On Stable Inversion for Linear Systems

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Abstract: System inversion is commonly used in many control applications, such as trajectory planning, tracking, optimal control, or feedforward control. In this paper, we introduce a generic stable inversion procedure for linear time invariant systems, using lumped and distributed delays. The solution is BIBO-stable and causal in the time domain. This solution is an approximated inverse of a plant: After a finite time, the error on this approximation will be identically zero. Constructive algebraic procedures and simulation results are carried out. 

Keywords: Linear systems, inversion, time-delay systems, distributed delay, trajectory planning, stability, causality.

1. INTRODUCTION

This paper addresses stable inversion for SISO linear time-invariant systems. In control theory, inversion of dynamics is perhaps one of the most common idea to design and synthesize controllers. Inversion principles can be found in feedforward control, trajectory planning, sizing of dynamical systems, decoupling, or disturbance rejection. For a given plant $T(s)$, the basic inversion problem aims to find $C(s)$ such that $T(s)C(s) = 1$. In inversion, stability problems appear for nonminimum phase systems or strictly proper systems. For instance, the plant

$$T(s) = \frac{s - 1}{s + 1}$$

does not have a stable inverse, since the unstable zero $z_0 = +1$ yields an unstable inverse $C(s)$. From the earliest work of Silverman (1969), various extensions were proposed to address this stability problem. Among others, we can cite Garcia et al. (1995) where the stability problem on partial model matching was addressed or Devasia et al. (1996) where stable noncausal feedforward was studied for nonlinear systems. Based on this last paper, some extensions for the considered stability were also proposed in Hunt et al. (1996). It was more recently pointed out the equivalence of such noncausal solution with the two-sided Laplace transform for linear systems in Sogo (2010). Other attempts were done for 2D systems in Loiseau and Brethé (1997), in Di Loreto (2006) for model matching of time-delay systems, or in Goodwin et al. (2000) using high gain feedback.

In this paper, we address a BIBO-stable inversion for a SISO linear system. The inverse $C(s)$ we define is not an exact inverse, but satisfies an identity of the form

$$E(s) = 1 - T(s)C(s),$$

where the inverse error has a finite time support. This inverse ensures an exact inversion or an exact trajectory tracking after a finite time. Such an inverse is causal and proper, in the sense that we can realize it using elementary operations, namely integration, addition, scalar multiplication and delay. It permits to overcome a computation of successive derivatives of a desired trajectory, as well as to use BIBO-stability instead of more conservative stability notions. The solution for this approximated inversion always exists, giving us a generic solution to many control problems. Constructive methods for the computation of such a solution are described, and some examples with numerical simulations illustrate the results. The key tools for the stable inversion we address are distributed delays, which are convolution operators with finite support kernels.

The paper is structured as follows. In Section 2, we introduce some mathematical and algebraic preliminaries we will use through the paper. Section 3 is devoted to define the exact stable inversion problem. Hence, we analyze and solve the finite time stable inversion for convolution kernel in Section 4. In the last section, we solve the approximated problem for trajectory planning.

2. ALGEBRAIC PRELIMINARIES

An input-output causal convolution system is a dynamical system described by an equation of the form

$$y(t) = (f * u)(t) = \int_0^t f(\tau)u(t - \tau) d\tau,$$

(1)

where $u$ and $y$ are called the input and output, respectively, while $f$ is called the kernel, and is locally integrable over $R_+$. Denote $\mathbb{I}_{a,b}$ any finite closed interval in $R_+$, for some bounded reals $a$ and $b$, $0 \leq a < b$. We define $\mathcal{X}(\mathbb{I}_{a,b})$ as the set of real valued functions $g(\cdot)$ of the form

$$g(t) = \begin{cases} g_{a,b}(t), & t \in \mathbb{I}_{a,b} \\ 0, & \text{elsewhere} \end{cases}$$

(2)

where

$$g_{a,b}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} t^i \lambda_i^j, \quad \lambda_i \in C,$$

(3)

for some $c_{ij}$ and $\lambda_i$ in $C$, the sum being finite. In other words, $g_{a,b}$ is a finite linear combination of exponential-
polynomials type functions, and it is in particular a continuous function in $\mathbb{I}_{u_1}$. 

