On structure of the value function

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Abstract: The value function (or the optimal result function) arising in optimal control problems with the Bolza pay-off functional are actual to optimal control theory and applications. See, Bellman (1957), Pontryagin, etc (1962), Krasovskii (1968), Warga (1977), Bardi (1997).

We consider controlling systems of the form

\[ \dot{x}(t) = f(t, x, u), \quad t \in [0, T], \quad x \in \mathbb{R}^n, \]

\[ x(t_0) = x_0, \quad (t_0, x_0) \in [0, T] \times \mathbb{R}^n. \]

Here the state \( x \in \mathbb{R}^n \), values of controls \( u \) belong to a
given compact set \( P \subset \mathbb{R}^m \), a time interval \([0, T]\) is fixed.

We put the set \( U_{[t_0, T]} \) of admissible open-loop controls
equal to the set of all measurable functions \( u(\cdot) : [t_0, T] \mapsto P \).

We estimate quality of an open-loop control at the initial state \((t_0, x_0) \in [0, T] \times \mathbb{R}^n\) with the help of the Bolza pay-off functional:

\[ I_{t_0, x_0} (u(\cdot)) = \sigma(x(T)) + \int_{t_0}^{T} g(t, x(t), u(t)) dt, \]

where \( x(\cdot) = x(\cdot, t_0, x_0, u(\cdot)) : [t_0, T] \mapsto \mathbb{R}^n \) is a trajectory of system (1) started at the state \((t_0, x_0) \) under an
admissible control \( u(\cdot) \in U_{[t_0, T]} \).

The symbol \( \Pi_T \) denotes the strip

\[ \Pi_T = \{(t, x) : t \in [0, T], x \in \mathbb{R}^n\}. \]

The optimal result \( \varphi(t_0, x_0) \) at the initial state \((t_0, x_0) \in \Pi_T \) is defined by the equality:

\[ \varphi(t_0, x_0) = \inf_{u(\cdot) \in U_{[t_0, T]}} I_{t_0, x_0} (u(\cdot)). \]

The function

\[ (t_0, x_0) \mapsto \varphi(t_0, x_0) : \Pi_T \mapsto \mathbb{R} \]
is called the value function of optimal control problem (1)–(3) in strip \( \Pi_T \ni (t_0, x_0) \).

The value function takes important place in solving problems (1)–(2). According to (3), the function estimates possibilities of controls at each initial state \((t_0, x_0) \in \Pi_T \). Moreover, the value function is a key element of constructions of optimal open-loop and closed-loop controls. See, for example, Pontryagin, etc (1962), Krasovskii (1968), Subbotin (1995).

The value function is nonsmooth, as a rule. We study the structure of the value function to use the information in constructions of optimal feedbacks, in the future.

2. APPLICATIONS OF THE HAMILTON-JACOBI EQUATIONS THEORY

It is well-known (Bellman (1957)), that the value function \( \varphi(t, x) \) of problem (1)–(3) satisfies the following boundary condition (at all \( x \in \mathbb{R}^n \)), and the Hamilton-Jacobi-Bellman equation (at points of differentiability)

\[ \varphi_t(t, x) + H(t, x, \varphi_x(t, x)) = 0, \quad \varphi(T, x) = \sigma(x), \]

\[ t \in [0, T], \quad x \in \mathbb{R}^n. \]

Here

\[ \varphi_t(t, x) = \partial \varphi(t, x)/\partial t, \]

\[ \varphi_x(t, x) = (\partial \varphi(t, x)/\partial x_1, \ldots, \partial \varphi(t, x)/\partial x_n), \]

the Hamiltonian \( H(t, x, s) \) of the problem (1)–(3) has the form

\[ H(t, x, s) = \min_{u \in P} \langle s, f(t, x, u) + g(t, x, u) \rangle, \]

where the symbol \( \langle s, f \rangle \) denotes the inner product of vectors \( s \) and \( f \).

We consider the boundary problem (4) under the following assumptions about the input data \( H(t, x, s) \) and \( \sigma(x) \):
A1 the Hamiltonian $H(t,x,s)$ is twice continuously differentiable in all variables, at any $(t,x,s) \in (0,T) \times \mathbb{R}^n$;
A2 the boundary function $\sigma(x)$ is continuously differentiable;
A3 the functions $\partial H(t,x,s)/\partial x_i$, $\partial H(t,x,s)/\partial s_j$, $i,j \in \{1,\ldots,n\}$, possess sublinear growth in $x$ and $s$, with a constant $K > 0$.

It is well known (see, for example, Subbotin (1995)) that the classical, smooth solution of problem (4) can be obtained with the help of the Cauchy method of characteristics.

