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Estimation of the domain of attraction for non-polynomial systems: A novel method

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Abstract: Until now, several methods have been developed to compute the domains of attraction for stationary points of polynomial non-linear systems. For the case of non-polynomial systems, though, this question is still open. In this paper a new method, based on Lyapunov's stability theory and the theorem of Ehlich and Zeller, is presented for the computation of domains of attraction of non-polynomial systems with quadratic Lyapunov functions. Unlike other methods, which use a polynomial approximation for the non-polynomial terms, we compute upper bounds on the interpolation error for each of the non-polynomial terms. Then, the theorem of Ehlich and Zeller is adapted to non-polynomial systems. Our method yields, for a given Lyapunov function, upper and lower bounds for the level curve enclosing the domain of attraction.

Keywords: Stability analysis, Stability domains, Nonlinear systems, Lyapunov methods, Lyapunov stability

1. INTRODUCTION

It is easy to see that the stability analysis of dynamical nonlinear systems is important. Unlike for linear systems, where a well-established mathematical theory is available, it is still a challenge to analyze the stability of nonlinear systems.

Such systems, which are described by

$$\dot{x} = f(x)\ ,\ x(0) = x^0\ , \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state vector and $f(x)$ is a nonlinear function of the state vector, can have one or several equilibrium points. In the following we assume that 0 is an equilibrium point.

The set of all initial conditions from which the trajectory of the system converges to the equilibrium point is called domain of attraction and is defined by

$$\Omega = \left\{ x^0 \,\middle|\, \lim_{t\to\infty} x(t) = 0,\ x(0) = x^0 \right\}. \tag{2}$$

Since usually no algebraic description for Ω is available (cf. Khalil [2002]) the aim of current research is to compute an inner approximation.

Therefore the theory of Lyapunov [1992] is used which says, that if a positive definite function $V(x)$ exists such that the time derivative $\dot{V}(x) = \frac{\partial f}{\partial x}^T f(x)$ is negative definite on a neighborhood of the origin, the system is called asymptotically stable and $V(x)$ is called a Lyapunov function. So we can define a subset Ω_c of the domain of attraction by

$$\Omega_c := \{x | V(x) \le c\}\,, \tag{3}$$

if $V(x)$ is a Lyapunov function on Ω_c. To compute the optimum value c^*, which ensures that Ω_{c^*} is the largest subset of the domain of attraction for the given Lyapunov function, the optimization problem

$$c^* = \min_{\substack{x\neq 0 \\ \dot{V}(x)=0}} V(x) \tag{4}$$

needs to be solved. Since the origin is very attractive to standard solvers, but excluded by the constraints, this optimization problem is very hard to solve.

In the past, several approaches to calculate an inner approximation of the domain of attraction were presented using the definitions above. Most of them can be applied to polynomial systems only, e.g. methods based on SOS relaxations which lead to LMIs (Parillo [2000], Tibken [2000], Hachicho and Tibken [2002], Chesi et al. [2003, 2005], Tan and Packard [2006]), BMI methods (cf. Fan and Tibken [2005]), methods using simulations and SOS programming (Topcu et al. [2008]) and methods based on the use of Chebychev points (Tibken et al. [1999], Tibken and Dilaver [2002], Dilaver [2008]).

Techniques for non-polynomial systems are rare: In Chesi [2005, 2009], an LMI technique which substitutes the non-polynomial terms with their Taylor expansions, was proposed. An interval arithmetic approach was presented in Warthenpfuhl et al. [2010].

In this paper we present a branch-and-bound method using a theorem of Ehlich and Zeller [1964] and its extensions in Ruttmann [1982] and Gärtel [1987].

2. CALCULATION OF BOUNDS

2.1 Theorem of Ehlich and Zeller

In the following $I = [\underline{x}, \overline{x}]$ denotes a nonempty compact interval with $I \subset \mathbb{R}$. For an algebraic variable $x \in I$ we define the set $X(N,I)$ of $N \in \mathbb{N}$ Chebychev points in I as

$$X(N,I) := \{x_j, j = 1, ..., N\}\,, \text{ where}$$

$$x_j = \frac{\underline{x} + \overline{x}}{2} + \frac{\overline{x} - \underline{x}}{2} \cos\left(\frac{(2j-1)\,\pi}{2N}\right).$$

