Polynomial-input-type final-state control taking account of input saturation

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Abstract: A final-state control method based on a polynomial input with a time-symmetry assumption was proposed. However, the method can not consider the saturation of control input. When a positioning time is shorten, it is inevitable to consider the input saturation to achieve a faster positioning time. In this study, we propose a polynomial-input-type final-state control taking account of input saturation. The effectiveness is shown by simulations.

Keywords: Polynomial-input-type final-state control, input saturation, linear matrix inequality

1. INTRODUCTION

For information devices such as hard disk drives, it is important to achieve fast and accurate positioning of the recording head. Since the system has mechanical vibration modes at high frequency, the feedforward input must be obtained so that the mechanical vibration modes are not excited while minimizing access time. Thus, various attempts have been made to design feedforward inputs (Yamaguchi et al., 2007).

We developed a design method for feedforward inputs that does not excite high frequency resonance modes based on frequency-shaped final-state control (FFSC) method, and the effectiveness of the FFSC method has been shown by simulation and experiment by applying it to track-seeking systems and the effectiveness has been shown (Hirose et al., 2008).

In the FFSC method, the feedforward input must be stored in memory since it is obtained as a time-series data. Therefore, a larger memory size is required if the step number of input is increased. To overcome the problem, we proposed a FFSC method based on a polynomial input (Hirata and Ueno, 2009). Since the polynomial-input-type FFSC (PFFSC) method only requires storing the coefficients, the memory size is drastically reduced. Moreover, we proposed a PFFSC method that assumes a time symmetry of input to reduced the order of the polynomial (Hirata and Ueno, 2010).

In practical control applications, the magnitude of the input is limited. Thus, the control input must be generated so as not to saturate. Especially, when the positioning time is shorten, it is inevitable to consider the input saturation to achieve a faster positioning time. This paper proposes a PFFSC method that can take into account the input saturation. In particular, we show that the time-symmetry assumption is effective for the PFFSC method with input saturation. The effectiveness of the proposed method is shown by simulations.

2. FINAL-STATE CONTROL TAKING ACCOUNT OF INPUT SATURATION

For a controllable discrete-time system of m-th order

\[ x[k + 1] = Ax[k] + Bu[k] \] (1)

\[ y[k] = Cx[k], \] (2)

let us consider the problem of determining an N-step \((N \geq m)\) control input \(u[k]\) that drives an initial state \(x[0]\) to a final state \(x[N]\). By substituting \(k = 0, 1, \ldots, N - 1\) into Eq.(1) recursively we have

\[ Y = \Sigma U \] (3)

where

\[ \Sigma = \left[ A^{N-1}B \ A^{N-2}B \ \cdots \ B \right] \] (4)

\[ U = \left[ u[0] \ u[1] \ \cdots \ u[N - 1] \right]^T \] (5)

\[ Y = x[N] - A^N x[0]. \] (6)

It is obvious that the input vector \(U\) satisfying the constraint Eq.(3) is not unique, then the following performance index is introduced:

\[ J = U^T QU, \quad Q > 0. \] (7)

The minimization problem of Eq.(7) subject to the constraint Eq.(3) can be formulated by linear-matrix inequalities (LMIs). Since \(\Sigma\) is guaranteed to be full rank by the controllability of \((A, B)\), \(\Sigma^\dagger \in \mathbb{R}^{N \times (N - m)}\) and \(\Sigma^\dagger \in \mathbb{R}^{N \times m}\) satisfying \(\Sigma \Sigma^\dagger = O\) and \(\Sigma^\dagger \Sigma = I\) can be introduced. Then \(\tilde{U}\) satisfying

\[ U = \left[ \Sigma^\dagger \ \Sigma^\dagger \right] \tilde{U} \] (8)

is introduced. Now substituting Eq.(8) into Eq.(3), we have \(Y = [I \ O] \tilde{U}\). Therefore, \(\tilde{U}\) can be represented by \(Y\) and a free parameter \(q \in \mathbb{R}^{(N - m) \times 1}\) as follows:
Thus we have
\[
U = \begin{bmatrix} \Sigma^\dagger \Sigma^\perp \end{bmatrix} \begin{bmatrix} Y \\ q \end{bmatrix}.
\] (9)

By introducing \( M \) as
\[
M = \begin{bmatrix} (\Sigma^\dagger)^T \\ (\Sigma^\perp)^T \end{bmatrix} Q \begin{bmatrix} \Sigma^\dagger \Sigma^\perp \end{bmatrix},
\]
the performance index \( J \) can be described as follows:
\[
J = U^T Q U = Y^T M_{11} Y + 2 Y^T M_{12} q + q^T M_{22} q
\]
The inequality \( J < \gamma \) for a given \( \gamma \) can be represented by a LMI as follows:
\[
\begin{bmatrix} \gamma - 2 Y^T M_{12} q - Y^T M_{11} Y q^T \\ q^T \end{bmatrix} > 0
\] (10)