Consider the characteristic system for problem (4)
\[
\dot{x} = H_x(t,\tilde{x},\tilde{s}), \quad \dot{s} = -H_s(t,\tilde{x},\tilde{s}), \quad \tilde{x} = (\tilde{s}, H_x(t,\tilde{x},\tilde{s})) - H(t,\tilde{x},\tilde{s}),
\]
and the boundary conditions
\[
x(T) = \xi, \quad s(T) = \sigma_x(\xi), \quad \tilde{z}(T) = \sigma(\xi), \quad \xi \in \mathbb{R}^n. \tag{6}
\]

Here
\[
H_x(t,\tilde{x},\tilde{s}) = (\partial H/\partial s_1,\ldots,\partial H/\partial s_n),
\]
\[
H_s(t,\tilde{x},\tilde{s}) = (\partial H/\partial x_1,\ldots,\partial H/\partial x_n),
\]
\[
\sigma_x(\xi) = (\partial \sigma(\xi)/\partial x_1,\ldots,\partial \sigma(\xi)/\partial x_n),
\]
\[
\xi \in \mathbb{R}^n \text{ is } n\text{-dimensional parameter.}
\]

Conditions A1–A3 provide existence, uniqueness and extendability on the interval $[0,T]$ for trajectories $\tilde{x}(\cdot,\xi)$, $\tilde{s}(\cdot,\xi)$, $\tilde{z}(\cdot,\xi)$, of system (5)–(6), for each $\xi \in \mathbb{R}^n$. The trajectories of the Hamiltonian characteristic system are called the characteristics of problem (4).

The Cauchy method of characteristics provides the construction of the classical solution of problem (4) only in domains where the phase components $\tilde{x}(\cdot,\xi)$ of characteristics (5)–(6) are not intersecting.

If the input data $H(t,x,s)$, $\sigma(x)$ are smooth, but nonlinear, then the phase components $\tilde{x}(\cdot,\xi)$ of characteristics (5)–(6) can be intersected close to initial manifold (6). The problem (4) under assumptions A1–A3 doesn’t have the global classical solution in $\Pi_T$, as a rule.

2.1 Generalized solutions of the Hamilton-Jacobi-Bellman equations

Let us remember the following notions of nonsmooth analysis and viability theory (see, for example, Clarke (1983), Subbotin (1995)) to consider notions of generalized solutions to problem (4) in the whole strip $\Pi_T$.

**Definition 1.** A set $\partial^+\varphi(t,x) \in \mathbb{R}^{n+1}$ is called the superdifferential of the function $\varphi(\cdot,\cdot) : \Pi_T \rightarrow \mathbb{R}$ at the point $(t,x) \in (0,T) \times \mathbb{R}^n$ if, for some $\varepsilon > 0$, it satisfies the relations
\[
\partial^+\varphi(t,x) = \{(\alpha,p) \in \mathbb{R}^{n+1} : \forall|\delta t| + |\delta x| \leq \varepsilon, \varphi(t+\delta t,x+\delta x) - \varphi(t,x) \leq \alpha\delta t + p\delta x + o(|\delta t| + |\delta x|)\}
\]
where $o(|\delta t| + |\delta x|)/(|\delta t| + |\delta x|) \rightarrow 0$, $\alpha|\delta t| + |\delta x| \rightarrow 0$.

**Definition 2.** The graph of a continuous function $(t,x) \rightarrow \varphi(t,x) : \Pi_T \rightarrow \mathbb{R}$ is weakly invariant relative to system (5)–(6) if for any $(t_0,x_0) \in \Pi_T$ there exists $\xi \in \mathbb{R}$, such, that solutions $\tilde{x}(\cdot,\xi)$, $\tilde{z}(\cdot,\xi)$ of (5)–(6) satisfy the equalities
\[
\tilde{x}(t_0,\xi) = x_0, \quad \tilde{z}(t_0,\xi) = \varphi(t_0,x_0);
\]
\[
\tilde{z}(t,\xi) = \varphi(t,\tilde{x}(t,\xi)), \quad \forall t \in [t_0,T].
\]

Consider the following notion of a generalized solution to problem (4) in $\Pi_T$.