For any continuous function f defined on an interval I the norm

$$\|f\|^I := \max_{x \in I} |f(x)|\,,$$

is the usual maximum norm. Let $\mathcal{P}_m$ be the set of polynomials p in one variable with $\deg(p) = m$. Then the following equation

$$\|p\|^I = C\left(\frac{m}{N}\right) \|p\|^{X(N,I)} \text{ with } N > m \text{ and} \tag{5}$$

$$C\left(\frac{m}{N}\right) := \left[\cos\left(\frac{m\pi}{2N}\right)\right]^{-1}$$

is valid for every $p \in \mathcal{P}_m$ and every nonempty compact interval I. This result was given by Ehlich and Zeller [1964]. For the minimum and maximum of a polynomial p on I we use the notation

$$p^I_{\min} := \min_{x \in I} p(x) \text{ and } p^I_{\max} := \max_{x \in I} p(x),$$

respectively. Using (5), the inequalities

$$p^I_{\max} \leq \frac{1}{2}\left\{\left(C\left(\frac{m}{N}\right)+1\right) p^{X(N,I)}_{max} - \left(C\left(\frac{m}{N}\right)-1\right) p^{X(N,I)}_{min}\right\} \quad \text{and} \tag{6}$$

$$p^I_{\min} \geq \frac{1}{2}\left\{\left(C\left(\frac{m}{N}\right)+1\right) p^{X(N,I)}_{min} - \left(C\left(\frac{m}{N}\right)-1\right) p^{X(N,I)}_{max}\right\}, \tag{7}$$

which are valid for every $p \in \mathcal{P}_m$ and $N > m$, are given by Ruttmann [1982].

The equation (5) and the inequalities (6) and (7) are valid for polynomials in one variable. They can be extended to polynomials of n variables if we use the following replacements. First of all, we replace the interval I by $\hat{I} = [\underline{x}_1, \overline{x}_1] \times ... \times [\underline{x}_n, \overline{x}_n]$, which represents an interval vector. We introduce the abbreviation m^p_h for the degree of p with respect to the h-th variable x_h and define the set of Chebychev points by

$$X\left(\hat{N}, \hat{I}\right) := X(N_1, [\underline{x}_1, \overline{x}_1]) \times ... \times X(N_n, [\underline{x}_n, \overline{x}_n]),$$

where N_h is the number of Chebychev points for the h-th variable x_h in the interval $[\underline{x}_h, \overline{x}_h]$. Then the inequalities

$$p^{\hat{I}}_{\max} \leq \frac{1}{2}\left\{(K+1)\, p^{X(\hat{N},\hat{I})}_{max} - (K-1)\, p^{X(\hat{N},\hat{I})}_{min}\right\} \tag{8}$$

$$p^{\hat{I}}_{\min} \geq \frac{1}{2}\left\{(K+1)\, p^{X(\hat{N},\hat{I})}_{min} - (K-1)\, p^{X(\hat{N},\hat{I})}_{max}\right\} \tag{9}$$

with

$$K = \prod_{h=1}^{n} C\left(\frac{m^p_h}{N_h}\right)$$

are valid under the conditions $N_h > m^p_h, h = 1, ..., n$.

The results above can be extended to rational functions $r(x)$ of n variables with

$$r(x) = \frac{p(x)}{q(x)} \quad \text{and} \quad p, q \in \mathcal{P}_m, q(x) \neq 0 \quad (x \in \hat{I}).$$

If we define

$$\kappa := \frac{|q|^{X(\hat{N},\hat{I})}_{\max}}{|q|^{X(\hat{N},\hat{I})}_{\min}} \quad \text{and} \quad \hat{K} := \frac{K+1+(K-1)\,\kappa}{K+1-(K-1)\,\kappa}$$

then the inequalities

$$K+1-(K-1)\,\kappa > 0 \quad \text{and} \quad N_h > m^p_h, h = 1, ..., n$$

are fulfilled, where $m_h = \max\left(m^p_h, m^q_h\right))$ and N_h are the degree and the number of Chebychev points in terms of the variable x_h respectively.

Then the inequalities

$$r^{\hat{I}}_{\max} \leq \frac{1}{2}\left\{\left(\hat{K}+1\right) p^{X(\hat{N},\hat{I})}_{max} - \left(\hat{K}-1\right) p^{X(\hat{N},\hat{I})}_{min}\right\} \tag{10}$$

and

$$r^{\hat{I}}_{\min} \geq \frac{1}{2}\left\{\left(\hat{K}+1\right) p^{X(\hat{N},\hat{I})}_{min} - \left(\hat{K}-1\right) p^{X(\hat{N},\hat{I})}_{max}\right\} \tag{11}$$

are valid, which was shown by Gärtel [1987].