In order to account for input saturation, we consider the minimization problem of Eq.(7) subject to the constraint
\[
z_{\text{min}} < z[k] < z_{\text{max}}, \quad (k = 0, \ldots, N - 1)
\] (11)
where \( z[k] \) is defined by
\[
z[k] = C_z x[k] + D_z u[k].
\] (12)
It is assumed that \( z_{\text{max}} \) and \( z_{\text{min}} \) are a positive and negative values, respectively. In Eq.(12), matrices \( C_z \) and \( D_z \) are design parameters to define \( z[k] \) from \( x[k] \) and \( u[k] \).

By introducing the following matrices for simple notation
\[
Z = [z[0] \ z[1] \ \cdots \ z[N-1]]^T
\]
\[
\Phi_2 = \begin{bmatrix} C_z \\ C_z A \\ \vdots \\ C_z A^{N-1} \end{bmatrix},
\]
\[
\Omega_2 = \begin{bmatrix} D_z & 0 & \cdots & 0 \\ C_z B & D_z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_z A^{N-2} B & \cdots & C_z B & D_z \end{bmatrix},
\]
\[
Z \text{ can be described as an affine function of } q \text{ as} \]
\[
Z(q) = \Phi_2 x[0] + \Omega_2 U
\]
\[
= \Phi_2 x[0] + \Omega_2 \begin{bmatrix} \Sigma^\dagger & \Sigma^\perp \end{bmatrix} \begin{bmatrix} Y \\ q \end{bmatrix}
\] (14)
(15)

By defining a row vector \( \delta_i \) whose \( i \)-th element is one and the other elements are zero, Eq.(11) is represented by \( N \) LMIs as
\[
\begin{bmatrix} \delta_i Z(q) - z_0 \\ \tilde{z} \end{bmatrix}^T > 0
\] (16)

where \( i = 1, \ldots, N \) and
\[
z_0 = (z_{\text{max}} + z_{\text{min}})/2, \quad \tilde{z} = (z_{\text{max}} - z_{\text{min}})/2.
\]

The control input that drives \( x[0] \) to \( x[N] \) minimizing \( J \) subject to the constraint Eq.(11) can be obtained by minimizing \( \gamma \) subject to Eq.(10) and Eq.(16). This method is referred to as a conventional method.

3. POLYNOMIAL-INPUT-TYPE FSC TAKING ACCOUNT OF INPUT SATURATION

3.1 Proposed method 1

In this section, the polynomial-input-type FSC method proposed in (Hirata and Ueno, 2009) is extended to treat input saturation.

It is assumed that the FSC input \( u[k] \) is generated by a polynomial as follows:
\[
u[k] = \alpha_0 + \alpha_1 k + \cdots + \alpha_{p-1} k^{p-1}
\] (17)

where \( p \) is a number of terms, \( \alpha_i, \ (i = 0, \ldots, p-1) \) are coefficients. By substituting Eq.(17) into \( U \) of Eq.(5) we have
\[
U = \Gamma \alpha
\] (18)

where
\[
\begin{bmatrix} 1 & 0^1 & \cdots & 0^{p-1} \\ 1 & 1^1 & \cdots & 1^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (N-2)^1 & \cdots & (N-2)^{p-1} \\ 1 & (N-1)^1 & \cdots & (N-1)^{p-1} \end{bmatrix}
\]
\[
\alpha = [\alpha_0 \ \alpha_1 \ \cdots \ \alpha_{p-1}]^T.
\] (19)

It is easy to show that \( \Gamma \) is full rank. By substituting Eq.(18) into Eq.(3) we have
\[
Y = \bar{\Sigma} \alpha, \quad \bar{\Sigma} = \Sigma \Gamma.
\] (21)

On the other hand, the performance index \( J \) of Eq.(7) becomes
\[
J_p = U^T Q U = \alpha^T \tilde{Q} \alpha
\] (22)
where \( \tilde{Q} = I^T Q I \) and \( \tilde{Q} > 0 \).

Since Eq.(21) and Eq.(22) correspond to Eq.(3) and Eq.(7), respectively, this problem can also be formulated by LMIs.

