialising existing results on time-varying extensions of Bode’s sensitivity integrals to the case of systems defined on a finite horizon. Two simulation examples will be finally used to illustrate tradeoffs between dynamic performance and optimality in the framework of guidance and control problems.

Keywords: Tracking; Time-varying systems; Nonlinear systems.

1. INTRODUCTION

A significant number of control applications require that one or more of the system’s output variables track a specified trajectory. The trajectory is usually designed by solving a corresponding optimal control problem, the formulation of which embodies the main requirements associated with the desired operation for the system. It is then left to a suitably designed tracking controller to actually guarantee that the error between the outputs of the system and their desired time evolutions is sufficiently small. The design of the tracking controller will therefore depend on the linearised model of the system computed along the reference trajectory, so one may wonder whether there is a connection between properties of the desired trajectories and corresponding properties of the linear time-varying (LTV) linearised model, which might render the design of the tracking controller "easier" or "harder" in some sense. In this paper the goal is to associate to each trajectory an index that takes into account the limitations on the dynamic performance that can be obtained following such trajectories. In fact, it could happen that for a given trajectory the cost function takes the optimal value but for the same trajectory the dynamic performance gets worse compared to the other designed trajectories. Hence, in this case, a trade-off is necessary and both indices (the cost function and the index associated with the limitation on the dynamic performance) have to be considered. Such an index will be defined by specialising existing results on time-varying extensions of Bode’s sensitivity integrals (see Iglesias [2002]) to the case of systems defined on a finite horizon. The paper is organized as follows. First of all, in Section 2, the problem of tradeoff between dynamic performance and optimality, is illustrated. In Section 3 the relevant background on Bode’s sensitivity integral in time-invariant systems and its extensions for time-varying systems is reported. In Section 4 the analogue of Bode’s sensitivity integral for time-varying systems proposed in Iglesias [2002] is considered and a new result on a finite horizon is shown. Finally, an example of tradeoff between dynamic performance and optimality in the framework of guidance and control problems is reported in Section 5.

2. PROBLEM STATEMENT

For the sake of clarity, the problem of finding a tradeoff between dynamic performance and optimality is first formulated with reference to the problem of choosing a suitable operating (equilibrium) point for a given nonlinear system. To this goal, consider the system

\[ \dot{x} = f(x, u) \]

\[ y = g(x, u), \]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \), and assume that a single equilibrium point exists corresponding to each constant input \( \bar{u} \):

\[ 0 = f(\bar{x}, \bar{u}) \]

\[ \bar{y} = g(\bar{x}, \bar{u}). \]

Introduce now a cost function

\[ J = h(\bar{y}, \bar{u}), \]

defining a criterion for the choice of an optimal operating point for system (1)-(2); \( J \) is clearly a function of the input \( \bar{u} \). If now a linear regulator is to be designed to keep the system as close as possible to the chosen equilibrium point, the linear time-invariant (LTI) model...
\[ \delta x = A \delta x + B \delta u \quad (6) \]
\[ \delta y = C \delta x + D \delta u, \quad (7) \]
will be considered, where \([A, B, C, D]\) and, as a consequence, the poles and zeros of the system, are functions of the input \(u\). As is well known (see Section 3 for details), on the basis of Bode’s sensitivity integrals the presence of right half plane (RHP) poles and zeros in the loop transfer function of a LTI control system leads to (computable) limitations on dynamic performance. Hence, for different choices of the input \(u\), one can find different values of Bode’s sensitivity integral but also different values of the cost function \(J\), so a tradeoff between performance and optimality is necessary.