The presented inequalities can now be extended to trigonometric polynomials with degree m which are defined as

$$p(\varphi) := \sum_{k=0}^{m} \sum_{i=0}^{k} a_{ki} \sin^i(\varphi) \cos^{k-i}(\varphi),$$

where $\varphi \in [0, 2\pi]$ is the angle and a_k, b_k are coefficients of the trigonometric polynomial. The set of trigonometric Chebychev points is defined as

$$\phi(N) = \left\{\frac{(j-1)\pi}{N}, \quad j = 1, \ldots, 2N\right\}$$

and if $N > m$, the inequalities

$$p^{[0,2\pi]}_{\max} \leq \frac{1}{2}\left\{\left(C\left(\frac{m}{N}\right)+1\right) p^{\phi(N)}_{max} - \left(C\left(\frac{m}{N}\right)-1\right) p^{\phi(N)}_{min}\right\} \quad \text{and} \tag{12}$$

$$p^{[0,2\pi]}_{\min} \geq \frac{1}{2}\left\{\left(C\left(\frac{m}{N}\right)+1\right) p^{\phi(N))}_{min} - \left(C\left(\frac{m}{N}\right)-1\right) p^{\phi(N)}_{max}\right\}, \tag{13}$$

are valid, which was shown by Gärtel [1987].

If bounds of a trigonometric polynomial on a subinterval of $\varphi \in [0, 2\pi]$ are required, (12) and (13) cannot be used, since these are only defined for the full interval $[0, 2\pi]$. Thus, a new variable $t = \tan\frac{\varphi}{2}$, is introduced and the interval is shifted w.l.o.g. to $\varphi \in [-\pi, \pi]$ and we observe that

$$\sin(\varphi) = \frac{2t}{1+t^2} \quad \text{and} \quad \cos(\varphi) = \frac{1-t^2}{1+t^2} \tag{14}$$

are valid. Hence, the range for t is $(-\infty, \infty)$. Now the interval $(-\infty, \infty)$ is split as $(-\infty, -1] \cup [-1, 1] \cup [1, \infty)$ and for the two outer intervals the new variable $s = \frac{1}{t}$ is introduced by which the two intervals are transformed into $[-1, 0]$ and $[0, 1]$ and then can be combine to $[-1, 1]$. A trigonometric polynomial on a subinterval is thus transformed into a rational function of t or s, respectively. Thus, (10) and (11) can be applied.

2.2 *Guaranteed bounds for the non-polynomial terms*

In this section we describe how the theorems and bounds presented in section 2.1 can be applied towards the computation of an estimation of the domain of attraction. We use a quadratic Lyapunov function (Chesi et al. [2005]) given by

$$V(x) = x^T Q x \tag{15}$$

where Q is a positive definite symmetric matrix. Hence, the set Ω_c (3) is an ellipsoid and compact. The time derivative is calculated by $\dot{V}(x) = 2f^T(x)Qx$.

Using the Cholesky decomposition $Q = L^T L$, the variables are changed according to $z = Lx$ which leads to

$$\hat{V}(z) = V(L^{-1}z) = z^T z \text{ and } \dot{\hat{V}}(z) = \dot{V}(L^{-1}z)\ .$$

The time derivative can be written as

$$\dot{\hat{V}}(z) = \dot{\hat{V}}_0(z) + \sum_{i=1}^{r} \hat{q}_i(z)\hat{f}_i(z), \tag{16}$$

where $\dot{\hat{V}}_0(z)$ and $\hat{q}_i(z)$ are polynomials in z and $\hat{f}_i(z)$ are the non-polynomial terms. Since the results from 2.1 cannot be applied to the non-polynomial terms, we assume that polynomial approximations p_{d_i} for these non-polynomial terms on intervals $z^I = [\underline{z}, \overline{z}]$ exist. The upper bound μ_{d_i} for the interpolation error is computed as follows:

