A game-theoretic model of interactions between agents with different beliefs in the form of linear cognitive maps

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Abstract: A game-theoretic model of conflicts of interests between agents in semistructured situations is considered. Linear cognitive maps are used as a model of agent beliefs about situations. Cognitive maps of different agents may vary.

1. INTRODUCTION

Cognitive maps were previously introduced by Axelrod (1976) to clarify and improve decision making process. A cognitive map is a weighted digraph-based mathematical model of a decision maker belief system about some limited domain, such as a policy problem. Cognitive map nodes correspond to situation concepts. Concepts are interpreted as variables those values may vary. Weighted edges are interpreted as direct causal links from one concept to another. Analysis of possible situation developments depending on the control (in terms of an influence on some concepts) is one of the possible applications of cognitive maps. Both direct (situation development prediction with the fixed control) and inverse (search of the appropriate control) cognitive analysis problems are considered for this purpose.

The game-theoretic model of interactions between several agents at a dynamic system in the form of a situation cognitive map was generally considered by Novikov (2008b). Since any cognitive map is a mathematical model of the belief system for the specific problem domain, we can address the problem of interactions between agents with different belief systems. This means that different agents operate under different cognitive maps during a situation analysis and strategy choice. Given the fact that each agent can realize the difference in situation understanding among agents, it is reasonable to consider the possibility of their reflexive perception. In other words, each agent not only realizes the situation development, but also understands what other agents think about the situation. In this paper we consider a conflict interaction between agents; generally each agent has his own cognitive map (see Fig. 1).

In this case different agents may predict the results of mutual actions in different ways. This induces the particularity of their interaction. In particular, in a two-agent game with one target concept and different desirable values of this concept there exist the possibility of the mutual agent influence such that each agent thinks that he entirely succeeds. In other words, differences in beliefs can compensate a significant dissimilarity in targets and cause a complete agreement, where the agreement can't exist in case of equal beliefs.

Fig. 1. Cognitive map-based belief systems of agents regarding the situation

2. SITUATION CONTROL AT A LINEAR COGNITIVE MAP-BASED MODEL

A linear cognitive map is called a weighted digraph, if its nodes (concepts) and edges (causal links) meet the conditions stated below, and the aforementioned rule regarding node value dynamics is given. By \(M = \{1, \ldots, m\}\) denote the set of all concepts. A causal concept is a concept where an edge starts; an effect concept is a concept where an edge ends. Thereafter let an adjacency matrix of the digraph \(W\) be a matrix with elements \(w_{ij} \in R\), if elements of the matrix correspond to weights of graph edges, which define types and strengths of causal links. Strength of the causal link from the \(j\)-th causal concept to the \(i\)-th effect concept is equal to the absolute value of the edge weight \(|w_{ij}|\). The sign of the edge weight corresponds to the link type: if \(w_{ij} > 0\), then the causal link from the \(j\)-th concept to the \(i\)-th one is positive, if \(w_{ij} < 0\), then the causal link is negative (Roberts (1976)).

All results were obtained for discrete time and the zero-time initial state. An pulse process of a cognitive map is defined by the rule (1) with the initial concept vector \(x(0) = (x_1(0), x_2(0), \ldots, x_m(0))\), \(x(0) \in R^m\), and the vector \(p(0) = (p_1(0), \ldots, p_m(0))\).
$p_2(0), \ldots, p_m(0)$, $p(0) \in R^m$ of an external pulse to each node at the zero time point (Roberts (1976)).

$$x(t+1) = x(t) + p(t), \quad \text{where}$$

$$p(t) = \left\{ \begin{array}{l} p_i(0), \quad t = 0 \\ \sum_{j=0}^n w_{ij} \cdot p_j(t-1), \quad t = 1, 2, 3, \ldots \end{array} \right. \quad (1)$$

Let us fix the discrete time point $T$ ($T > 0$). Then the concept vector $x(T)$ is defined by the expression:

$$x(T) = x(0) + p(0) + p(1) + \ldots + p(T-1) = x(0) + p(0) + p(0)W + \ldots + p(0)W^{T-1} = x(0) + p(0)(E + W + \ldots + W^{T-1}) = x(0) + p(0)\cdot Q.$$