Consider now a more general scenario in which system (1)-(2) is required to operate by tracking a trajectory, to be designed by solving a corresponding optimal control problem, such as, e.g., determining the trajectory-control pair, \([\tau_0, \tau_f] \in \tau \rightarrow \{x \in \mathbb{R}^n, u \in \mathbb{R}^m\}\) and the final time \(\tau_f\) that minimize the cost function
\[ J(x(\cdot), u(\cdot), \tau_f) = \int_{\tau_0}^{\tau_f} F(x(\tau), u(\tau))d\tau \quad (8) \]
subject to the dynamic constraints
\[ \dot{x}(\tau) = f(x(\tau), u(\tau)), \quad \forall \tau \in (\tau_0, \tau_f), \quad (9) \]
the end point constraints
\[ e_L < e(x(\tau_0), x(\tau_f)) \leq e_U, \quad (10) \]
where \(e\) is the tracking error, and the box constraints
\[ x_L \leq x(\tau) \leq x_U, \quad u_L \leq u(\tau) \leq u_U \quad (11) \]
\[ \tau_0 \leq \tau \leq \tau_f, \quad \tau_L \leq \tau \leq \tau_U. \quad (12) \]
As in the previous case, when one turns to the problem of designing a controller to track the desired trajectory, the linearised model
\[ \delta \dot{x} = A(t) \delta x + B(t) \delta u \quad (13) \]
\[ \delta y = C(t) \delta x + D(t) \delta u \quad (14) \]
comes into play. Note that the dynamics of the linearised model is relevant only over the time interval \([\tau_0, \tau_f] \in \tau\) during which the tracking problem has to be analysed. The problem addressed in this paper is the one of defining a measure of achievable dynamic performance in tracking the given optimal trajectory. Such a measure can provide quantitative information to arrive at a performance/optimality tradeoff in the design of the entire system. As we will rely extensively on Bode’s sensitivity integrals and their extensions for time-varying systems proposed in Iglesias [2002], in the following Section a short overview of such topics will be provided.

3. BACKGROUND: PERFORMANCE LIMITATIONS IN LINEAR CONTROL SYSTEMS
As discussed in Stein [2003], a frequency domain quantification of control difficulty is captured by the Bode integrals [Bode [1945], Freudenberg and Loose [1985]]:
\[ B = \int_0^\infty \ln |S(j\omega)|d\omega = 0 \quad (15) \]
\[ B = \int_0^\infty \ln |S(j\omega)|d\omega = \pi \sum_{i=1}^{N_p} \text{Re}(p_i), \quad (16) \]
where \(S\) is the sensitivity function of a SISO feedback system and \(p_i\) are the \(N_p\) RHP-poles of the plant. The integrals state that the log of the magnitude of the sensitivity function, integrated over frequency, is zero for stable plants and positive for unstable ones. It becomes larger as the number of unstable poles increases and/or as the poles move farther into the right-half plane.

So for open-loop stable systems, the average sensitivity improvement a feedback loop achieves over frequency is exactly offset by its average sensitivity deterioration. For open-loop unstable systems, things are worse because the average deterioration is always larger than the improvement. Sensitivity improvements in one frequency range must be paid for with sensitivity deteriorations in another frequency range, and the price is higher if the plant is open-loop unstable.

All the extensions to Bode’s sensitivity integral (15) present in the literature rely on integrals of the system sensitivity transfer function. Extending these results to classes of systems that do not allow for straightforward definitions of transfer function operators, for example time-varying or nonlinear systems, presents several difficulties. For discrete-time time-varying systems, an extension of Bode’s integral was proposed in [2001]. What makes this extension possible is the connection that exists between the logarithmic integral found in Bode’s relationship and a cost function used in control theory, the entropy. In Glover and Doyle [1988] it was shown that in the LTI case, the minimum entropy cost function is equivalent to that considered in stochastic risk-sensitive control problems. This relationship holds for both discrete and continuous time systems that admit a state-space realization (see Peters and Iglesias [1999]). These connections serve to provide a time-domain interpretation to the Bode’s sensitivity integral and allow extensions to time-varying systems. In Iglesias [2002] a generalization of Bode’s integral relationship to continuous time, time-varying systems is presented. The generalization of Bode’s integral is made for the class of time-varying systems which possess an exponential dichotomy (see Section 4 for details). It is shown that the sensitivity function is constrained, on average, by the spectral values in the dichotomy spectrum of the antistable component of the open-loop dynamics. For regular systems the Lyapunov exponents of the open-loop system, natural generalizations of the real part of the eigenvalues, coincide with the spectral values.