Definition 1. A distributed delay is a causal convolution system of the form (1), where kernel $f$ lies in $\mathcal{K}(\mathbb{I}_{u_1}, \mathbb{I}_{u_2})$, for some bounded real numbers $0 \leq u_1 < u_2$.

In other words, a distributed delay can be written like an input-output convolution operator of the form

$$y(t) = (f * u)(t) = \int_{-\infty}^{t} f(t - \tau)u(\tau)d\tau$$

with notations introduced in (1). The set of distributed delays, as in Definition 1, is denoted by $\mathcal{G}$. Any distributed delay $\mathcal{G}$ admits a Laplace transform, corresponding to the finite Laplace transform of its kernel $f \in \mathcal{K}(\mathbb{I}_{u_1}, \mathbb{I}_{u_2})$.

$$\hat{y}(s) = \hat{f}(s)\hat{u}(s), \quad \hat{f}(s) = \int_{-\infty}^{t} f(t - \tau)e^{-st}d\tau, \quad (4)$$

where $\hat{f} \in \mathcal{G}$, which denotes the Laplace transform of $f$, is an entire function, i.e. homomorphic on the whole complex plane.

To distinguish stable solutions from unstable ones, we introduce two subspaces in direct sum in $\mathcal{K}(\mathbb{I}_{u_1}, \mathbb{I}_{u_2})$, namely $\mathcal{K}_c(\mathbb{I}_{u_1}, \mathbb{I}_{u_2})$ and $\mathcal{K}_d(\mathbb{I}_{u_1}, \mathbb{I}_{u_2})$, consisting of linear combinations of exponential-polynomials type functions on some finite interval as in (2), with $\Re \lambda_i < 0$ and $\Re \lambda_i > 0$, respectively, for all $i \geq 0$.

A distributed delay being given, it is of interest to decompose it on some more simple generators for the ring $\mathcal{G}$. For this, we define the complex valued function $\theta_{\lambda}(\cdot) \in \mathcal{K}(\mathbb{I}_{0}, \mathbb{I}_{u_2})$, for some $\lambda \in \mathbb{C}$ and $\vartheta > 0$, by

$$\theta_{\lambda}(t) = \begin{cases} e^{\lambda t}, & t \in [0, \vartheta] \\ 0, & \text{elsewhere}. \end{cases} \quad (5)$$

Its Laplace transform writes

$$\hat{\theta}_{\lambda}(s) = \frac{1 - e^{-(s-\lambda)\vartheta}}{s - \lambda}, \quad (6)$$

which is an entire function even in $s = \lambda$ where $\theta_{\lambda}(\lambda) = \vartheta$, and is then BIBO-stable for any $\lambda \in \mathbb{C}$. We call $\theta_{\lambda}(\cdot)$ the elementary distributed delay. The $k$-th derivative $\theta_{\lambda}^{(k)}(s)$ of $\theta_{\lambda}(s)$ yields

$$\hat{\theta}_{\lambda}^{(k)}(s) = \int_{0}^{\vartheta} (-\tau)^{k}e^{-(s-\lambda)\tau}d\tau, \quad (7)$$

which is still in $\mathcal{G}$, and corresponds to the Laplace transform of the function $\theta_{\lambda}^{(k)}(t) = (-\tau)^{k}e^{\lambda t}$ for $t \in [0, \vartheta]$, and 0 elsewhere. From Brethélé and Loiseau (1997) or Lu et al. (2010), we have the following result.

Lemma 2. Any element in $\mathcal{G}$ can be decomposed into a finite sum of Laplace transforms of elementary distributed delays and its successive derivatives.