**Definition 3.** A continuous function $(t,x) \rightarrow \varphi(t,x) : \Pi_T \rightarrow \mathbb{R}$ is called the global generalized solution of problem (4) in $\Pi_T$ if
\begin{itemize}
  \item the boundary condition
  \[
  \varphi(T,x) = \sigma(x), \quad \forall x \in \mathbb{R}^n
  \]
  is satisfied;
  \item for all $(t,x) \in (0,T) \times \mathbb{R}^n$ the superdifferential $\partial^+\varphi(t,x)$ is nonempty and bounded, i.e.
  \[
  \exists R(t,x) \in (0,\infty) : \forall h = (\alpha,s) \in \partial^+\varphi(t,x) \Rightarrow \|h\| \leq R(t,x);
  \]
  \item the graph of $\varphi(t,x)$ is weakly invariant relative to system (5)–(6).
\end{itemize}

The symbol $\|h\|$ denotes the Euclidean norm of the vector $h \in \mathbb{R}^n$.

It is proven (see, Subbotina (2009), Kolpakova (2010)), that definition 3 is equivalent to the well-known definitions of minimax (Subbotin (1995)), and viscosity (Crandall, Lions (1983)) solutions to problem (4). The equivalence and results of the theory of minimax and viscosity solutions to the Hamilton-Jacobi equations imply the following assertion.

**Theorem 1.** If assumptions A1–A3 are true, then there exists a function $\varphi(\cdot,\cdot) : \Pi_T \rightarrow \mathbb{R}$ satisfying definition 3 of the global generalized solution to problem (4) in $\Pi_T$. The solution $\varphi(\cdot,\cdot)$ is unique and coincides with the value function of problem (1)–(3).

2.2 Properties of the value function $\varphi(t,x)$

According to theorem 1, the structure of the value function $\varphi(t,x)$ of problem (1)–(3) is defined via definitions 1–3.

So, the following properties of the value function are obtained in the framework of the theory of minimax and viscosity solutions to problem (4) (see, Crandall, Lions (1983), Subbotin (1995), Subbotina (2006), Bardi (1997), Evans (1998)).

**Theorem 2.** If assumptions A1–A3 are valid in problem (4), then the value function $\varphi(t,x)$ of problem (1)–(3) has the properties:
\begin{itemize}
  \item the value function $\varphi(t,x)$ is locally Lipshitz continuous;
  \item for any $(t,x) \in \Pi_T$, the representative formula
  \[
  \varphi(t,x) = \min_{\xi \in \mathbb{R}^n} \{\tilde{z}(t,\xi) : \tilde{x}(t,\xi) = x\}, \tag{7}
  \]
  is true;
\end{itemize}
for any \((t,x) \in (0,T) \times \mathbb{R}^n\), the superdifferential 
\(D^+\varphi(t,x)\) is not empty and has the form
\[
D^+\varphi(t,x) = \text{co}\{(H(t,x,\tilde{s}(t,\xi)),\tilde{s}(t,\xi)) : \tilde{x}(t,\xi) = x, \}
\]
\[\varphi(t,x) = \tilde{z}(t,\xi)\}.
\]
Here \(\tilde{x}(\cdot,\xi), \tilde{s}(\cdot,\xi), \tilde{z}(\cdot,\xi)\) are solutions of the characteristic system \((5)-(6)\).

Let us introduce the following definition.

**Definition 4.** The set \(Q \subset \Pi_T\) is called the singular set of the value function \(\varphi(\cdot,\cdot)\) of problem \((1)-(3)\) iff it contains all points \((t,x)\in \Pi_T\) where the function \(\varphi(\cdot,\cdot)\) is not differentiable.

According to theorem 2 and the Rademacher’s theorem, the local Liptshitz continuous value function is differentiable almost everywhere.

Let \(Q_T = \{x \in \mathbb{R}^n : (t',x) \in Q\}, \ t' \in [0,T]\).

Definition 2–4, and uniqueness of solutions of the Hamiltonian characteristic system \((5)-(6)\), for any parameter \(\xi \in \mathbb{R}^n\), imply that the singular set \(Q\) of the value function \(\varphi(\cdot,\cdot)\) has the following properties.

**Theorem 3.** Let assumptions A1–A3 be true. A point \(x \in Q_T\), iff there exist such parameters
\[\xi_1 \in \mathbb{R}^n, \ \xi_2 \in \mathbb{R}^n, \ \xi_1 \neq \xi_2,\]
that
\[\tilde{x}(t',\xi_1) = \tilde{x}(t',\xi_2) = x, \ \tilde{z}(t',\xi_1) = \tilde{z}(t',\xi_2) = \varphi(t',x);\]
\[\tilde{s}(t',\xi_1) \neq \tilde{s}(t',\xi_2),\]
\[\tilde{z}(t,\xi_1) = \varphi(t,\tilde{x}(t,\xi_1), \ \tilde{z}(t,\xi_1) = \varphi(t,\tilde{x}(t,\xi_1),\]
\[\tilde{s}(t,\xi_1) \in \partial^+_\varphi(t,\tilde{x}(t,\xi_1), \ \tilde{s}(t,\xi_2) \in \partial^+_\varphi(t,\tilde{x}(t,\xi_2),\]
for all \(t \in [t',T]\).