Where $E$ is an identity matrix. Let a matrix $Q = E + W + \ldots + W^{T-1}$ be a matrix of an influence reachability by the time $T$ for the adjacency matrix $W$. Then the sum of the consequent increments for the concept $x_j$ is as follows:

$$\sum_{t=0}^T p_j(t) = \sum_{t=0}^T q_{ij} \cdot p_i(0). \quad (2)$$

Where $q_{ij}$ are elements of the matrix $Q$. Let us consider the problem of an semistructured situation control at a linear cognitive map-based model. Let control actions be external pulses to each node at the zero-time point $p(0)$; where $p(0) = 0$, if there is no control to the node $j$. A control effect is a set of all concept values at the time point $T$:

$$x_j(T) = x_j(0) + \sum_{t=0}^T p_j(t), \quad j \in M \quad (3)$$

A control target is defined by desirable values for all or some concepts $x(T) = (x_1(T), x_2(T), \ldots, x_n(T)), x(T) \in R^m$. In this case search of the control reduced to solution of the equation system $p(0) \cdot Q = x(T) \cdot x(0)$. (Roberts (1976))

3. DESCRIPTION OF A GAME-THEORETIC MODEL

Let us consider a problem of a situation control with a multiple agent influence at the zero time point. An agent choice of a control action is determined by the rule of a rational choice (Myerson (1991)). In this case an inter-agent game occurs, i. e. an agent interaction where the utility of each agent depends on his choice of his own action (strategy) as well as on the action choice of other agents. By $N = \{1, \ldots, n\}$ denote the set of all agents. Each agent $i \in N$ can be formally represented by the three parameters $<S_i, f_i, C_i>$:

1) the set of strategies $S_i$ (the ability of control actions on the situation);
2) the utility function for the agent $f_i$ (the target of control);
3) the linear cognitive map $C_i$ (the agent knowledge about the situation).

Let us consider each parameter in detail. Let linear cognitive maps of agents $C_1, C_2, \ldots, C_n$ have the equally ordered set of concepts $M = \{1, \ldots, m\}$, but concept causal links may vary from agent to agent. Thus, cognitive maps of different agents have different adjacency matrices of digraphs $W(1), W(2), \ldots, W(m)$ in general case. However, some agents (or all agents) may have the same adjacency matrix $W(1) = W(0)$. Discrete time equally flows for all agents at cognitive map-based models. The initial time point is fixed and equal to zero. All agents know the initial concept vector $x(0) \in R^n$.

The agent with the number $i \in N$ has a nonempty subset of concepts $M_i \subseteq M$ he can control. Let $M_i$ be a set of controlled concepts of the $i$-th agent. For any two agents $i, j \in N$: $M_i \cap M_j = \emptyset$ and $M = \cup_{i=1}^n M_i \subseteq M$. By $m_i$ denote the number of concepts at the set $M_i$.

A control action of each agent is contained in a vector of mutual control actions $p(0) = (p_1(0), p_2(0), \ldots, p_m(0))$. Let the strategy $s_i$ of the $i$-th agent be a vector of ordered components of the vector $p(0)$ with indices from the set $\{k_1, k_2, \ldots, k_n\} = M$: $s_i = \{p_{k_1}(0), p_{k_2}(0), \ldots, p_{k_m}(0)\}$. Each agent defines only “his” components of the vector $p(0)$ during the influence on the situation. If there are no agent who influences on the concept, then the corresponding component is null: $(\forall j \in M \setminus \cup_{i=1}^n M_j), p_j(0) = 0$.

Control actions require some expense of limited resources. Let us impose the basic restrictions to control actions for each concept in the form of the interval of acceptable values: $(\forall j \in \cup_{i=1}^n M_j)\ \ p_j(0) \in [p_j^\text{min}, p_j^\text{max}]$, where $p_j^\text{min}, p_j^\text{max} \in R$. Then the set of $i$-th agent strategies $S_i$ can be represented as the Cartesian product $m_i$ of intervals $[p_i^\text{min}, p_i^\text{max}] \times [p_i^\text{min}, p_i^\text{max}] \times \ldots \times [p_i^\text{min}, p_i^\text{max}]$. Let the hypercube $(s_1, s_2, \ldots, s_n) \in S_1 \times \ldots \times S_n$ be the set of all agent strategies $S_i \times \ldots \times S_n$.