Let us define \( \bar{\Sigma}^\dagger \in \mathbb{R}^{p \times (p-m)} \) and \( \bar{\Sigma}^\dagger \in \mathbb{R}^{p \times m} \) that satisfy \( \bar{\Sigma} \bar{\Sigma}^\dagger = O \) and \( \bar{\Sigma}^\dagger \bar{\Sigma} = I \) for the full-rank matrix \( \Sigma \). Thus the coefficient vector \( \alpha \) can be described by a free parameter \( \tilde{q} \in \mathbb{R}^{(p-m) \times 1} \) as
\[
\alpha = [\bar{\Sigma}^\dagger \ \bar{\Sigma}^\perp] \begin{bmatrix} Y \\ \tilde{q} \end{bmatrix}.
\] (23)

It is obvious that Eq.(23) corresponds to Eq.(9) if \( \alpha \) is regarded as \( U \). Thus we have the following LMI to satisfy \( J_p < \gamma \):
\[
\begin{bmatrix} \gamma - 2 Y^T \bar{M}_{12} \tilde{q} - Y^T \bar{M}_{11} Y \tilde{q}^T \\ \tilde{q} \end{bmatrix} > 0
\] (24)
When the input is saturated, it resides between \( u_{\text{min}} \) and \( u_{\text{max}} \) and the intervals A–B and D–E become almost flat as shown in Fig.1. In FSC methods, an actual input \( u[k] \) is augmented by an integrator as shown in Fig.2 to evaluate its derivative for smooth feedforward input. Therefore, the input to the augmented system becomes almost zero at the intervals A–B and D–E as shown in Fig.3. However, it is difficult for a polynomial to generate such a flat input. Thus, local minima and maxima of the polynomial are repeated at the intervals. As a result, a high order polynomial will be required to obtain a good performance.

In order to solve this problem, a time-symmetry assumption of input is introduced in the proposed method 1. Under this assumption the polynomial only generate an input from \( t = 0 \) to the point C in Fig.3 (Hirata and Ueno, 2010), then the order of the polynomial might be reduced.

The input vector \( U \) with the time-symmetry assumption can be defined as:

\[
U = \begin{cases} 
    U_o, & \text{if } N = 2n - 1 \\
    U_e, & \text{if } N = 2n
\end{cases}
\]  

(27)

where \( N \) is an even or an odd number.

By substituting Eq.(17) into Eq.(27) we have

\[
U = \Gamma_s \alpha
\]  

(28)

where

\[
\Gamma_s = \begin{cases} 
    \Gamma_o, & \text{if } N = 2n - 1 \\
    \Gamma_e, & \text{if } N = 2n
\end{cases}
\]  

(29)

\[
\Gamma_o = \begin{bmatrix} 
    1 & 0^1 & \cdots & 0^{p-1} \\
    1 & 1^1 & \cdots & 1^{p-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & (n - 2)^1 & \cdots & (n - 2)^{p-1} \\
    1 & (n - 1)^1 & \cdots & (n - 1)^{p-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & 1^1 & \cdots & 1^{p-1} \\
    1 & 0^1 & \cdots & 0^{p-1}
\end{bmatrix}
\]
Fig. 4. Frequency response of the plant

\[
\Gamma_p = \begin{bmatrix}
1 & 0^1 & \ldots & 0^{p-1} \\
1 & 1^1 & \ldots & 1^{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (n-1)^1 & \ldots & (n-1)^{p-1} \\
1 & (n-1)^1 & \ldots & (n-1)^{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1^1 & \ldots & 1^{p-1} \\
1 & 0^1 & \ldots & 0^{p-1}
\end{bmatrix}
\]

Since Eq.(28) is similar to Eq.(18), \( J_p < \gamma \) is represented by the LMI of Eq.(24) in which \( \Gamma \) is replaced by \( \Gamma_p \).

As for the constraint to \( z[k] \), it is enough to impose the constrain to the first half of \( z[k] \), e.g., from \( z[0] \) to \( z[n-1] \), since the input is time symmetry. Therefore, the LMI of Eq.(26) is evaluated for \( i = 0, \ldots, n-1 \).

The method considering the time symmetry of input is referred to as a proposed method 2.

4. SIMULATIONS

4.1 Plant model

In order to verify the effectiveness of the proposed method 1 and 2, a mechanical system that has a ridged-body mode and two vibration modes are assumed.

\[
P(s) = \frac{k_0}{s^2} + \sum_{i=1}^{2} \frac{k_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}
\]

The plant parameters are defined as shown in Table 1 to have vibration modes at 1kHz and 2kHz. The sampling period \( \tau \) is assumed to be 25µs, and the bode plot of the discretized plant is shown in Fig.4.