4. BODE’S SENSITIVITY INTEGRAL FOR FINITE-HORIZON TIME-VARYING SYSTEMS

4.1 Preliminaries
The basic definitions and results presented in this subsection are recalled from Iglesias [2002]. Consider the \(n\)-dimensional LTV system
\[ \dot{x}(t) = A(t)x(t) \quad (17) \]
and denote by \(\Phi_A(t, \tau)\) its transition matrix, which can be written as
\[ \Phi_A(t, \tau) = X(t)X^{-1}(\tau) \quad (18) \]
where \(X(t)\) is the fundamental solution to the matrix differential equation
\[ \dot{X}(t) = A(t)X(t), \quad X(0) = X_0 \quad (19) \]
with $X_0$ invertible.

**Lemma 1.** The transition matrix for $A(t)$ satisfies

$$\ln \det \Phi_A(t, \tau) = \int_{\tau}^{t} \text{trace}[A(\sigma)]d\sigma$$

for every $t$ and $\tau$.

**Definition 1.** The matrix function $A(t)$ is uniformly exponentially stable, UES, (resp. antistable, UEA) if there exist positive constants $\gamma$, $\delta$ such that

$$\|\Phi(t, \tau)\| \leq e^{-\gamma(t-\tau)}$$

for all $t$ and $\tau$ such that $t \geq \tau$ (resp. $t \leq \tau$).

The linear system (17) is said to possess an exponential dichotomy if there exists a projection $P$ and real constants $\gamma > 0$, $\delta > 0$ such that

$$\|X(t)PX^{-1}(\tau)\| \leq e^{-\gamma(t-\tau)} \quad (\text{for } t \geq \tau)$$

$$\|X(t)(I-P)X^{-1}(\tau)\| \leq e^{\gamma(t-\tau)} \quad (\text{for } t \leq \tau).$$

The dichotomy spectrum $\mathcal{S}_{\text{dich}}$ of the system (17) is the set of real values $\lambda$ for which the translated systems

$$\dot{x}(t) = (A(t) - \lambda I)x(t)$$

fail to have an exponential dichotomy. In general, the spectrum $\mathcal{S}_{\text{dich}}$ is a collection of compact intervals

$$\mathcal{S}_{\text{dich}} = \bigcup_{i=1}^{m} [\bar{\lambda}_i, \bar{\lambda}_i],$$

where $m \leq n$ and $\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \cdots \leq \bar{\lambda}_m \leq \bar{\lambda}_m$. If each of these intervals is a point, the spectrum is known as point spectrum, each $\lambda_i$ in the point spectrum equals a Lyapunov exponent and the system is said to be regular.

Consider a LTV system $\Sigma_G$ admitting the state space representation

$$\Sigma_G := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} u(t) \\ 0 \end{bmatrix}$$

With this system an infinite-dimensional operator $G$ is associated mapping the input $u$ to output $y$. This operator has an integral representation

$$y(t) = \int_{-\infty}^{t} g(t, \tau)u(\tau)dt,$$

where the kernel $g(t, \tau)$ equals

$$g(t, \tau) = D\delta(t-\tau) + C(t)\Phi_A(t, \tau)B(\tau)$$

and zero otherwise.

**Lemma 2.** If $A(t)$ in (26) admits an exponential dichotomy, then there exists a bounded matrix function $T(t)$ with bounded inverse such that

$$\begin{bmatrix} \tilde{T}(t) + T(t)A(t)T^{-1}(t) \\ C(t)T^{-1}(t) \end{bmatrix} \begin{bmatrix} T(t)B(t) \\ D(t) \end{bmatrix} \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} = \begin{bmatrix} A_{us}(t) & 0 \\ 0 & A_{u}(t) \end{bmatrix}$$

where $A_{us}(t)$ is UES and $A_{u}(t)$ is UEA.

Consider now the open-loop system $\Sigma_L$ that has a uniformly stabilizable and detectable state space representation

$$\Sigma_L = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$$

The corresponding state space representation for the sensitivity system is

$$\Sigma_S = \begin{bmatrix} A(t) - B(t)C(t) & C(t) \\ -C(t) & I \end{bmatrix}.$$

**Assumption 1.** $A(t) - B(t)C(t)$ is UES. Equivalently, the input/output operator $S$ associated with $\Sigma_S$ is bounded.