It is known (c.f. Stoer and Bulirsch [2002]), that the following inequality is valid

$$\left|\hat{f}_i(z) - p_{d_i}(z)\right| \le \frac{\left|\hat{f}_i(\xi)\right|^{(d_i+1)}}{(d_i+1)!} \prod_{k=1}^{d_i+1} |z - z_k|\,, \tag{17}$$

where $\xi \in z^I$ and $z_k \in z^I$ are supporting points, for which the Chebychev points are used. To compute an upper bound for the polynomial

$$\prod_{k=1}^{d_i+1} |z - z_k|\,, \tag{18}$$

we transform the interval $[\underline{z}, \overline{z}]$ to $[-1, 1]$ by the simple linear transformation $z = ay + b$. Since $\underline{z} = -a + b$ and $\overline{z} = a + b$ we can compute a and b and obtain

$$|z - z_k| = |ay + b - ay_k - b| = a\,|y - y_k| = \frac{\overline{z} - \underline{z}}{2}\,|y - y_k|$$

which can be used in (18)

$$\prod_{k=1}^{d_i+1} |z - z_k| = \left(\frac{\overline{z} - \underline{z}}{2}\right)^{d_i+1} \prod_{k=1}^{d_i+1} |y - y_k|\,. \tag{19}$$

Since the y_k are Chebychev points on $[-1, 1]$, the upper bound (c.f. Stewart [1996]) for

$$\prod_{k=1}^{d_i+1} |y - y_k| \text{ is } \frac{1}{2^{d_i+1}},$$

which leads to

$$\left|\hat{f}_i(z) - p_{d_i}(z)\right| \le \underbrace{\frac{\left|\hat{f}_i(\xi)\right|^{(d_i+1)}}{(d_i+1)!} \left(\frac{\overline{z} - \underline{z}}{4}\right)^{d_i+1}}_{\mu_{d_i}}\ . \tag{20}$$

In the following we rewrite $\dot{\hat{V}}(z)$ as $\dot{\tilde{V}}(r, \varphi)$ by using the n-dimensional polar coordinates r and $\varphi = (\varphi_1, \ldots, \varphi_{n-1})$ which are introduced by

$$\begin{aligned}
z_1 &= r\cos(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{n-1})\\
z_2 &= r\sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{n-1})\\
z_3 &= r\cos(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{n-1})\\
&\vdots\\
z_{n-1} &= r\cos(\varphi_{n-2})\sin(\varphi_{n-1})\\
z_n &= r\cos(\varphi_{n-1})
\end{aligned}$$

with $r \in [0, \infty)$, $\varphi_1 \in [0, 2\pi]$ and $\varphi_2, \ldots, \varphi_{n-1} \in [0, \pi]$ to

$$\dot{\tilde{V}}(r, \varphi) = \dot{\tilde{V}}_0(r, \varphi) + \sum_{i=1}^{r} \tilde{q}_i(r, \varphi)\tilde{f}_i(r, \varphi)\,, \tag{21}$$

where $\dot{\tilde{V}}_0(r, \varphi)$ and the $\tilde{q}_i(r, \varphi)$ are trigonometric polynomials and the $\tilde{f}_i(r, \varphi)$ are non-polynomial terms.

In the following we concentrate on upper bounds, since lower bounds can be derived analogously. By adding and subtracting polynomial interpolations $p_{d_i}(r, \varphi)$ for $\tilde{f}_i(r, \varphi)$ we get

$$\begin{aligned}
\dot{\tilde{V}}(r, \varphi) &= \dot{\tilde{V}}_0(r, \varphi) + \sum_{i=1}^{r} \Big[\tilde{q}_i(r, \varphi)\\
&\quad \Big(\tilde{f}_i(r, \varphi) + p_{d_i}(r, \varphi) - p_{d_i}(r, \varphi)\Big)\Big]\\
&= \dot{\tilde{V}}_0(r, \varphi) + \sum_{i=1}^{r} \tilde{q}_i(r, \varphi)p_{d_i}(r, \varphi)\\
&\quad + \sum_{i=1}^{r} \tilde{q}_i(r, \varphi)\Big(\tilde{f}_i(r, \varphi) - p_{d_i}(r, \varphi)\Big)\,.
\end{aligned}$$

Thus, by (20), we have the upper bound

$$\dot{\tilde{V}}(r, \varphi) \le \underbrace{\dot{\tilde{V}}_0(r, \varphi) + \sum_{i=1}^{r} \tilde{q}_i(r, \varphi)p_{d_i}(r, \varphi)}_{\dot{\tilde{V}}_p(r, \varphi)} + \sum_{i=1}^{r} |\tilde{q}_i(r, \varphi)|\,\mu_{d_i}$$

where $\dot{\tilde{V}}_p(r, \varphi)$ is a trigonometric polynomial. It is easily possible to compute the upper bound $|\tilde{q}_i(r, \varphi)|$ via the Ehlich-Zeller-inequalities. Due to the fact that $\dot{\tilde{V}}_p(r, \varphi)$ is a polynomial, we could also apply the Ehlich-Zeller-inequalities here, but the evaluation of $\dot{\tilde{V}}_p(r, \varphi)$ at a Chebychev point r_k, φ_k is not easily possible because the p_{d_i} have not been constructed yet.