In other words, for any element $\hat{g} \in \mathcal{G}$, there exist complex polynomials, in the variable $e^{-s\vartheta}$, say $\hat{g}_k \in \mathbb{C}[e^{-s\vartheta}]$, and $\lambda_i \in \mathbb{C}$, in finite number, such that

$$\hat{g}(s) = \sum_{i,k} \hat{g}_k(s)e^{-s\vartheta}\hat{\theta}_{\lambda_i}^{(k)}(s).$$

Lemma 2 allows to conclude that any element $\hat{g} \in \mathcal{G}$ can be put into an entire fraction

$$\hat{g}(s) = \frac{n(s, e^{-\vartheta s})}{d(s)}, \quad (8)$$

where $n(s, e^{-\vartheta s}) \in \mathbb{R}[s, e^{-\vartheta s}]$ is a real quasipolynomial with respect to the algebraically independent variables $s$ and $e^{-\vartheta s}$, while $d(s) \in \mathbb{R}[s]$. This fraction has no pole, and has strictly negative degree with respect to $s$. Note that, like in (8), when no confusion is possible, we will denote $d(s)$ the elements in factorization instead of $d(s)$. Numerical approximation of distributed delays with kernels in $\mathcal{K}_c(\mathbb{I}_{u_1}, \mathbb{I}_{u_2})$ or in $\mathcal{K}_d(\mathbb{I}_{u_1}, \mathbb{I}_{u_2})$ was solved in Zhong (2004) or in Mirkin (2004). See also Lu et al. (2010).

Quasipolynomials in $\mathbb{R}[s, e^{-\vartheta s}]$ are elements of the form

$$N(s) = \sum_{i=0}^{p} \sum_{j=0}^{q} N_{ij}s^ie^{-j\vartheta s}, \quad (9)$$

where coefficients $N_{ij}$ lie in $\mathbb{R}$. The degree with respect to $s$ of $N(s)$ is $\deg_s N(s) = p$. The quasipolynomial $N(s)$ is said to be monic if its leading coefficient in $s$ is 1, that is, $\sum_{j=0}^{q} N_{ij}s^j e^{-j\vartheta s} = 1$. The zeros of $N(s)$ are the complex numbers $s_0$ such that $N(s_0) = 0$.

According to Pontryagin (1955), we say that $N(s)$ in $\mathbb{R}[s, e^{-\vartheta s}]$ is stable if there exists $\varepsilon < 0$ such that for all $s \in \mathbb{C}$, with $\Re s \geq \varepsilon$, $N(s) \neq 0$.

Since $\mathbb{R}[s, e^{-\vartheta s}]$ is a unique factorization domain, every nonzero quasipolynomial $N(s)$ can be decomposed into a product of irreducible factors. In particular $N(s)$ can be decomposed in a unique way by

$$N(s) = N_u(s)N_d(s), \quad (10)$$

where $N_u(s)$ is stable and monic, while no nontrivial factor of $N_d(s)$ is stable.

Actually, operations defined by $\mathcal{G}$ and $\mathbb{R}[s, e^{-\vartheta s}]$ include distributed delays, lumped delays, scalar multiplication and successive derivatives. In order to get a closed ring under these operations, and following Brethélé and Loiseau (1997), we define the ring

$$\mathcal{E} = \mathcal{G} + \mathbb{R}[s, e^{-\vartheta s}]. \quad (11)$$

Elements in $\mathcal{E}$ are called quasipolynomials. For instance, the element

$$p(s) = s + 1 + e^{-s} + \frac{1 - e^{-s}}{s}$$

is the sum of a quasipolynomial and a distributed delay, and then is in $\mathcal{E}$.

