Research of Fixed Strike Lookback Put Option on Extremes in Diffusion Model (B,S) - Financial Market

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Abstract: The problem under consideration is that of risk hedging in the financial market by means of the put option which belongs to the options extremes class when there is a capital inflow in the form of dividends by an underlying asset. The formulas defining costs of options and also evolution in time of portfolios and capitals, i.e. hedging strategy and corresponding to them are obtained. Some properties of decisions are investigated.

1. INTRODUCTION

An option is one the most common secondary (derivative) securities in financial markets since it gives you the right rather than obligation to exercise (Hull, 2002; Wilmut, 2000). An essential disadvantage of standard call and put options with the payoff functions of the form

\[ f_c^r(S_t) = \max\{0, S_t - K\} \quad \text{and} \quad f_p^r(S_t) = \min\{0, K - S_t\} \]

is the fact that they do not take into account the price history of the risk (basic) asset \( S_t \) in the interval \( t \in [0, T] \), where \( t = 0 \) is the moment of conclusion of the contract, and \( T \) is fixed instant at which the option is to be exercised. Such a disadvantage is excluded by history dependent options. An important particular case of this options class is options on extremes which are based on the use of extreme values \( S_t \) in the interval \( t \in [0, T] \) (Buchen et al., 2005; Conze et al., 1991; Hull, 2002; Zhang, 2000). In (Conze et al., 1991) the price of call and put options of such a type with the payoff functions

\[ f_c^r(S_t) = S_t - \min_{0 \leq s \leq T} S_s \quad \text{and} \quad f_p^r(S_t) = \max_{0 \leq s \leq T} S_s - S_t \]

has been found. The research of the portfolio (hedging strategy) and the capital for these options has demonstrated their degeneracy because a capital is formed only on the basis of a risk asset. The solution to the problem is using the specified value \( K \) as a striking price just as for standard options. In (Anikina et al., 2007) has been researched the problem of hedging for call option with the payoff function of the form

\[ f_c^r(S_t) = \max_{0 \leq s \leq T} S_s - K \].

In this work, we study hedging for put options with the payoff functions of the form

\[ f_p^r(S_t) = \left\{ K - \min_{0 \leq s \leq T} S_s \right\} \quad \text{or} \quad f_p^r(S_t) = \left\{ K - \max_{0 \leq s \leq T} S_s \right\} \]

which enables one to sell an underlying asset at the striking price \( \min_{0 \leq s \leq T} S_s \) or \( \max_{0 \leq s \leq T} S_s \) within the fixed time interval \( t \in [0, T] \), when there is a capital inflow in the form of dividends.

We denote the mathematical expectation by \( E\{\cdot\} \), the normal (Gaussian) density with the parameters \( a \) and \( b \) by \( N(a;b) \), the indicator of the event \( A \) by \( I[A] \), and \( \Phi(x) = \left[ 1/\sqrt{2\pi} \right] \times \int_{-\infty}^{x} \exp\left( -y^2/2 \right) dy \).

2. STATEMENT OF THE PROBLEM

Consideration of the problem is carried out in standard probability space \( \Omega, F_t, F = \{ F_t \}_{t \in [0, T]} \) (Shiryaev, 1998; Shiryaev et al., 1994). Risk and non-risk assets are circulating in financial market, current prices of which \( S_t \) and \( B_t \) during the time interval \( t \in [0, T] \) are defined by the equations

\[ dS_t = S_t \left( \mu dt + \sigma dW_t \right), \quad dB_t = rB_t dt, \]

where \( W_t \) is the Wiener process, \( \sigma > 0, r > 0, \sigma < 0, B_0 \), their solutions looks like

\[ S_t = S_0 \exp \left( \left[ \mu - \frac{\sigma^2}{2} \right] t + \sigma W_t \right), \]

\[ B_t = B_0 \exp \left[ rt \right]. \]

Let consider that current value of the investor capital \( X_t \) is defined in the form \( X_t = \beta B_t + \gamma S_t \), where \( \pi_t = (\beta_t, \gamma_t) \) is pair of \( F_t \) - measurable processes, composing portfolio of the investor securities. Dividends are paid for shareholding in accordance with the process \( D_t \) at the rate \( \delta, S_t \), which is
proportionate to the risk part of the capital with the coefficient \( \delta \), i.e. \( dD_t = \delta_t S_t dt \), \( 0 \leq \delta < r \).