Table 1. Parameters of the plant model

<table>
<thead>
<tr>
<th>( k_0 )</th>
<th>( k_1 )</th>
<th>( k_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.5 \times 10^4 )</td>
<td>( 0.25 \times 10^4 )</td>
<td>( -1.5 \times 10^4 )</td>
</tr>
<tr>
<td>( \zeta_1 )</td>
<td>( 2 \times 10^{-3} )</td>
<td>( 10 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>( 2\pi \times 1000 )</td>
<td>( 2\pi \times 2000 )</td>
</tr>
</tbody>
</table>

4.2 Feedforward input design

For feedforward input design, a ridged-mode model is used. In order to take into account the smoothness and the frequency components of the input, the following performance index is introduced for the augmented system in Fig.2 (Hirata et al., 2002).

\[
J_w = \sum_{k=0}^{N-1} u^2[k] + \sum_{i=1}^{l} q_i |\hat{U}_r(\omega_i)|^2
\]

where

\[
\hat{U}_r(\omega_i) = \int_0^{N\tau} u_c(t)e^{-j\omega_i t} dt.
\]

Each \( \omega_i \) is selected to be frequency between 1kHz ±2% and 2kHz ±2% in order to reduce the frequency components of the input around vibration modes of the plant. The variation range of each resonance frequency is divided into 10 frequencies to specify \( \omega_i \). The weight \( q_i \) for \( \omega_i \) is selected to be \( 1 \times 10^7 \) for all frequency points. The performance index of Eq.(30) can be represented by a quadratic form as \( J_w = U^T Q_u U \) (Hirata et al., 2002).

As an example, the step number of \( N = 80 \) and the number of terms of \( p = 7 \) are assumed, and the initial and final states are selected to be a zero velocity at the origin and a zero velocity at \( r_m = 1 \), respectively.

The frequency-shaped final-state control input obtained by using Eq.(30) without input constraint is shown in Fig.5.
In order to constrain the actual input $u_c[k]$ as $|u_c[k]| < u_{max}$, $C_z$ and $D_z$ in Eq.(12) are selected so that $z[k] = u_c[k]$ holds. The maximum absolute value $u_{max}$ is determined to be 80% of the maximum absolute value of the FFSC input without input constraint, and $u_{max} = 79$ is determined in this simulation. Then, $z_{max} = u_{max}$ and $z_{min} = -u_{max}$ are defined in Eq.(11) to obtain feedforward inputs for all methods.

For simple notation, the results of the conventional method, the proposed method 1, and the proposed method 2 are referred to as LMI FFSC, LMI PFFSC (LMI Polynomial-input-type FFSC), and LMI PFFSC-TS (LMI PFFSC with Time Symmetry), respectively.

Figs.5-7 show the time responses and their spectra of LMI FFSC, LMI PFFSC, and LMI PFFSC-TS inputs. In these figures, the result of FFSC without input constraint is shown by dashed line for comparison. Fig.5 shows that the LMI FFSC method satisfies the constraint of $|u_c[k]| < 79$ and the spectrum around 1kHz and 2kHz is well reduced.

The value of the performance index $J_p$ of the LMI PFFSC and LMI PFFSC-TS will be larger than that of $J$ of LMI FFSC because the class of input is restricted to polynomial. Therefore, the relative error $J_{RE}$ is introduced to evaluate the gap between $J_p$ and $J$:

$$J_{RE} = \frac{J_p - J}{J} \times 100 \%$$

The value of $J_{RE}$ for each method is shown in Table 2. The LMI PFFSC method shows significant performance degradation of $J_{RE} = 525\%$ while the LMI PFFSC-TS achieves a small performance degradation of $J_{RE} = 29.5$. Since $J_{RE}$ of the LMI FFSC method is 25.8, the performance degradation of the LMI PFFSC-TS method is kept to be minimum.

4.3 Simulations

Residual vibrations are evaluated when the obtained feedforward inputs are imposed to the plant. The results are shown in Fig.8. For comparison, the result of the FFSC without input constraint is also shown by dashed line. From Fig.8(b), the result of LMI PFFSC shows large residual vibration. This is because that the spectrum of the LMI

Table 2. Performance degradation

<table>
<thead>
<tr>
<th>Methods</th>
<th>$J_{RE}$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMI FFSC (Conventional method)</td>
<td>25.8</td>
</tr>
<tr>
<td>LMI PFFSC (Proposed method 1)</td>
<td>525</td>
</tr>
<tr>
<td>LMI PFFSC-TS (Proposed method 2)</td>
<td>29.5</td>
</tr>
</tbody>
</table>
Fig. 8. Time responses of output

PFFSC input is not reduced around 1kHz and 2kHz. On the other hand, the result of the LMI PFFSC-TS achieves a small residual vibration, and the performance is almost the same that of the LMI FFSC. Since the feedforward input of the LMI PFFSC-TS is generated by a polynomial with \( p = 7 \), the required memory size is reduced.

5. CONCLUSION

In this paper, we have proposed two PFFSC methods considering input saturation. The effectiveness has been evaluated by applying them to the plant having a ridged-body mode and two mechanical vibration modes. From simulation results, it has been confirmed that the time-symmetry assumption in the proposed method 2 plays an important role to achieve a small residual vibration. The proposed method can easily apply to various positioning systems such as HDD and Galvano scanner.

REFERENCES