We have seen in (8) that any Laplace transform of distributed delay writes like $G(s) = N(s)D^{-1}(s)$, where $N(s) \in \mathbb{R}[s, e^{-\vartheta s}]$, $D(s) \in \mathbb{R}[s]$, and $\deg D(s) - \deg N(s) < 0$. Similarly, using a common multiple denominator in (11), any quasipolynomial $P(s) \in \mathcal{E}$ is written in the form

$$P(s) = N(s)D^{-1}(s), \quad (12)$$

where $N(s) \in \mathbb{R}[s, e^{-\vartheta s}]$, and $D(s) \in \mathbb{R}[s]$. Element $P(s)$ has no pole. The degree of a quasipolynomial is defined by $\deg P(s) = \deg N(s) - \deg D(s)$, and can be either positive or negative.

The ring of quasipolynomials is a Bézout ring. Coprimeness of quasipolynomials is also well defined, in the sense of 2D-coprimeness. Then, we say that $N(s)$ and $D(s)$ in $\mathcal{E}$ are coprime if they have no nontrivial common factor.
This comes down, in fact, to the existence of $X(s)$ and $Y(s)$ in $\mathcal{S}$ such that
\[ X(s)N(s) + Y(s)D(s) = 1. \] (12)

The realization of a fraction of pseudopolynomials by a quadruple of matrices $(A, B, C, D)$, defined as convolution operators with Laplace transforms in $\mathcal{G} + \mathcal{R}[e^{-\sigma t}]$, is equivalent to require an input-output proper transfer function in the following sense.

**Definition 3.** A rational fraction $T = ND^{-1}$, where $N, D \in \mathcal{S}$, is said to be proper if the denominator $D(s)$ is monic, and the degree condition
\[ \deg_s D(s) \geq \deg_s N(s) \]
holds. If $\deg_s D(s) > \deg_s N(s)$, we say that $N(s)$ and $D(s)$ are strictly proper.

For stable inversion, we will be interested with the set of pseudopolynomials in $\mathcal{S}$ with degree less than zero. In other words, this set, denoted by $\mathcal{P}_\mathcal{S}$, is defined by
\[ \mathcal{P}_\mathcal{S} = \mathcal{G} + \mathcal{D}, \] (13)
where $\mathcal{D}$ is the set of elements of the form
\[ d(t) = \sum_{k=0}^{r} d_k \delta(t - k\tau), \] (14)
with $d_k \in \mathbb{R}$, $r \in \mathbb{N}$ being finite, and $\delta(t)$ denotes the Dirac distribution at time $t$. Elements in $\mathcal{P}_\mathcal{S}$ have no pole, since they are pseudopolynomials. They are proper in the sense of Definition 3. They correspond in the time-domain to the truncation over a finite–time interval of rational elements in the Callier–Desoer algebra $\mathcal{S}$ of Callier and Desoer (1978).

The Banach algebra $\mathcal{S}$ is the set of elements $f$ of the form
\[ f(t) = \begin{cases} f_a(t) + f_{pa}(t), & t \geq 0 \\ 0, & t < 0 \end{cases} \]
where $f_a$ is locally integrable over $\mathbb{R}_+$, i.e. $f_a$ is a complex valued function $f_a(\cdot) \in L^1_1(\mathbb{R}_+)$, and
\[ \|f_a\|_{L^1_1} = 0. \quad \|f_a(t)\| dt \to \infty. \]
The complex-valued distribution $f_{pa}$ stands for the purely atomic part, and writes
\[ f_{pa}(t) = \sum_{n=0}^{\infty} f_n \delta(t - t_n), \]
where $f_n \in \mathbb{C}$, $n = 0, 1, ..., 0 = t_0 < t_1 < t_2 < ...$, and $\sum_{n \geq 0} |f_n| \to \infty$.

### 3. STABLE INVERSION PROBLEM

To address stable inversion, we need first to define precisely the stability we consider, namely stability in the sense of Bounded Input–Bounded Output.

**Definition 4.** A convolution system (1) is said to be BIBO-stable if its kernel $f$ lies in $\mathcal{S}$.

For linear systems, and more generally for proper fractions in the sense of Definition 3, BIBO-stability is equivalent to require that all the poles of the plant are in the open left-half complex plane. Hence it corresponds to the stability notion introduced for quasipolynomials.