The problem is to form the portfolio (the hedging strategy) \( \pi_t = (\gamma_t, \beta_t) \) so that the evolution of the capital \( X_t^\pi \) find the options price in accordance with the payoff functions (1.1) or (1.2), as well as the hedging strategy and the corresponding capital, which ensures the fulfillment of the payment guarantee liability

\[
X_t^\pi = f_t(S).
\]  
(2.3)

3. PRELIMINARY RESULTS

Proposition 1 (Shiryaev, 1998; Shiryaev et al., 1994). Let us a transformation of measure \( \rho_t \) in measure \( \rho_t^\mu-X_0 \) is given by

\[
Z_t^\mu = \exp \left\{-\frac{\mu - r + \delta}{\sigma} W_t - \frac{1}{2} \left( \frac{\mu - r + \delta}{\sigma} \right)^2 t \right\}.
\]

Then, we have

\[
\text{Law}(S_t^\mu, r, \delta) = \text{Law}(S_t, \rho_t \rho_t^\mu),
\]

i.e. properties of process \( S_t(\mu, r, \delta) \) defined by equation

\[
dS_t(\mu, r, \delta) = S_t(\mu, r, \delta)(r - \delta) dt + \sigma dW_t(\mu, r, \delta)
\]  
(3.1)

with regard to measure \( \rho_t^\mu \) are coinciding with properties of process \( S_t(\rho_t) \) defined by equation

\[
dS_t(\rho_t) = S_t(\rho_t)(r - \delta) dt + \sigma dW_t(\rho_t)
\]

(3.2)

with respect to the measure \( \rho_t \), where

\[
W_t(\mu, r, \delta) = W_t + (\mu - r + \delta) t / \sigma
\]

is Wiener process with respect to \( \rho_t^\mu \).

Proposition 2 (Shiryaev et al., 1994). Let

\[
m_t = \min_{0 \leq t \leq T} \rho_t = \min_{0 \leq t \leq T} (\sigma W_t + h t),
\]

\[
\xi_t = W_t + \frac{h t}{\sigma}, \quad h = r - \delta - \frac{\sigma^2}{2}.
\]  
(3.3)

Then \( p^\mu(t, x) = \partial P [m_t \leq x] / \partial x \) for \( x \leq 0 \) and \( h \in R \) has the form

\[
p^\mu(t, x) = \frac{1}{\sigma \sqrt{2 \pi}} \exp \left\{ \frac{- (x - ht)^2}{2 \sigma^2 t} \right\} + \frac{2h}{\sigma^2} \exp \left\{ \frac{2hx}{\sigma^2} \right\} \Phi \left\{ \frac{(x + ht)}{\sigma \sqrt{t}} \right\} + \frac{1}{\sigma \sqrt{2 \pi}} \exp \left\{ \frac{- (x - ht)^2}{2 \sigma^2 t} \right\}.
\]  
(3.4)

Proposition 3 (Shiryaev et al., 1994). Let

\[
M_t = \max_{0 \leq t \leq T} \rho_t = \max_{0 \leq t \leq T} (\sigma W_t + h t).
\]

Then \( p^\mu(t, x) = \partial P [M_t \leq x] / \partial x \) for \( x \leq 0 \) and \( h \in R \) has the form

\[
p^\mu(t, x) = \frac{1}{\sigma \sqrt{2 \pi}} \exp \left\{ \frac{- (x - ht)^2}{2 \sigma^2 t} \right\} - \frac{2h}{\sigma^2} \exp \left\{ \frac{2hx}{\sigma^2} \right\} \Phi \left\{ \frac{(x + ht)}{\sigma \sqrt{t}} \right\} + \frac{1}{\sigma \sqrt{2 \pi}} \exp \left\{ \frac{- (x - ht)^2}{2 \sigma^2 t} \right\}.
\]  
(3.5)

Proposition 4. Let \( X \sim N(a; d) \). Then, the following property is valid

\[
E[\exp(cX)I[X \leq b]] = \exp \left\{ \frac{ca + c^2 d}{2} \right\} \Phi \left\{ \frac{b - (a + cd)}{\sqrt{d}} \right\}.
\]  
(3.7)