**Assumption 2.** The open-loop dynamics $A(t)$ admits an exponential dichotomy with rank($P$) = $n_s$ ($n_s$ fundamental solutions are UES, whereas $n_n = n - n_s$ are UEA). Moreover, the antistable component has dichotomy spectrum

$$\Lambda_u = [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4] \cup \cdots \cup [\lambda_m, \lambda_m]$$

with dimension $n_1, \ldots, n_m$ and $\sum_{i=1}^{m} n_i = n_u$.

**Assumption 3.** The open-loop system has relative degree (in the sense of Ilchmann and Mueller [2007]) of at least 2, i.e., $C(t)B(t) = 0$ for all $t$.

Consider the differential Riccati equation (DRE)

$$-\dot{X}(t) = A'(t)X(t) + X(t)A(t) - X(t)B(t)B'(t)X(t).$$

The above DRE has a stabilizing solution if $X(t) = X'(t) \geq 0$ satisfies (33), $X(t)$ is bounded, and $A(t) - B(t)B'(t)X(t)$ is UES.

**Lemma 3.** Suppose that $A(t)$ admits an exponential dichotomy and that the pair $(A(t), B(t))$ is uniformly stabilizable. Then (33) has a stabilizing solution $X(t)$ and

(i) If $A(t)$ is UES, then $X(t) \equiv 0$ for all $t$.

(ii) If $A(t)$ is UEA, then $\exists \epsilon > 0$ such that $\epsilon I \leq X(t)$ for all $t$.

(iii) If $A(t)$ is UES, then $X(t) \equiv 0$ for all $t$.

**Corollary 1.** Under the assumptions of Lemma 3, the sensitivity operator has an inner/outer factorization $S = S_{o}S_{s}$ (with $S_{o}$ and $S_{o}^{-1}$ bounded and $\|S_{o}w\|_{2} = \|w\|_{2}$ for any $w \in L^{2}$) and the two factors have state space representations

$$\Sigma_{S_{o}} = \begin{bmatrix} A(t) - B(t)B'(t)X(t) & B(t) \\ -B'(t)X(t) & I \end{bmatrix}$$

and

$$\Sigma_{S_{s}} = \begin{bmatrix} A(t) - B(t)C(t) & B(t) \\ B(t)X(t) - C(t) & I \end{bmatrix}.$$ 

**Theorem 1.** Suppose that the system $\Sigma_{L}$ satisfies Assumptions 1-3. Then

$$0 < \sum_{i=1}^{n_s} \lambda_i \leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{0}^{T} \text{trace}[s_{o}(t, t)]dt \leq \sum_{i=1}^{n_n} \lambda_i,$$

where $\lambda_i$ and $\bar{\lambda}_i$, $i = 1, \ldots, m$, are the spectral values of the antistable component of $A(t)$ and $s_{o}(t, \tau)$ is the kernel of $S_{o}$ given by

$$s_{o}(t, \tau) = [B'(t)X(t) - C(t)]\Phi_{A-BC}(t, \tau)B(\tau), \quad t \geq \tau,$$

and zero otherwise.
Corollary 2. If, in addition to the assumptions of Theorem 1, the antistable component of the open-loop dynamics \( A_\alpha(t) \) is regular, then
\[
\lim_{T \to \infty} \frac{1}{2T} \int_0^T \text{trace}[s_\alpha(t, t)] dt = \frac{1}{m} \sum_{i=1}^m \lambda_i(T),
\]  
where \( \lambda_i \) are the Lyapunov exponents of \( A_\alpha(t) \).

Consider now the quantity
\[
B_\ell = \frac{1}{2T} \int_0^T \text{trace}[s_\ell(t, t)] dt,
\]
and let
\[
B_\infty := \lim_{T \to \infty} \frac{1}{T} \int_0^T B_\ell dt = \pi \lim_{T \to \infty} \frac{1}{2T} \int_0^T \text{trace}[s_\ell(t, t)] dt.
\]

Then it is shown in Iglesias [2002] that
\[
B_\infty = \pi \lim_{T \to \infty} \frac{1}{T} \int_0^T \text{trace}[A_\alpha(t)] dt = \frac{1}{m} \sum_{i=1}^m \lambda_i(T),
\]  
where \( B_\infty \) is a time-varying analogue of Bode’s sensitivity integral (16), considered in the time-domain on infinite horizon. Note that, in the LTI case, \( B_\infty \) depends on the unstable eigenvalues of the open-loop dynamics while, in the LTV case, \( B_\infty \) depends on the Lyapunov exponents of the antistable component of the open-loop dynamics. In fact Lyapunov exponents, in regular LTV systems, can be seen as a generalization of the real part of the eigenvalues of LTI systems.