Exact stable inversion can be stated as follows. Let $T$ be a transfer function of a given linear plant. Such a plant admits a coprime factorization over $\mathbb{R}[s]$, say $(N, D)$, such that
\[ T(s) = N(s)D^{-1}(s), \]
and that verify a Bézout type identity, that is there exist $X(s), Y(s)$ over $\mathbb{R}[s]$ such that $XN + YD = 1$. Exact stable inversion is a control problem which aims to find a plant $R(s)$, which is stable in the sense of Definition 4, such that
\[ T(s)R(s) = 1. \] (15)

This problem is illustrated in Fig. 1. In other words, if $\hat{y}(s) = T(s)\hat{u}(s)$, this problem is equivalent to have a stable plant $R(s)$ such that $\hat{u}(s) = R(s)\hat{v}(s)$, where $\hat{v}$ corresponds, if a stable solution exists, to $y$. From Definition 4, stability means that for any bounded function $y(t), t \geq 0$, the output function $u(t)$ is also bounded.

![Fig. 1. Exact stable inversion](image)

The main constraint for the existence of a solution comes from stability condition. Such a constraint is synthesized in the following obvious result, noting that any element in $\mathcal{S}$ is in fact proper in the sense of Definition 3.

**Theorem 5.** (Loiseau and Brethé, 1997) There exists a solution to the exact stable inversion if and only if $T$ has no unstable zeros, and
\[ \deg_s N(s) = \deg_s D(s). \]

For instance, the plants
\[ \frac{s-1}{s+1}, \quad \frac{1}{s-1} \]
have no solution to the exact stable inversion problem, since for the first one there is an unstable zero, while for the second the inverse is not proper.

From Theorem 5, it appears that exact stable inversion is too restrictive. Among possible extensions, we may define an approximation of the inverse. The key idea is to find a solution such that, in the time-domain, after a finite time, this approximated inverse coincides with exact inverse. This deep approximated inverse need to be characterized by a stable and proper precompensator.

Unstable dynamics are constrained to lie only in this finite time interval, the other dynamics being stable. This will give us a BIBO–stable solution. In other words, exact stable inverse is obtained after a finite time interval. This problem, called hereafter the finite time stable inversion, is defined and solved in the next section.

### 4. FINITE TIME STABLE INVERSION

In this section we define the finite time stable inversion, and propose a general procedure for the computation of a precompensator solution.

This inverse problem can be stated as follows. Let $T(s)$ be a given plant. Find a stable precompensator $R(s)$ such that the error on inversion
\[ \hat{e}(s) = 1 - T(s)R(s) \] (16)
is the Laplace transform of a causal function with finite support. Such a precompensator $R(s)$ is said to be a solution to the finite time stable inversion problem, while $\hat{e}(s)$ is called the inversion error. This problem is illustrated in Fig. 2 and 3.

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6653
Among numerous classes where the inversion error \( \hat{e}(s) \) may be defined, we consider only a class of rational elements, in the sense that algebraic procedures can be handled for its computation. For this, we impose that \( e(t) \) be in the set \( \mathcal{P}_E \) defined in (13). Any element in \( e \in \mathcal{P}_E \) is of the form

\[
e(t) = g(t) + \sum_{k=0}^{r} e_k \delta(t - k\vartheta),
\]

(17)

where \( g \in \mathcal{G} \) is a distributed delay, and \( e_k \in \mathbb{R} \), for \( k = 0, \ldots, r \). In the Laplace domain, this is equivalent to say that any \( \hat{e}(s) \in \mathcal{P}_E \) can be written like

\[
\hat{e}(s) = \hat{g}(s) + \sum_{k=0}^{r} e_k e^{-k\vartheta s}.
\]

(18)

Fig. 2. Finite time stable inversion

Elements in \( \mathcal{P}_E \) are proper in the sense of Definition 3, and BIBO-stable, since they have a finite support. An illustration is made in Fig. 3, where delayed Dirac distributions arise at time \( t_i \), for \( i = 1, 2, 3 \), which correspond to multiple commensurate delays of \( \vartheta > 0 \). After a finite time \( t_f \), the inversion error \( e(t) \) will be identically zero. The time interval \([0, t_f]\) enclose the support of \( e(t) \). Its length is related to the support of the distributed delay \( g(t) \) and \( r\vartheta \).