To avoid the actual construction of the p_{d_i} we add and subtract the non-polynomial terms $\tilde{f}_i(r_k, \varphi_k)$ and can then compute upper bounds for each evaluation $\dot{\tilde{V}}_p(r_k, \varphi_k)$.

$$\begin{aligned}
\dot{\tilde{V}}_p(r_k, \varphi_k) &= \dot{\tilde{V}}_0(r_k, \varphi_k) + \sum_{i=1}^{r} \tilde{q}_i(r_k, \varphi_k)p_{d_i}(r_k, \varphi_k)\\
&= \dot{\tilde{V}}_0(r_k, \varphi_k) + \sum_{i=1}^{r} \Big[\tilde{q}_i(r_k, \varphi_k)\\
&\quad \Big(\tilde{f}_i(r_k, \varphi_k) + p_{d_i}(r_k, \varphi_k) - \tilde{f}_i(r_k, \varphi_k)\Big)\Big]\\
&\le \dot{\tilde{V}}_0(r_k, \varphi_k) + \sum_{i=1}^{r} \tilde{q}_i(r_k, \varphi_k)\tilde{f}_i(r_k, \varphi_k)\\
&\quad + \sum_{i=1}^{r} |\tilde{q}_i(r_k, \varphi_k)|\,\mu_{d_i}\\
&= \dot{\tilde{V}}(r_k, \varphi_k) + \sum_{i=1}^{r} |\tilde{q}_i(r_k, \varphi_k)|\,\mu_{d_i}
\end{aligned}$$

If we define

$$\tilde{q}_{iM} := \max|\tilde{q}_i(r_k,\varphi_k)| \text{ and } \mu := \sum_{i=1}^{r} \tilde{q}_{iM}\cdot \mu_{d_i}$$

we have

$$\dot{\tilde{V}}_p(r_k,\varphi_k) \le \dot{\tilde{V}}(r_k,\varphi_k) + \mu\ .$$

Likewise we get

$$\dot{\tilde{V}}_p(r_k,\varphi_k) \ge \dot{\tilde{V}}(r_k,\varphi_k) - \mu\ .$$

Thus we have

$$\begin{aligned}\max \dot{\tilde{V}}_p(r_k,\varphi_k) &\le \max \dot{\tilde{V}}(r_k,\varphi_k) + \mu \text{ and}\\ \min \dot{\tilde{V}}_p(r_k,\varphi_k) &\ge \min \dot{\tilde{V}}(r_k,\varphi_k) - \mu\ .\end{aligned}$$

Now we can apply the Ehlich-Zeller-inequalities

$$\begin{aligned}\dot{\tilde{V}}_p(r,\varphi) &\le \frac{1}{2}\Big\{(K+1)\max \dot{\tilde{V}}_p(r_k,\varphi_k)\\ &\quad -(K-1)\min \dot{\tilde{V}}_p(r_k,\varphi_k)\Big\}\\ &\le \frac{1}{2}\Big\{(K+1)(\max \dot{\tilde{V}}(r_k,\varphi_k)+\mu)\\ &\quad -(K-1)(\min \dot{\tilde{V}}(r_k,\varphi_k)-\mu)\Big\}\\ &\le \frac{1}{2}\Big\{(K+1)\max \dot{\tilde{V}}(r_k,\varphi_k)\\ &\quad -(K-1)\min \dot{\tilde{V}}(r_k,\varphi_k)\Big\} + K\mu\ .\end{aligned}$$

Since $\sum_{i=1}^{r} |\tilde{q}_i(r,\varphi)|\,\mu_{d_i} \le \mu$, we can write

$$\begin{aligned}\dot{\tilde{V}}(r,\varphi) &\le \frac{1}{2}\Big\{(K+1)\max \dot{\tilde{V}}(r_k,\varphi_k)\\ &\quad -(K-1)\min \dot{\tilde{V}}(r_k,\varphi_k)\Big\} + (K+1)\mu \qquad (22)\\ &= \overline{\dot{\tilde{V}}}{}^{I}_{\max}\ .\end{aligned}$$