4.2 Main result

Consider equation (40) and note that if the LTV system is defined only on a finite horizon, \( t \in [0, T] \), the average of \( B_\ell \) is taken only over a finite interval so that
\[
B_{\ell T} := \frac{1}{T} \int_0^T B_\ell dt = \pi \frac{1}{2T} \int_0^T \text{trace}[s_\ell(t, t)] dt.
\]

On the other hand, from the theory of Lyapunov exponents we have that
\[
\frac{1}{T} \int_0^T \text{trace}[A_\alpha(t)] dt = \frac{1}{m} \sum_{i=1}^m \lambda_i(T),
\]
where \( \lambda_i(T) \) is the \( i \)-th truncated Lyapunov exponent at time \( T \). At this point the aim is to understand what is the relation between Bode’s sensitivity integral on a finite horizon \( B_{\ell T} \) and the sum of the truncated Lyapunov exponents associated with \( A_\alpha(t) \). Such a connection is provided by the following result.

Theorem 2. Consider a LTV system \( \Sigma_L \) defined on a finite horizon, \( t \in [0, T] \), and suppose that such system satisfies Assumptions 1-3, then:
\[
B_{\ell T} = \pi \frac{1}{2T} \int_0^T \text{trace}[s_\ell(t, t)] dt
= \pi \left( \frac{1}{m} \sum_{i=1}^m \lambda_i(T) + \frac{1}{2T} \ln \left( \frac{\det X_u(T)}{\det X_u(0)} \right) \right),
\]  
where \( X_u \) is defined in Lemma 3.

Theorem 2 differs from Theorem 1 in Iglesias [2002] in that the computation of Bode’s sensitivity integral is taken only over a finite horizon. It results that, while the integral on infinite horizon is equal to the sum of the truncated Lyapunov exponents associated with \( A_\alpha \) (multiplied by \( \pi \)), on finite horizon, instead, the truncated Lyapunov exponents represent only a lower bound on Bode’s sensitivity integral. Theorem 2 can be derived along the lines of the proof of Theorem 1, see Iglesias [2002], as follows.

Proof. Consider (37), then
\[
\begin{align*}
\ln [\det X_u(T)] &= [B'(t)X(t) - C(t)]B(t) \\
&= B'(t)X(t)B(t) \\
&= B'(t)X_u(t)B_u(t),
\end{align*}
\]
where (45) follows from Assumption 3 and (46) from Lemma 3. Moreover
\[
\begin{align*}
\text{trace}[B_u'(t)X_u(t)B_u(t)] &= \text{trace}[X_u(t)B_u(t)B_u'(t)] \\
&= \text{trace}[B_u(t)B_u'(t)X_u(t)].
\end{align*}
\]

The DRE for \( X_u \) is
\[
\dot{X}_u(t) = -A_u'(t)X_u(t) + X_u(t)B_u(t)B_u'(t)X_u(t),
\]
where \( X_u(t) = \Phi_{-A_u - B_uX_u}(t, t)X_u(\tau)\Phi_{A_u}(\tau, t). \) (48)