4. MAIN RESULTS

Let

\[
d_1(t) = \left( \frac{r - \delta + \sigma}{\sigma} \right) \sqrt{T - t} = \ln(K/S_t) / \sigma \sqrt{T - t} = y_1(t),
\]

\[
d_2(t) = \left( \frac{r - \delta - \sigma}{\sigma} \right) \sqrt{T - t} = \ln(K/S_t) / \sigma \sqrt{T - t} = y_2(t) = \ln(K/S_t) / \sigma \sqrt{T - t} = \ln(K/S_t) + \left( \frac{r - \delta + \sigma^2}{2} \right) (T - t)
\]

\[
y_3(t) = \ln(K/S_t) + \left( \frac{r - \delta - \sigma^2}{2} \right) (T - t)
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\]

\[
y_3(t) = \ln(K/S_t) + \left( \frac{r - \delta - \sigma^2}{2} \right) (T - t)
\]

and \( d_1, d_2, y_1, y_2, y_3 \) are defined by formulas (4.1) at \( t = 0 \).
Theorem 1. The cost of option price is defined by formula

\[
P_T = K \exp(-rT) - S_0 \left[ (1 + \alpha^{-1}) \exp(-\delta T) \Phi(-d_1) + \left( 1 - \alpha^{-1} \right) \exp(-rT) \Phi(d_2) \right]
\]

if \( S_0 \leq K \);

\[
P_T = K \exp(-rT) -
\]

\[
- S_0 \left[ (1 + \alpha^{-1}) \exp(-\delta T) \Phi(y_1) - \left( 1 - \alpha^{-1} \right) \exp(-rT) \left( \frac{K}{S_0} \right) \Phi(y_2) \right]
\]

if \( S_0 > K \).

Proof. Since the payoff function \( f_T(S) \) of form (1.1) is natural (Shiryaev, 1998; Shiryaev et al., 1994), we have

\[
E^{\omega - r T} f_T(S(\omega, r, \delta)) = E[f_T(S(r, \delta))].
\]

Then, according with [5, 6], we obtain

\[
P_T = \exp(-rT) E^{\omega - r T} f_T(S(r, \delta)).
\]

By (2.2), (3.2)-(3.4), we have

\[
P_T = \exp(-rT) E^{\omega - r T} \left( K - S_0 \exp(m_T) \right) ^ = = \exp(-rT) F_T(S_0),
\]

\[
F_T(S_0) = E \left( K - S_0 \exp(m_T) \right) ^ =
\]

\[
= \int_{-\infty}^{0} \left( K - S_0 \exp(x) \right) \left( \exp(x) \right) \exp(-rT) F_T(S_0) \, dx.
\]

Using (3.5) in (4.8), we obtain

\[
\alpha^{-1} \exp(-r(T-t)) \Phi(d_2(t))
\]

if \( S_0 \leq K \); if \( S_0 > K \).

\[
\gamma^*(t) = \left[ (1 + \alpha^{-1}) \exp(-\delta(T-t)) \Phi(-d_1(t)) + \left( 1 - \alpha^{-1} \right) \exp(-r(T-t)) \Phi(d_2(t)) \right]
\]

if \( S_0 \leq K \); if \( S_0 > K \).

Theorem 2. A capital \( X^*_t \) and hedging strategy \( \pi^*_t = (\gamma^*_t, \beta^*_t) \) are defined by the formulas

\[
X^*_t = K \exp(-r(T-t)) -
\]

\[
- S_0 \left[ (1 + \alpha^{-1}) \exp(-\delta(T-t)) \Phi(-d_1(t)) + \left( 1 - \alpha^{-1} \right) \exp(-r(T-t)) \Phi(d_2(t)) \right]
\]

if \( S_0 \leq K \); if \( S_0 > K \).

\[
\gamma^*(t) = \left[ (1 + \alpha^{-1}) \exp(-\delta(T-t)) \Phi(-d_1(t)) + \left( 1 - \alpha^{-1} \right) \exp(-r(T-t)) \Phi(d_2(t)) \right]
\]

if \( S_0 \leq K \); if \( S_0 > K \).
\[ \beta^*_t = \frac{1}{B_t} \exp\left\{-r(T-t)\right\} \times \]
\[ \times \left[ K \Phi(y_1(t)) + S \left( \frac{K}{S} \right)^{\alpha} \Phi(y_2(t)) \right] \]  
(4.19)

if \( S_0 > K \).