In the same way, we can derive the lower bound

$$\begin{aligned}\dot{\tilde{V}}(r,\varphi) &\ge \frac{1}{2}\Big\{(K+1)\min \dot{\tilde{V}}(r_k,\varphi_k)\\ &\quad -(K-1)\max \dot{\tilde{V}}(r_k,\varphi_k)\Big\} - (K+1)\mu \qquad (23)\\ &= \underline{\dot{\tilde{V}}}{}^{I}_{\min}\ .\end{aligned}$$

3. ALGORITHM

In order to compute a subset of the domain of attraction the following algorithm performs a bisection with respect to r, i.e. we keep track of two values $\underline{r}$ and $\overline{r}$ (initial values can be computed using a method similar to Tibken et al. [1999]) which are a lower and upper bound for the optimum value $r^\star$. We then test if the midpoint $r_{\text{test}} = \frac{\underline{r}+\overline{r}}{2}$ is a new upper or lower bound. To check this, we have to keep track of all angle variables $\varphi_1,\ldots,\varphi_{n-1}$ and their respective intervals. Based on the previously defined transformations we have the interval $[-1,1]$ for φ_1 and the interval $[0,1]$ for the remaining angles. All of these intervals exist two times, once for t_l and once for s_l. The variable t_l is used to compute $\sin\varphi_l$ and $\cos\varphi_l$ via

$$\sin\varphi_l = \frac{2t_l}{1+t_l^2},\quad \cos\varphi_l = \frac{1-t_l^2}{1+t_l^2}$$

and the variable s_l is used to compute $\sin\varphi_l$ and $\cos\varphi_l$ via

$$\sin\varphi_l = \frac{2s_l}{1+s_l^2},\quad \cos\varphi_l = -\frac{1-s_l^2}{1+s_l^2}\ .$$

In order to keep track of the bisections with respect to the angle variables, a typical list element I_j consists of a sequence of $n-1$ intervals Φ_l, a sequence of $n-1$ boolean variables γ_l indicating t_l if 0 and s_l if 1, and a flag describing the state of decision with respect to the sign of $\dot{\tilde{V}}(r,\varphi)$. In formulas this means

$$I_j = \big(\Phi_{1j},\Phi_{2j},\ldots,\Phi_{(n-1)j},\gamma_{1j},\ldots\gamma_{(n-1)j}, flag_j\big)$$

The list $\mathcal{L}$ is initialized with the following 2^{n-1} permutations of the intervals for t_l and s_l

$$([-1,1],[0,1],\ldots,[0,1],\gamma_1,\ldots,\gamma_{n-1},3)$$

for all $\gamma_l \in \{0,1\}$, $l = 1,\ldots,n-1$.

Furthermore, we define the constant ε as our termination criterion, $N_{\max}$ as the maximum number of Chebychev points and $d_{\max}$ as the maximum degree of the approximation polynomial in our algorithm.

Step 1) If $\overline{r}-\underline{r} < \varepsilon$ the algorithm stops. Otherwise, the radius interval $[\underline{r},\overline{r}]$ is bisected into $[\underline{r}, r_{\text{test}}]$ and $[r_{\text{test}},\overline{r}]$ with $r_{\text{test}} = \frac{\underline{r}+\overline{r}}{2}$.

Step 2) The steps 2a) to 2c) are executed for every permutation I_j in the list $\mathcal{L}$.

Step 2a) The initial degree d_j of the approximation polynomials for I_j is set to a value which is at least 2 but smaller than or equal to $d_{\max}$. N_j is set to a value greater than $\max\Big(\deg(\dot{\tilde{V}}_0(r,\varphi)), \max_i(\deg(\tilde{q}_i(r,\varphi)) + d_j)\Big)$.

Step 2b) The bounds of $\dot{\tilde{V}}$ on I_j are computed by application of (22) and (23) on I_j.

Step 2c) The element I_j of the list $\mathcal{L}$ is classified into one of the following cases:

i) $\overline{\dot{\tilde{V}}}{}^{I_j}_{\max} > 0$ and $\max \dot{\tilde{V}}^{X(N_j,I_j)} > 0$:

$flag_j$ is set to 1.

ii) $\overline{\dot{\tilde{V}}}{}^{I_j}_{\max} < 0$:

$flag_j$ is set to 2.

iii) otherwise:
as long as $d_j \le d_{\max}$ and $N_j \le N_{\max}$:
if $-\frac{1}{2}(K-1)\underline{\dot{\tilde{V}}}{}^{I_j}_{\min} > (K+1)\mu$ we increment d_j.
Otherwise, we increment N_j.