The symbol \(\partial^+_\varphi(t,x)\) denotes the projection of the set \(\partial \varphi(t,x)\) to the space \(\mathbb{R}^n \ni p\), namely,
\[\partial^+_\varphi(t,x) = \{p \in \mathbb{R}^n : \exists \alpha \in R \ \ (\alpha,p) \in \partial \varphi(t,x)\}.
\]

3. APPLICATIONS OF THE THEORY OF QUASILINEAR EQUATION

3.1 The initial value problem for quasilinear equation

Consider optimal control problem \((1)-(3)\) for the case of one-dimensional state space \(R \ni x\). Let a function \(\varphi(t,x)\) be the value function of problem \((1)-(3)\).

The value function is defined for all \((t,x)\in \Pi_T\). According to theorem 1, the function is the unique minimax and viscosity solution to the boundary problem for Hamilton-Jacobi-Bellman equation \((4)\).

Consider also the initial value problem for one-dimensional quasilinear equation

\[
w_\tau - H_x(\tau,x,w) = 0, \ w(0,x) = \sigma_x(x), \ \tau \in [0,T], \ x \in R.
\]

The input data of problem \((13)\) can be considered as a result of differentiation in \(x\) of the input data for problem \((4)\), and after the time transformation
\[\tau = T - t, \ t \in [0,T].\]

It is well known, that problem \((13)\) doesn’t have a continuous global solution, as a rule.

Consider the characteristic system for problem \((13)\)
\[
\dot{x} = -H_x(\tau,x,\dot{s}), \ \dot{s} = H_\tau(\tau,x,\dot{s}),
\]
and the boundary conditions
\[\dot{x}(0,\xi) = \xi, \ \dot{s}(0,\xi) = \sigma_x(\xi), \ \xi \in R.\]

We introduce the following notion.

**Definition 5.** A function \(w(\cdot,\cdot) : \Pi_T \rightarrow R\) is called the global generalized solution of problem \((13)\) in stripe \(\Pi_T\), if the following conditions are satisfied:

**B1** for each point \((\tau_1,x_1)\in \Pi_T\), where the function \(w(\cdot,\cdot)\) is continuous, there exists the unique characteristic \((\tilde{x}(\cdot,\xi), \tilde{s}(\cdot,\xi))\) defined by \((14)-(15)\), such, that
\[\tilde{x}(\tau_1,\xi) = x_1, \ \tilde{s}(\tau,\xi) = w(\tau,\tilde{x}(\tau,\xi)), \ 0 \leq \tau \leq \tau_1;
\]

**B2** if the point \((\tau_1,x_1)\in \Pi_T\) is the point of discontinuity of the function \(w(\cdot,\cdot)\), then there exist at least two characteristics \((\tilde{x}(\cdot,\xi), \tilde{s}(\cdot,\xi))\) and \((\tilde{x}(\tau,\xi), \tilde{s}(\tau,\xi))\), such, that
\[\tilde{x}(\tau_1,\xi_1) = x_1, \ \tilde{s}(\tau,\xi_1) = w(\tau,\tilde{x}(\tau,\xi_1)), \ 0 \leq \tau \leq \tau_1;
\]

**B3** for any closed curve \(C \subset \Pi_T\), the equality is true
\[
\int_C H(\tau,x,w(\tau,x))d\tau = w(\tau,x)dx = 0.
\]

Functions satisfying definition 5 can be discontinuous functions with finite jumps. Two generalized solutions of problem \((13)\) satisfying definition 5 are equal, if they are equal at all points of continuity for each of the solutions.

Let us show a link between the global generalized solution of the problem \((13)\) satisfying definition 5, and the generalized solution introduced by O.A. Oleinik (Oleinik (1954)).

Consider a domain \(G_0 = G_0([a,b]) \subset \Pi_T\) of the form:
\[G_0 = \{(\tau,x) \in \Pi_T : \tau \in [0,T^*], \ \tilde{x}(\tau,a) \leq x \leq \tilde{x}(\tau,b),\}
\]
where \(\tilde{x}(\tau,a), \tilde{x}(\tau,b)\) are solutions of characteristic system \((14)\) satisfying the boundary conditions
\[\tilde{x}(0,a) = a, \ \tilde{x}(0,b) = b, \ T^* = \min_{\tau \in [0,T]} \{\tau^* : \tilde{x}(\tau^*,a) \leq \tilde{x}(\tau^*,b)\}, \text{ and } T^* = T.
\]

Remember the Oleinik’s definition.