Fig. 3. Time-domain evolution of the inverse error \( e(t) \).

Using this class, finite time stable inversion comes down to find a stable and proper precompensator \( R(s) \) such that

\[
\hat{e}(s) = 1 - T(s)R(s) \in \mathcal{P}_E.
\]

(19)

We give next a complete characterization of a solution to the above problem, with a constructive proof.

**Theorem 6.** The problem of finite time stable inversion always admits a proper and stable solution.

**Proof.** The proof is by construction. Take any coprime factorization \((N, D)\) of \( T(s) \) in \( \mathbb{R}[s] \). We first decompose \( N \) into a stable–unstable factorization, like in (10), that is

\[
N(s) = N_u(s)N_u(s),
\]

where \( N_u \) is monic and has stable zeros, while \( N_u \) has only unstable zeros.

The second step is a particular decomposition of (1). Take any arbitrary element \( Q \in \mathcal{P}_E \), satisfying

\[
\deg Q(s) \leq \min(\deg N_u(s) - \deg D(s), -\deg N_u(s)).
\]

(20)

Define \( X = Q \) and \( Y = 1 - QN_u \). Then we have

\[
X(s)N_u(s) + Y(s) = 1.
\]

(21)

This means that \( N_u(s) \) and 1 are obviously coprime over \( \mathcal{P}_E \).

The third and last step is the construction of the precompensator. For this, define

\[
R(s) = D(s)N_u^{-1}(s)X(s).
\]

(22)

From (20), we see that \( R(s) \) is proper. Since \( N_u^{-1} \) is stable and \( X \in \mathcal{P}_E \), we see that \( R(s) \) is stable. Furthermore, using (21) and (22), the inversion error is

\[
\hat{e}(s) = 1 - T(s)R(s) = X(s)N_u(s) + Y(s) - T(s)R(s) = Y(s).
\]

The inversion error is \( Y(s) = 1 - X(s)N_u(s) \), and using (20), we see that such a solution satisfies

\[
\deg Y(s) \leq 0,
\]

and is in \( \mathcal{P}_E \). Hence \( e(t) \in \mathcal{P}_E \), which completes the proof.

The above theorem gives a complete answer to the finite time stable inversion for a linear plant, for which a solution always exists and can be algebraically computed. Obviously, such a solution is highly non unique, from the free parameter \( Q \) in \( \mathcal{P}_E \). Note also that the plant \( T(s) \) can be stable or unstable, as proper or strictly proper. In practice, if there is at least one unstable zero in the plant, \( Q \) will be a distributed delay, with an arbitrary negative degree given by (20).

**Example 7.** Consider the plant

\[
T(s) = \frac{s - 1}{s + 1},
\]

where \( N_u(s) = s - 1 \), \( N_s(s) = 1 \) and \( D(s) = s + 1 \). Take for instance

\[
X(s) = \frac{1 - e^{-(s-1)\vartheta} - \vartheta}{s - 1} \in \mathcal{P}_E.
\]

A precompensator that solves the finite time stable inversion is

\[
R(s) = (s + 1) \frac{1 - e^{-(s-1)\vartheta} - \vartheta}{s - 1}.
\]

since the inversion error is \( \hat{e}(s) = Y(s) = e^{-(s-1)\vartheta} \). Note that there is no unstable pole-zero cancellation, since +1 is not a pole of \( X(s) \). In the time domain, this inverse error is

\[
e(t) = e^{\vartheta}(t - \vartheta),
\]

i.e. the error is a delayed Dirac at time \( t_f = \vartheta \). Obviously, other choices for \( X(s) \) are possible, leading to another inverse error with finite time support. A simulation is shown in Fig. 4, where the inversion error is plotted with \( \vartheta = 1 \).