Consider \( \tau = 0 \) and \( t = T \) and take the logarithm of the determinant of both sides of (48)
\[
\ln \det X_u(T) = \ln \det \Phi_{-A_u - B_uX_u}(T, 0) + \ln \det X_u(0) + \ln \det \Phi_{A_u}(0, T). \]  
Applying Lemma 1 to the transition matrices in (49) yields
\[
\ln \det \Phi_{-A_u - B_uX_u}(T, 0) = -\int_0^T \text{trace}[A_u(t)] dt + \int_0^T \text{trace}[B_u(t)B_u'(t)X_u(t)] dt
\]
and
\[
\ln \det \Phi_{A_u}(0, T) = -\int_0^T \text{trace}[A_u(t)] dt.
\]
Thus
\[
\ln \det X_u(T) - \ln \det X_u(0) = 2\int_0^T \text{trace}[A_u(t)] dt + \int_0^T \text{trace}[B_u(t)B_u'(t)X_u(t)] dt.
\]  
Finally
\[
B_{\ell T} = \pi \frac{1}{2T} \int_0^T \text{trace}[s_\ell(t, t)] dt
= \pi \frac{1}{2T} \int_0^T \text{trace}[B_u(t)B_u'(t)X_u(t)] dt
= \pi \left( \frac{1}{T} \int_0^T \text{trace}[A_u(t)] dt + \frac{1}{2T} \ln \left( \frac{\det X_u(T)}{\det X_u(0)} \right) \right)
= \pi \left( \frac{1}{m} \sum_{i=1}^m \lambda_i(T) + \frac{1}{2T} \ln \left( \frac{\det X_u(T)}{\det X_u(0)} \right) \right).
\]  

Remark 1. In the case of LTI systems (33) reduces to an algebraic Riccati equation and its solution is constant, so $X_u(0) = X_u(T)$, the second term in the right hand side of (52) is zero and then
\[ B_T = \pi \sum_{i=1}^{m} \lambda_i(T) = \pi \sum_{i=1}^{N_v} \text{Re}(p_i) = B \]
as expected.

Remark 2. In the case of $T \to \infty$, since $X_u(t)$ is bounded from above and from below (from Lemma 3), the logarithm in equation (52) is finite and then for $T \to \infty$ the second term in the right hand side of the equation is zero. Hence
\[ B_T = \pi \lim_{T \to \infty} \frac{1}{2T} \int_0^T \text{trace}[s_u(t,t)]dt \]
\[ = \pi \left( \lim_{T \to \infty} \frac{1}{2T} \int_0^T \text{trace}[A_u(t)]dt \right) \]
\[ + \lim_{T \to \infty} \frac{1}{2T} \ln \left( \frac{\det X_u(T)}{\det X_u(0)} \right) \]
\[ = \pi \sum_{i=1}^{N_v} \lambda_i = B_\infty \]
again, as expected.

Bode’s finite horizon sensitivity integral computed in Theorem 2 can be used as a performance index in the framework of optimality and performance tradeoffs. In particular the truncated Lyapunov exponents, that represent a lower bound on the Bode’s sensitivity integral, can be used as a quantitative bound on the achievable closed-loop performance.

5. NUMERICAL EXAMPLES

In this Section the problem of trading performance and optimality in guidance and control law design is illustrated with two numerical examples. The second order system
\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = x_1 x_2 + u \]
\[ y = x_1 \]
is considered, for which the goal is to design a feasible trajectory imposing initial and final conditions on the states and also optimizing a suitable cost function (guidance problem). Moreover a controller has to be designed such that the system tracks the trajectory returned by the guidance system (control problem). The guidance and control problems are resolved considering the flatness property of the system (see Fliss et al. [1995] for the definition of flatness) as presented, for example, in Desiderio and Lovera [2010a] for the guidance problem and Desiderio and Lovera [2010b] for the control problem. Choose $z = x_1$ as flat output, then
\[ x_1 = z \]
\[ x_2 = \dot{x}_1 = \dot{z} \]
\[ u = x_2 - x_1 x_2 = \dot{z} - z \dot{z}. \]
Let $z^*$ be the desired polynomial trajectory
\[ z^* = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 = x_1^* \]
then
\[ \ddot{z}^* = 5 a_5 t^4 + 4 a_4 t^3 + 3 a_3 t^2 + 2 a_2 t + a_1 = x_2^* \]
and
\[ \dddot{z}^* = 20 a_5 t^3 + 12 a_4 t^2 + 6 a_3 t + 2 a_2. \]
Let $a = [a_5, a_4, a_3, a_2, a_1, a_0]$ and consider the following constrained optimization problem:
\[ \min_a J = \int_0^T (\alpha (x_1^2 + x_2^2) + \rho \dot{u}^2) dt \]
subject to
\[ x_1(0) = x_{10} \quad x_2(0) = x_{20}, \]
\[ x_1(T) = x_{1T} \quad x_2(T) = x_{2T}, \]
with $T$ fixed. Note that from (63) two coefficients are easily computed
\[ a_0 = x_{10} \quad a_1 = x_{20}, \]
so the optimization problem is solved respect to four coefficients ($a_5, a_4, a_3, a_2$) only and the constraint (64).