**Definition 5’.** A function \(w(\cdot,\cdot) : G_0 \rightarrow R\) is called the generalized solution of problem \((13)\) in domain \(G_0\), if the following conditions are satisfied:

**B1’** for each point \((\tau_1,x_1)\in G_0\), where the function \(w(\cdot,\cdot)\) is continuous, there exists the unique characteristic \((\tilde{x}(\cdot,\xi), \tilde{s}(\cdot,\xi))\) defined by \((14)-(15)\), such, that
\[\xi \in [a,b], \ \tilde{x}(\tau_1,\xi) = x_1, \ \tilde{s}(\tau,\xi) = w(\tau,\tilde{x}(\tau,\xi)), \ 0 \leq \tau \leq \tau_1;
\]

**B2’** if the point \((\tau_1,x_1)\in G_0\) is the point of discontinuity of the function \(w(\cdot,\cdot)\), then there exist at least two
characteristics defined by (14)–(15): \((\hat{x}(\tau, \xi_1), \hat{s}(\tau, \xi_1))\) and \((\hat{x}(\tau, \xi_2), \hat{s}(\tau, \xi_2))\), such that
\[\xi_1 \in [a, b], \xi_2 \in [a, b], \xi_1 \neq \xi_2,\]
\[\hat{x}(t_1, \xi_1) = x(t_1, \xi_2) = x_1,\]
\[\hat{s}(\tau, \xi_1) = w(\tau, \hat{x}(\tau, \xi_1)), \quad 0 \leq \tau \leq t_1,\]
\[\hat{s}(\tau, \xi_2) = w(\tau, \hat{x}(\tau, \xi_2)), \quad 0 \leq \tau \leq t_1;\]
B3’ the equality
\[\int_C H(\tau, x(w(\tau, x))) d\tau - w(\tau, x) dx = 0 \quad (18)\]
is true, for any closed curve \(C' \subset G_0\) coinciding with the boundary of a domain \(G' \subset G_0\) of the form
\[G' = \{(\tau, x) \in G_0 : \tau \in [0, t'], t' \subset [0, T]\},\]
\[\hat{x}(\tau, \xi_1) \leq x \leq \hat{x}(\tau, \xi_2),\]
where
\[a \leq \xi_1 < \xi_2 \leq b, \quad \hat{x}(t', \xi_1) = \hat{x}(t', \xi_2) = x',\]
characteristics \(\hat{x}(\tau, \xi_1), \hat{x}(\tau, \xi_2)\) satisfying condition B2’.

One can see that a generalized solution of problem (13) satisfying definition 5’ is not global, it is defined in domain \(G_0 \subset \Pi_T\). Conditions B1, B2, and B1’, B2’ coincide in domain \(G_0\). Condition B3’ follows from B3.

The following assertion is true (see, Oleinik (1954)).

Theorem 4. If assumptions A1 – A3 are true, then there exists the generalized solution \(w(\cdot, \cdot) : G_0 \to R\) satisfying definition 5’, and it is unique. The solution has the following properties:

- all break points of the generalized solution \(w(\cdot, \cdot)\) lie on not more than countable set of lines \(\tau \to x(\tau)\) satisfying the Rankine—Hugoniot condition

\[\frac{dx}{d\tau} = \frac{H(\tau, x(x(\tau)), w+(\tau, x(\tau))) - H(\tau, x(\tau), w-(\tau, x(\tau)))}{w+(\tau, x(\tau)) - w-(\tau, x(\tau))} \quad (20)\]

where
\[w+(\tau, x(\tau)) = \lim_{\tau \to (\tau, x(\tau)) +} w(\tau, x),\]
\[w-(\tau, x(\tau)) = \lim_{\tau \to (\tau, x(\tau)) -} w(\tau, x);\]

- the condition
\[w+(\tau, x(\tau)) < w-(\tau, x(\tau)) \quad (21)\]
is valid on the Rankine—Hugoniot lines \(\tau \to x(\tau)\).

Let us consider the unique minimax solution \(\varphi(t, x)\) of problem (4) (the boundary problem for Hamilton-Jacobi equation).

We introduce a function \(w^*(\cdot, \cdot) \to R\) satisfying the inclusion
\[w^*(\tau, x) \in \partial^+ \varphi(t, x), \quad \forall (\tau, x) \in \Pi_T, \quad t = T - \tau \quad (22)\]

According to theorem 2, the function \(w^*(\tau, x)\) satisfies the equality
\[w^*(\tau, x) = \varphi(t, x), \quad t = T - \tau,\]
almost everywhere in \(\Pi_T\).