**Example 8.** Let \( T(s) \) be the stable strictly proper plant

\[
T(s) = \frac{1}{s + 1},
\]

with \( N_u = 1 \), \( N_s = 1 \) and \( D(s) = s + 1 \). Take

\[
X(s) = \frac{1 - e^{-\vartheta(s+1)}}{s + 1} \in \mathcal{P}_E.
\]

Using the proper and stable precompensator \( R(s) = 1 - e^{-(s+1)\vartheta} \), the inversion error is

\[
\hat{e}(s) = 1 - \frac{1 - e^{-\vartheta(s+1)}}{s + 1} \in \mathcal{P}_E.
\]
In the time-domain, the error writes
\[ e(t) = \begin{cases} \delta(t) - e^{-t}, & t \in [0, \vartheta] \\ 0, & \text{elsewhere} \end{cases} \]

Simulation results of the inversion error in time are shown in Fig. 5, for three different values of \( \vartheta \).

Fig. 5. Kernel of the inversion error in time for \( \vartheta = 1 \) (continuous), \( \vartheta = 0.5 \) (dashed) and \( \vartheta = 0.1 \) (dashed-dot).

Previous results on stable inversion give generic solutions, and are concerned with the stable inversion of the convolution kernel, or in other words, for a Dirac distribution input. The inversion error kernel is defined to have a finite time support. When another arbitrary bounded input is taken, the output error will be only bounded, but has no particular property in the time-domain, except the case when this input has a finite support, in which case the output will have finite support. For instance, for any constant persistent input, an inversion error in \( \mathcal{G} \) goes to a finite constant value. In practice, inversion is understood from an input–output point of view. So we may have some arbitrary input, and we shall obtain an inversion error with finite support. This aim, of course, comes down to take into account the exogenous model of the input signal into the plant \( T(s) \). So we address this inversion in the next section, and illustrate it on the trajectory planning.

5. TRAJECTORY PLANNING

Let be a given bounded desired trajectory \( y_d(t), t \geq 0 \). We want to find, with stability, an input \( u_d(t), t \geq 0 \), such that \( y_d(t) = T(t) * u_d(t) \), where \( T(t) \) denotes the impulse response of \( T(s) \). As for the stable kernel inversion, this problem appears to be too restrictive. Following the case of kernel inversion, we propose here to solve the following less restrictive problem.

We may find a proper and stable precompensator \( R(s) \), such that the output of the plant \( y(t) \) satisfies, for some \( t_f \geq 0 \),
\[ y(t) = y_d(t), \ \forall t \geq t_f. \]  (23)

For this, take \( v(t) = y_d(t), t \geq 0 \). The inversion error is
\[ e(t) = y_d(t) - y(t), \ t \geq 0, \]  (24)
where
\[ y(t) = T(t) * R(t) * y_d(t), \ t \geq 0. \]  (25)

In the Laplace domain, this identity is of the form
\[ \tilde{e}(s) = T_m(s) - T_e(s)R(s), \]  (26)
where \( T_m(s) = \tilde{y}_d(s) \), and \( T_e(s) = T(s)\tilde{y}_d(s) \).

There is a matching problem, where \( T_m \) is a given model we try to realize using a feedforward control. The stable trajectory planning (23) is then equivalent to require that \( \tilde{e}(s) \) be in \( \mathcal{D}_e \), as for the finite time stable inversion. This problem is illustrated in Fig. 6, where \( e \) denotes a Dirac distribution input at time \( t = 0 \).

Fig. 6. Trajectory planning as a model matching problem.

The trajectory planning problem is similar to the kernel inversion problem. We include in the plant the exogenous model of the desired output. In practice, this model will be incorporated into the proper precompensator \( R \). This form is strongly related to the internal model principle.