**Proof.** According to (Shiryaev, 1998; Shiryaev et al., 1994), we have

\[ X^*_t = \exp\left\{-r(T-t)\right\} E\left[f_t(S(r, \delta)) \mid S_t \right] = \exp\left\{-r(T-t)\right\} F_{r,t}(S_t), \]
(4.20)

Thus, formulas (4.14), (4.17) arise from (4.2), (4.3) with replacements \( S_0 \to S_t, \ T \to (T-t) \). In accordance with (Shiryaev, 1998; Shiryaev et al., 1994), we have

\[ y^*_t = \frac{\partial X^*_t}{\partial S} \bigg|_{S=S_t}, \]
(4.21)

\[ \beta^*_t = \frac{X^*_t - y^*_t S_t}{B_t}. \]
(4.22)

The use of (4.21) in (4.14) brings us to (4.15). Formula (4.16) arises from (4.22), (4.14), (4.15).

From (4.17), we have

\[ \frac{\partial X^*_t}{\partial S} = -\left(1 + \alpha^{-1}\right) \exp\left\{-\delta(T-t)\right\} \Phi(y_1(t)) + \]
\[ + \left(1 - \alpha^{-1}\right) \exp\left\{-r(T-t)\right\} \Phi(y_2(t)) \left( \frac{K}{S} \right)^{\alpha} + \Psi, \]
(4.23)

\[ \Psi = \Psi_1 + s \alpha^{-1} \Psi_2, \]
(4.24)

\[ \Psi_1 = K \exp\left\{-r(T-t)\right\} \frac{\partial \Phi(y_1(t))}{\partial S} - \]
\[ -s \exp\left\{-\delta(T-t)\right\} e^{\sigma(T-t)} \frac{\partial \Phi(y_1(t))}{\partial S}, \]
(4.25)

\[ \Psi_2 = \exp\left\{-r(T-t)\right\} \left( \frac{K}{S} \right)^{\alpha} \frac{\partial \Phi(y_2(t))}{\partial S} - \]
\[ - \exp\left\{-\delta(T-t)\right\} \frac{\partial \Phi(y_1(t))}{\partial S}, \]
(4.26)

From (4.1), we obtain

\[ y_1(t) = y_1(t) - \sigma \sqrt{T-t}, \]
(4.27)

\[ y_1(t) = y_1(t) - 2 \frac{(r-\delta)}{\sigma} \sqrt{T-t}. \]
(4.28)

From definition of \( \Phi(x) \) follows that

\[ \frac{\partial \Phi(y_1(t))}{\partial S} = -\frac{1}{s \sigma \sqrt{2\pi(T-t)}} \exp\left\{-\frac{y_1^2(t)}{2\sigma^2}\right\}, \]
(4.29)

\[ \frac{\partial \Phi(y_2(t))}{\partial S} = -\frac{1}{s \sigma \sqrt{2\pi(T-t)}} \exp\left\{-\frac{y_2^2(t)}{2\sigma^2}\right\}. \]
(4.30)

Using (4.27) and (4.29), we obtain

\[ \frac{\partial \Phi(y_1(t))}{\partial S} = -\frac{1}{s \sigma \sqrt{2\pi(T-t)}} \exp\left\{-\frac{y_1^2(t)}{2\sigma^2}\right\}. \]
(4.31)

Using (4.28) and (4.30), we obtain

\[ \frac{\partial \Phi(y_2(t))}{\partial S} = -\frac{1}{s \sigma \sqrt{2\pi(T-t)}} \exp\left\{-\frac{y_2^2(t)}{2\sigma^2}\right\}. \]
(4.32)

The use of (4.29), (4.31) in (4.25) and (4.30), (4.32) in (4.26) brings us to \( \Psi_1 = 0, \ \Psi_2 = 0 \). Then, according to (4.24), \( \Psi = 0 \) and (4.18) arises from (4.21), (4.23), and (4.19) arises from (4.22), (4.17), (4.18).