Let us prove the assertion.

Theorem 5. The function \(w^*(\cdot, \cdot) \to R\) satisfying (22) is the global generalized solution of problem (13).

Proof. We check that the function \(w^*(\tau, x)\) defining by (22) satisfies definition 5.

Consider the restriction \(w_0(\tau, x)\) of the function \(w^*(\tau, x)\) to a domain \(G_0\) of the form (17).

At first, check condition B1’, B2’ for the function \(w_0(\tau, x)\). From the equality
\[w_0(\tau, x) = w^*(\tau, x), \quad \forall (\tau, x) \in G_0,\]
and theorem 2, we get that the function \(w_0(\tau, x)\) is continuous at points of differentiability of the minimax solution \(\varphi(t, x)\) of problem (4).

Let \((\tau', x') \in G_0\) be a point, where the function \(w_0(\tau, x)\) is continuous, and consider \(t' = T - \tau'\). It follows from theorems 2 and 3, there is that there is a characteristic \(\tilde{x}(t, \xi)\) defined by (5)–(6) and arrived to a point of differentiability \((t', x')\) of the minimax solution \(\varphi(t, x)\) of problem (4), so, that
\[\tilde{x}(t', \xi) = x', \quad \tilde{s}(t, \xi) \in \partial^+_x \varphi(t, \tilde{x}(t, \xi)), \quad \forall t \in [t', T]. \quad (23)\]
The characteristics \(\tilde{x}(t, \xi)\) coincides with the characteristic \(\hat{x}(t, \xi)\) defined by (14)–(15), so, that, for all \(\tau \in [0, T]\), the relations hold
\[\hat{x}(t, \xi) = \tilde{x}(t, \xi), \quad t = T - t, \quad (\tau, \hat{x}(t, \xi)) \in G_0. \quad (24)\]

Now consider a point \((\tau', x') \in G_0 \cap Q\), where the function \(w_0(\tau, x)\) is discontinuous, and \(t' = T - \tau'\). According to theorems 2 and 3, there are at least two characteristics \(\tilde{x}(t, \xi), \tilde{x}(t, \xi)\) defined by (5)–(6), \(\xi_1 \neq \xi_2\), and arrived to the point \((t', x') \in Q\), where the superdifferential \(\partial^+_x \varphi(t, x)\) is not a singleton. It is true, for \(i = 1, 2\), that
\[\tilde{x}(t, \xi) = x', \quad \tilde{s}(t, \xi) \in \partial^+_x \varphi(t, \tilde{x}(t, \xi)), \quad \forall t \in [t', T]. \quad (25)\]
The characteristics \(\tilde{x}(t, \xi), \tilde{x}(t, \xi)\) coincide with characteristics \(\hat{x}(t, \xi), \hat{x}(t, \xi)\) defined by (14)–(15) so, that, for all \(\tau \in [0, T]\), \(i = 1, 2\), the relations hold
\[\hat{x}(t, \xi) = \tilde{x}(t, \xi), \quad t = T - t, \quad (\tau, \hat{x}(t, \xi)) \in G_0. \quad (26)\]

It means that the function \(w_0(\tau, x)\) satisfies conditions B1’, B2’ and B1, B2 in \(G_0\).

Let us check condition B3’ in \(G_0\).

We calculate the integral
\[\int_{C'} w_0(\tau, x) dx - H(t, x, w_0(\tau, x)) d\tau, \quad (27)\]
where the curve \(C'\) has the form (19).

Transforming this integral we get the result
\[\int_{t_1}^{t_2} \int_0^{\hat{s}(\tau, \xi_1)} H_1(\tau, \hat{x}(\tau, \xi_1), \hat{s}(\tau, \xi_1)) + H(\tau, \hat{x}(\tau, \xi_1), \hat{s}(\tau, \xi_1)) d\tau + \int_0^{t_1} \hat{s}(\tau, \xi_2) H_2(\tau, \hat{x}(\tau, \xi_2), \hat{s}(\tau, \xi_2)) + H(\tau, \hat{x}(\tau, \xi_2), \hat{s}(\tau, \xi_2)) d\tau = \]
\[\sigma(\xi_2) - \sigma(\xi_1) + \int_0^{t_1} \hat{s}(\tau, \xi_1) d\tau + \int_0^{t_1} \hat{s}(\tau, \xi_2) d\tau = 0\]
because of (9). Here \( \tilde{z}(\cdot, \xi), \tilde{s}(\cdot, \xi), \tilde{z}(\cdot, \xi) \) are solutions of the characteristic system (5)-(6). Hence, condition B3’ is satisfying.