In the following the results in terms of optimal cost and Lyapunov exponents are reported. Consider the initial and final conditions
\[ x_{10} = 200 \quad x_{20} = 45, \]
\[ x_{1T} = 0 \quad x_{2T} = 0 \]
with $T = 10$ s.

Case 1: $\alpha/\rho$ variable. In this case different values of $\alpha$, with respect to $\rho$, are considered. In Table 1 the optimal cost and the Lyapunov exponents with $\alpha = [10^4, 10^5, 10^6]$ and $\rho = 10$ are reported. The positive Lyapunov exponents associated with the antistable component of $A(t)$ are highlighted in bold. In the last column of the table the sum of the positive Lyapunov exponents multiplied by $\pi$ is reported; it is denoted with $B_{TL}$ as it represents a lower bound on the finite horizon Bode integral, see equation (52).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\rho$</th>
<th>$\alpha/\rho$</th>
<th>$J$</th>
<th>$\lambda_1(T)$</th>
<th>$\lambda_2(T)$</th>
<th>$B_{TL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^4$</td>
<td>10</td>
<td>$10^2$</td>
<td>1.11 $\times$ $10^3$</td>
<td>126.19</td>
<td>0.54</td>
<td>398.13</td>
</tr>
<tr>
<td>$10^5$</td>
<td>10</td>
<td>$10^3$</td>
<td>2.60 $\times$ $10^3$</td>
<td>84.61</td>
<td>0.68</td>
<td>267.95</td>
</tr>
<tr>
<td>$10^6$</td>
<td>10</td>
<td>$10^4$</td>
<td>8.57 $\times$ $10^3$</td>
<td>41.99</td>
<td>-0.98</td>
<td>131.91</td>
</tr>
</tbody>
</table>

Table 1. Case 1: optimal cost and $\lambda$-values with $\alpha/\rho = 10^2, 10^3, 10^4$.

Note that if the $\alpha/\rho$ ratio increases then the optimal cost increases but the attainable closed-loop performance, as expressed by $B_{TL}$, gets better. The value of $B_{TL}$ represents only a lower bound on Bode’s sensitivity integral, but it is sensible to claim that, if the second term in (52) is sufficiently small, then when $B_{TL}$ decreases the performance improves.

In Figure 1 the state and control variables for trajectories associated with $\alpha/\rho = 10^2, 10^3, 10^4$ are reported; the solid line is associated with the best optimal cost ($1.11 \times 10^3$), the dashed line with the second optimal cost ($2.60 \times 10^3$), while the dotted line with the third optimal cost ($8.57 \times 10^3$). The second trajectory (reported with the dashed line) can be a good compromise between optimality and attainable dynamic performance.
In this particular example, with $\alpha/\rho = 10^2$, the best dynamic performance is obtained with a polynomial of degree four. In the other two cases considered before, $\alpha/\rho = 10^3, 10^5$, not reported for brevity, the best dynamic performance is obtained respectively in the case $k = 4$ and $k = 5$. This shows that the Lyapunov exponents should be taken into account for the choice of the trajectory to consider, extra respect to the optimality of the trajectory.

In Figure 2 the state and control variables for trajectories associated with $\alpha/\rho = 10^2$ and $k = 5, 4, 3$ are reported: the solid line is associated with the best optimal cost ($1.11 \times 10^9$), the dashed line with the second optimal cost ($1.75 \times 10^9$), while the dotted line with the third optimal cost ($2.237 \times 10^9$). The second trajectory (reported with the dashed line) could be the best choice in terms of optimality and dynamic performance obtainable.

### 6. Conclusions

The problem of trading dynamic performance and optimality in guidance and control law design is considered. To this purpose a time-domain analogue of Bode's sensitivity integral for finite horizon time-varying systems has been derived and its relation with the truncated Lyapunov exponents is shown. The application of this result in the quantification of the performance/optimality tradeoff in guidance and control problems has been demonstrated in a simulation example.

### REFERENCES