After that, we go back to step 2b).
If $d_j > d_{\max}$ or $N_j > N_{\max}$ the flag of the element I_j is set to 3.

Step 3) All elements I_j of the list $\mathcal{L}$ for which $flag_j = 3$ are removed from the list and bisected with respect to $t_i^{I_j}$ into smaller intervals which are added to the list $\mathcal{L}$. For each of these bisected intervals the steps 2) to 3) are executed as long as intervals I_j with $flag_j = 3$ are still in the list. To avoid a infinite runtime a limit of the number of

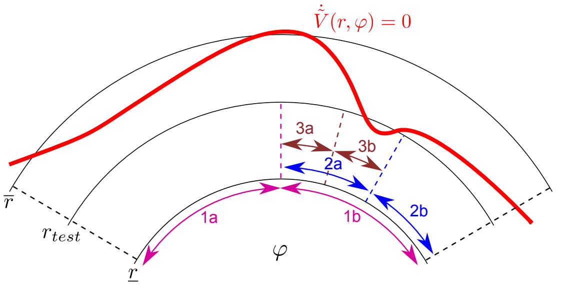

Fig. 1. Example for the occurring bisections

iterations can be specified. If the iteration limit is reached, these elements I_j are flagged with 1.

Step 4) The list contains only elements which are flagged with 1 or 2.

Step 4a) If all elements I_j are flagged with 2, we can be certain that $\dot{\tilde{V}} < 0$ between $\underline{r}$ and r_{test}. Thus $\underline{r}$ is set to r_{test} and the algorithm loops back to step 1).

Step 4b) If at least one elements is flagged with 1, we know that somewhere between $\underline{r}$ and r_{test} there is at least one Chebychev point with $\dot{\tilde{V}} \geq 0$. Thus, all elements with $flag_j = 2$ are removed from $\mathcal{L}$ since they are now irrelevant, and $\overline{r}$ is set to r_{test} for the remaining angle intervals. After that, the algorithm loops back to step 1).

We want to show the application of these steps for one iteration. Fig. 1 shows one iteration of the algorithm. For reasons of clarification the figure shows the bisections in terms of φ although in the algorithm itself the bisections are carried out in terms of $t_i^{I_j}$. We assume that we start with given boundaries $\underline{r}$, $\overline{r}$ and the list $\mathcal{L}$, which contains only two elements $1a$ and $1b$. Furthermore we assume that only for the elements $1a$, $2b$, $3a$ and $3b$ the application of the theorem of Ehlich and Zeller and the use of the Chebychev points allows us to classify them as flag values 1 or 2. In all other cases only a classification with a flag value of 3 is possible. The flag values are shown in Table 1.

We assume that $\overline{r} - \underline{r} > \varepsilon$. Therefore an iteration of the algorithm is required. At the beginning we take the first element $1a$, calculate $\overline{\dot{\tilde{V}}}_{\max}^{I_{1a}}$ and $\max \dot{\tilde{V}}^{X(N_{1a},I_{1a})}$. We might have to increase $d_{1a} \leq d_{\max}$ or $N_{1a} \leq N_{\max}$, but in the end $1a$ is marked with a flag value of 2. We repeat the same steps for the element $1b$, but this time the flag value is set to 3.

Table 1. Assumed flag values for elements shown in Fig. 1

element	flag	element	flag
$1a$	2	$2b$	2
$1b$	3	$3a$	2
$2a$	3	$3b$	1

Since $flag_{1b}$ is 3, in step 3) we remove $1b$ from $\mathcal{L}$ and bisect $1b$, as shown in Fig. 1, into $2a$ and $2b$. Repeating the steps done in 2) for $2a$ and $2b$, the list $\mathcal{L}$ contains the elements $1a$, $2a$ and $2b$ with $flag_{1a} = 2$, $flag_{2a} = 3$, and $flag_{2b} = 2$. Thus $2a$ is removed from the list and bisected into $3a$ and $3b$. Once again repeating step 2) for the new elements, we finally obtain $\mathcal{L}$ with elements $1a$, $2b$, $3a$ and $3b$ with $flag_{1a} = 2$, $flag_{2b} = 2$, $flag_{3a} = 2$ and $flag_{3b} = 1$.

As $\mathcal{L}$ contains one element with a flag value of 1, we remove all elements with a flag value of 2 in step 3). Thus $\mathcal{L}$ is reduced to a list containing only the element $3b$. The boundary $\overline{r}$ for the next iteration is set to the current value of r_{test}.
Theoretically the running time increases in the worst case, exponentially $\mathcal{O}(2^n)$ with the dimension of the state space n. However the examples, which we implemented shows that the behavior of the running time remain within acceptable limits.