Using theorem 4, we have got that the restriction \( w_0(\tau, x) \) of the function \( w^*(\tau, x) \) (22) to domain \( G_0 = G_0([a, b]) \) coincides with the unique generalized solution of problem (13) satisfying definition 5’.

Consider a sequence of domains \( G_k = G_k([a_k, b_k]) \) expanding to the stripe \( \Pi_T \), as \( a_k \to -\infty, b_k \to +\infty \).

Consider an expansion \( w(\tau, x) \) of function \( w_0(\tau, x) \) defining on \( G_k \) as the unique generalized solution of problem (13) satisfying definition 5’. This expansion exists and it is unique in the whole stripe \( \Pi_T \). Hence, it coincides with the function \( w^*(\tau, x) \) (22) at points of continuity \( w(\tau, x) = w^*(\tau, x), (\tau, x) \in \Pi_T \).

It follows from theorem 4 that the set of points of discontinuity of function \( w^*(\tau, x) \) consists of not more than countable set of lines \( \tau \to x(\tau) \) satisfying the Rankine—Hugoniot condition (20), and the condition (21)

\[
\begin{align*}
    w^*_+(\tau, x(\tau)) &< w^*_-(\tau, x(\tau)) \\
\end{align*}
\]

is valid on the Rankine—Hugoniot lines.

Let us prove that the function \( w^*(\tau, x) \) (22) satisfies condition B3.

Consider a closed curve \( C \). The curve \( C \) consists of not more than countable set of intervals \( \Gamma_i = \{ t = t(r), x = x(r), r \in [t_i, t_{i+1}] \} \), on which the function \( w^*(t(r), x(r)) \) is either continuous or discontinuous.

We calculate the integrals (27) along the intervals \( \Gamma_i \).

Let \( \Gamma_1 \) be an interval of the curve \( C \), where the function \( w^*(t(r), x(r)) \) is continuous. The equality \( w^*(t(r), x(r)) = \varphi_x(t(r), x(r)) \) is valid at the points of such curve \( \Gamma_i \). Then the following expression

\[
d\varphi(\tau, x) = w^*(\tau, x)dx - H(\tau, x, w^*(\tau, x))d\tau
\]

is true along \( \Gamma_i \).

So, we have

\[
\int_{\Gamma_i} w^*(\tau, x)dx - H(\tau, x, w^*(\tau, x))d\tau = \int_{t_{i+1}}^{t_i} d\varphi(t(r), x(r)) = \varphi(t_{i+1}, x_{i+1}) - \varphi(t_i, x_i).
\]

Let \( \Gamma_j \) be an interval of the curve \( C \), and the interval \( \Gamma_j = \{ t, x(t), t \in [t_j, t_{j+1}] \} \) satisfies the Rankine—Hugoniot condition (20)

\[
\frac{dx}{d\tau} = H(\tau, x(\tau), w_+(\tau, x(\tau))) - H(\tau, x(\tau), w_-(\tau, x(\tau))) = f(\tau).
\]

The function \( w^*(\tau, x) \) has a jump on this interval \( \Gamma_j \) of curve \( C \).

Using (20), we calculate the integrals

\[
\begin{align*}
    \int_{\Gamma_j} w^*_+(\tau, x)dx - H(\tau, x, w^*_+(\tau, x))d\tau = \\
    \int_{t_{j+1}}^{t_j} -w_+(\tau, x(\tau))H(\tau, x(\tau), w^*_+(\tau, x(\tau)))d\tau + \\
    \int_{t_j}^\tau w^*_+(\tau, x(\tau))d\tau.
\end{align*}
\]

We define the integral

\[
\int_{\Gamma_j} w^*(\tau, x)dx - H(\tau, x, w^*(\tau, x))d\tau = \int_{t_{j+1}}^{t_j} w^*_-(\tau, x(\tau))d\tau - H(\tau, x, w^*_-(\tau, x(\tau)))d\tau.
\]

Consider the derivative of function \( \varphi(\cdot, \cdot) \) at the point \( (\tau, x(\tau)) \) in the direction \( (1, f(\tau)) \). The derivative \( \frac{d\varphi(\tau, x)}{d(1, f(\tau))} \) has the form (see, Rockafellar (2005))

\[
\frac{d\varphi(\tau, x)}{d(1, f(\tau))} = \min_{\alpha, \beta} \left\{ \beta + \alpha f(\tau) \right\}.
\]