Theorem 9. The trajectory planning problem in (23) always admits a proper and stable solution.

Proof. To solve this problem, following Di Loreto (2006), we denote \( T_e = ND^{-1} \), \( T_m = N_mD_m^{-1} \), where the pairs \((N, D)\) and \((N_m, D_m)\) are coprime and defined over \( \mathbb{R}[s] \), or \( \mathcal{E} \), if the exogenous model of \( \tilde{y}_d \) includes some lumped or distributed delays. As in (10), we factorize the stable and unstable parts of \( N(s) \) like
\[ N(s) = N_u(s)N_u(s). \]
If \( N_u \) and \( D_m \) are coprime, there exist \( X(s) \) and \( Y(s) \) in \( \mathcal{E} \) such that
\[ X(s)N_u(s) + Y(s)D_m(s) = 1. \]  (27)
If this is not the case, we can divide by the common factor and will work on the quotients. So there is no restriction in this assumption. The model \( T_m \) being proper, \( D_m \) is monic. We can divide \( X(s)N_u(s) \) by \( D_m \). We see that there exists \( Q(s), P(s) \in \mathcal{E} \), with \( \deg Q, P < \deg D_m(s) \) such that
\[ X(s)N_u(s) = Q(s)D_m(s) + P(s). \]  (28)
Since $P(s)$ is non unique in this division, we can take it with arbitrary negative degree. Define $R(s) = D(s)N^{-1}(s)P(s)D^{-1}_m(s).$ This element is proper, and stable if $D_m$ is stable (this is precisely the trajectory we are planning). With this precompensator, the finite time model matching error will be

$$
\dot{e}(s) = T_m(s) - T_e(s)R(s) = Y(s)N_m(s) + N_u(s)Q(s).
$$

This error is in $\mathcal{D}$. Since $\dot{e} = T_m - T_eR$, and the right-hand side is proper, we deduce that $\dot{e}$ is proper. So it is in $\mathcal{P}_\mathcal{D}$, which concludes the proof. 

**Example 10.** Take $T(s) = s + 1$, and consider a desired output $y_d(t) = 1, t \geq 0.$ We have $T_m = s^{-1}, T_e = \frac{1}{s+1},$ that is $N_e = 1, N_m = 1, N_u = s - 1, D = s(s+1), D_m = s.$

Take

$$X(s) = \frac{1}{1-e^{-s}} \left[ 1 - e^{-(s-1)\vartheta} \right], Y(s) = -e^{s\vartheta} \left[ 1 - e^{-s\vartheta} \right],$$

that satisfy $X(s)N_u(s) + Y(s)D_m = 1.$ Division (28) yields $Q = 0$ and $P(s) = \frac{1-e^{s\vartheta}}{s+1} \left[ 1 - e^{-(s-1)\vartheta} \right].$ A precompensator takes the form

$$R(s) = \frac{1-e^{s\vartheta}}{s+1} \left( 1 - e^{-(s-1)\vartheta} \right),$$

while the tracking error is

$$\dot{e}(s) = -e^{s\vartheta} \left[ 1 - e^{-(s-1)\vartheta} \right].$$

In the time domain, this error is

$$e(t) = \left\{ \begin{array}{ll} -e^{t\vartheta}, & t \in [0, \vartheta] \\ 0 & \text{elsewhere} \end{array} \right. .$$

After a time $t_f = \vartheta$, we have $y(t) = y_d(t)$, using a precompensator $R(s)$ which is stable and proper. Simulation results are shown in Fig. 7.

6. CONCLUSION

In this paper, we define and solve an approximation problem on inversion for linear systems, for which the inversion error is required to be defined over a finite time support. This gives us a stable and proper solution to realize practical inversion. Solutions are causal, and are easily simulated. Prospects include the link with realizability theory, optimization, and the link with high gain feedback. Indeed, in some cases, if the finite time on which the error is defined goes to zero, the precompensator tends to take high gains, like purely distributional gains. When such a limit is valid or not requires some future works.

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