4. EXAMPLES

The effectiveness of our approach is shown by means of three benchmark examples. These examples were presented in Chesi [2005, 2009] and were also used in Warthenpfuhl et al. [2010]. Our results were computed with MATLAB 2008b on a standard PC and are not only consistent, but tighter than the inclusions given by Warthenpfuhl et al. [2010].

4.1 Example 1

Our first example is a simple pendulum system from Chesi [2005] with the state space description

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \sin x_1 \ .\end{aligned}$$

The Lyapunov function used here is $V(x) = 4x_1^2 + 2x_1x_2 + 3x_2^2$ which leads to the time derivative

$$\dot{V}(x) = 6x_1x_2 - 4x_2^2 + (-2x_1 - 6x_2)\sin x_1 \ .$$

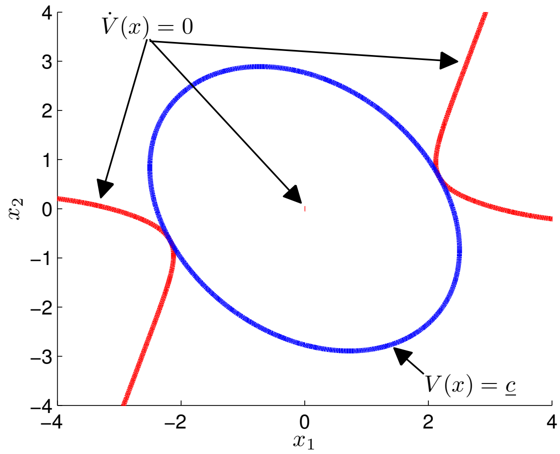

Fig. 2. Example 1: $\dot{V}(x) = 0$ (red) and $V(x) = \underline{c}$ (blue), which bounds the guaranteed estimation of the domain of attraction. The lower bound is very tight.

Applying our method to the problem we obtained the lower bound $\underline{c} = 23.00718671474091$ and the upper bound $\overline{c} = 23.00718671474093$ for c^*. The best lower bound from Chesi [2005] is $\underline{c} = 22.94$. Fig. 2 shows that our guaranteed estimation of the domain of attraction is almost optimal for the selected Lyapunov function.

4.2 Example 2

Our second example was also taken from Chesi [2005]. The state space representation is

$$\dot{x}_1 = -\frac{1}{4}x_1 + \ln(1 + x_2)$$
$$\dot{x}_2 = -\frac{3}{8}x_1 - \frac{1}{5}x_1x_2 + \left(\frac{1}{8}x_1 - x_2\right)\cos x_1$$

and the chosen Lyapunov function $V(x) = x_1^2 + x_2^2$.

With our approach the lower bound $\underline{c} = 0.273707536046659$ and the upper bound $\overline{c} = 0.273707536046660$ for c^* were computed, which are an improvement over the results from Chesi [2005], where a lower bound $\underline{c} = 0.2606$ was determined.

4.3 Example 3

In Chesi [2009] the following system was considered

$$\dot{x}_1 = -x_1 + x_2 + 0.5\,(e^{x_1} - 1)$$
$$\dot{x}_2 = -x_1 - x_2 + x_1x_2 + x_1\cos x_1$$

and the quadratic Lyapunov function $V(x) = x_1^2 + x_2^2$ was used. This yields the time derivative

$$\dot{V}(x) = 2x_1x_2^2 - 2x_1^2 - 2x_2^2 + 2x_1x_2\cos(x_1) - x_1 + x_1e^{x_1}.$$

the computed bounds for c^* are $\underline{c} = 0.321074071102361$ and $\overline{c} = 0.321074071102362$ are consistent with the lower bound $\underline{c} = 0.3210$ in Chesi [2009].

5. CONCLUSION

In this paper, a new method for computing domains of attraction for non-polynomial non-linear systems has been presented. This method is based on computing bounds for the error resulting from the polynomial interpolation of the non-polynomial terms. The theorem of Ehlich and Zeller was adapted for non-polynomial systems and used to compute upper and lower bounds enclosing the domain of attraction. The method was tested on three benchmark examples from literature. On these, it yielded better results than those presented in Chesi [2005, 2009] and Warthenpfuhl et al. [2010]. Future research will concentrate on adapting the method to work with higher-order and rational Lyapunov functions and on using it in controller design to maximize the domain of attraction of the controlled system.

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