From the condition (8) and equality \( w^*(\tau, x) = \varphi_x(\tau, x) \) valid for almost all \( (\tau, x) \in \Pi_T \), we get the representation

\[
\partial^+ \varphi(\tau, x(\tau)) = \min \left\{ \left( H(\tau, x(\tau), w^*_+(\tau, x(\tau))), w^*_+(\tau, x(\tau)), w^*_-(\tau, x(\tau)) \right), \left( H(\tau, x(\tau), w^*_+(\tau, x(\tau))), w^*_+(\tau, x(\tau)) \right) \right\}.
\]

Hence,

\[
\frac{d\varphi(\tau, x(\tau))}{d(1, f(\tau))} = H(\tau, x(\tau), w^*_+(\tau, x(\tau))) + f(\tau)w^*_+(\tau, x(\tau)) = \\
= \frac{-w^*_+(\tau, x(\tau))H(\tau, x(\tau), w^*_+(\tau, x(\tau)))}{w^*_+(\tau, x(\tau)) - w^*_-(\tau, x(\tau))} + \\
\frac{w^*_+(\tau, x(\tau))H(\tau, x(\tau), w^*_+(\tau, x(\tau)))}{w^*_+(\tau, x(\tau)) - w^*_-(\tau, x(\tau))}.
\]

The last expression coincide with (28).

So, we have

\[
\int_{t_{j+1}}^{t_j} -w^*_+(\tau, x(\tau))H(\tau, x(\tau), w^*_+(\tau, x(\tau)))d\tau + \int_{t_j}^{t_{j+1}} w^*_+(\tau, x(\tau))d\tau.
\]
\[
\int_{t_j}^{t_{j+1}} \left( w_+^*(\tau, x(\tau)) - w_-^*(\tau, x(\tau)) \right) d\tau,
\]
\[
= \varphi(t_{j+1}) - \varphi(t(j)).
\]
Summing integrals along all curves \( \Gamma_i, \Gamma_j \), we get
\[
\int_{\gamma} H(\tau, x, w^*(\tau, x)) d\tau - w^*(\tau, x) dx = 0.
\]

So, it is proven for the function \( w^*(\cdot, \cdot) \) defined by (22), that conditions \( B1-B3 \) in definition 5 are valid. Hence, the function \( w^*(\cdot, \cdot) \) is the global generalized solution of problem (13).

4. ON STRUCTURE OF THE VALUE FUNCTION.

We have considered the value function \( \varphi(t, x) \) of optimal control problem (1)–(3) as the unique minimax and viscosity solution of problem (4) in strip \( \Pi_T \), according to theorem 1. We have provided a description of the singular set \( Q \) of the value function in theorems 2 and 3. The set contains all points \((t, x)\) where the nonempty superdifferential \( \partial^* \varphi(t, x) \) is not a singleton.

In the case of one-dimensional state space, we have considered a selector \( w^*(t, x) \) defined by (22) of the superdifferential \( \partial^* \varphi(t, x) \). The singular set \( Q \) of the value function \( \varphi(t, x) \) in optimal control problem (1)–(3) coincides with the set all points \((t, x)\) \( \in \Pi_T \), where the function \( w^*(t, x) \) defined by (22) is discontinuous.

We have proven in theorem 5 that the function \( w^*(t, x) \) defined by (22) is equal to the global generalized solution to the auxiliary problem (13). The function \( w^*(\cdot, \cdot) : \Pi_T \to R \) satisfies definition 5’ in domains \( G_k = G([a_k, b_k]) \subset \Pi_T, a_k \to -\infty, b_k \to \infty \).

Theorem 4 provides the following new properties of the singular set \( Q \).

Theorem 6. If assumptions \( A1-A3 \) are valid in optimal control problem (1)–(3) and the state space is one-dimensional, then the singular set \( Q \) of the value function \( \varphi(t, x) \) consists of not more than countable number of lines \( t \to x_*(t) \) : \( 0 \leq t < t_* \leq T \) satisfying the Rankine-Hugoniot condition (20).

The inequality
\[
\lim_{x \to x_*(t) - 0} \varphi_x(t, x) < \lim_{x \to x_*(t) + 0} \varphi_x(t, x).
\]
is valid on the Rankine-Hugoniot lines \( t \to x_*(t) \).

5. CONCLUSION

The paper deals with structure of the value function in optimal control problem with the Bolza pay-off functional. We studied the set of points where the value function is not differentiable. The description of the set is important to constructions of optimal feedbacks. The presented link between generalized solutions to boundary problems for the Hamilton-Jacobi-Bellman equation and the quasilinear equation of the first order can be useful to study and construct solutions of optimal control problems and one dimensional conservations laws, too (see Kolpakova (2010)).

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REFERENCES