**Theorem 3.** The cost of option is defined by formula

\[ P_T = K \exp\left\{-r(T-t)\right\} \Phi(y_1(t)) - S_0 \left(1 + \alpha^{-1}\right) \exp\left\{-\delta(T-t)\right\} \times \]
\[ \times \left[ \Phi(y_1(t)) - \Phi(-d_1) \right] + \left(1 - \alpha^{-1}\right) \exp\left\{-r(T-t)\right\} \Phi(-d_2) + \]
\[ + \alpha^{-1} \exp\left\{-r(T-t)\right\} \left(\frac{K}{S}\right)^{\alpha} \Phi(y_2(t)) \]
(4.33)

if \( S_0 \leq K \); and \( P_T = 0 \), if \( S_0 > K \).

**Proof.** Since payoff function \( f_t(S) \) of the form (1.2) is natural (Shiryaev, 1998; Shiryaev et al., 1994), we have

\[ E^{\mu-r\delta} \left\{ f_t(S(r, \delta))\right\} = E\left[f_t(S(r, \delta))\right]. \]

Then, according to (Shiryaev, 1998; Shiryaev et al., 1994), we obtain

\[ P_T = \exp\left\{-r(T-t)\right\} E\left[f_t(S(r, \delta))\right]. \]

Formulas (2.2), (3.2)-(3.5) imply that

\[ P_T = \exp\left\{-r(T-t)\right\} E\left[K - S_0 \exp\left\{M_{r,t}\right\}\right] = \]
\[ = \exp\left\{-r(T-t)\right\} F_r(S_0), \]
(4.34)

\[ F_r(S_0) = E\left[K - S_0 \exp\left\{M_{r,t}\right\}\right] = \]
\[ = \int_0^\infty \left(K - S_0 \exp\left\{x\right\}\right) p^M(T,x) dx. \]
(4.35)

a) Case \( S_0 \leq K \). The use of (3.4) in (4.5) brings us to

\[ F_r(S_0) = K \int_0^{\delta} p^M(T,x) dx - S_0 \int_0^{\delta} \exp\left\{x\right\} p^M(T,x) dx = \]
\[ = F^r_2 - F^r_1 = K \delta - S_0 J, \]
(4.36)

\[ J = J_1 - J_2 + J_3, \]
(4.37)
\[ J = J_1 - J_2 + J_3 \]

\[ J_1 = \frac{1}{\sigma \sqrt{2\pi T}} \int_0^b \exp\left( x \right)\exp\left( -\frac{(x-hT)^2}{2\sigma^2 T} \right) dx, \]

\[ J_2 = \frac{2h}{\sigma^2} \int_0^b \left( 1 + \frac{2h}{\sigma^2} x \right) \Phi\left( -\frac{(x+hT)}{\sigma \sqrt{T}} \right) dx = \frac{2h}{\sigma^2} J_3, \]

\[ J_3 = \frac{1}{\sigma \sqrt{2\pi T}} \int_0^b \left( \frac{x+hT}{\sigma \sqrt{T}} \right)^2 \Phi\left( -\frac{(x-hT)}{2\sigma^2 T} \right) dx, \]

\[ F_T(S_0) = F_T^2 - F_T^1, \quad (4.42) \]

\[ F_T^1 = K \int_{-\infty}^b p(T,x)dx, \quad (4.43) \]

\[ F_T^2 = S_0 \int_{-\infty}^b \exp(x)p(T,x)dx, \quad (4.44) \]

\[ \text{where } b = \ln\left( \frac{K}{S_0} \right). \]

Using (3.4), (3.5) in (4.43), (4.44), we obtain

\[ F_T^1 = K \left[ \Phi(y_1) + \left( \frac{K}{S_0} \right)^{a-1} \Phi(y_2) \right], \]

\[ F_T^2 = S_0 \left[ (1+\alpha^{-1}) \exp\left( (r-\delta)T \right) \Phi(y_1) + (1-\alpha^{-1}) \left( \frac{K}{S_0} \right)^a \Phi(y_2) \right]. \quad (4.45) \]

Substitution of (4.45) in (4.42), (4.36) gives (4.35).

REFERENCES