Let us define the utility function $f_i(x_1(T), x_2(T), \ldots, x_n(T))$ on the result set for each agent. The control target of $i$-th agent is the maximization the function $f_i$. If the $i$-th agent want to unrestrictedly approximate the concept value $x_i$ to some advantageous value $x_i^\text{opt}$, then it is desirable for him to maximize the expression $-\langle x_i(T) - x_i^\text{opt}\rangle^2$. If the agent $i$ can define desirable values for several concepts, then the weighted sum should be maximized according to the above stated expressions for such concepts. Each coefficient is interpreted as an “importance percentage” of restrictions on the corresponding concept.

The utility function of the $i$-th agent is as follows:

$$f_i(x_1(T), x_2(T), \ldots, x_n(T)) = -\sum_{j=0}^T \gamma_j \cdot (x_j(T) - x_j^\text{opt})^2 \quad (4)$$

Where $\gamma_j$ is the “importance percentage” of the $j$-th concept value for the $i$-th agent, $\gamma_j \in [0, 1]$, the sum of all $\gamma_j$ at the right hand side of the expression (4) is equal to 1.

Let the target concept of the $i$-th agent be a concept with $\gamma_j \neq 0$ at the utility function (4).

After the definition of all game parameters, let us represent the game at the normal form:

$$\Gamma = \left\{ N, \left\{ S_i \right\}_{i=1}^n, \left\{ f_i \right\}_{i=1}^n, \left\{ C_i \right\}_{i=1}^n \right\} \quad (5)$$
Let us enumerate all hypotheses for the game (5):
1) concept sets of all agents are the same at cognitive maps;
2) sets of controlled concepts are defined and mutually disjoint for all agents;
3) each agent knows cognitive maps, utility functions and strategy sets of other agents;
4) each agent believes in adequacy of his cognitive map only;
5) points 1-4 are the common knowledge of all agents.

4. NASH EQUILIBRIUM SEARCH

Let us consider the game (5). We choose an agent \( i \in N \) and consider a situation from his point of view. Now we reconstruct his utility function \( f_i(x_i(T), x_s(T), ..., x_n(T)) \) into those one with explicit dependence of his payoff from actions of all agents \( g_s(s_1, s_2, ..., s_n) \). We substitute \( x_i(T) \) for the right hand side of expressions (3) and (2) in (4).

\[
\sum_{j \in M_i} \gamma_{ij} \left[ -(x_i(T) - x_i^*)^2 \right] = 
\]
\[
= -\sum_{j \in M_i} \gamma_{ij} \cdot (x_i(0) + \sum_{k=1}^{M_i} q_{ij} \cdot p_k(0) - x_i^*)^2 .
\]

Notice that we use elements \( r_{ijk}^{(0)} \) of the matrix of influence reachability by the time point \( T \) of the adjacency matrix of the digraph \( W^{(0)} \) during the substitution of the expression (2). It’s substantiated by the hypothesis 4 at the point 3 that each agent believes in adequacy of his cognitive map only. Thus, we obtain the target function (6) from the utility function (4).

\[
g_i = -\sum_{j \in M_i} \gamma_{ij} \cdot (c_i + \sum_{k=1}^{M_i} q_{ij}^o \cdot p_k(0) - x_i^*)^2 = 
\]
\[
= -\sum_{j \in M_i} \gamma_{ij} \cdot (c_i + q_{ij}^o \cdot s_j^* + q_{ij}^o \cdot s_j^*)^2 .
\]

Where \( c_i = x_i(0) - x_i^* \), \( r_{ij}^{(0)} \), \( r_{ij}^{(0)} \) are vectors of the correspondent coefficients \( r_{ijk}^{(0)} \) of the first and second sums respectively, \( s_j^* \) is the transposed strategy vector, and \( s_j^* \) is the transposed strategy vector of all agents other than \( i \)-th. The function (6) is strictly concave with respect to the strategy variable \( s_j^* \): \( \forall a \in (0, 1) \)
\[
g_i(\alpha \cdot s_j^* + (1 - \alpha) \cdot s_j^* + s_j^*) > 
\]
\[
> \alpha \cdot g_i(s_j^* + s_j^*) + (1 - \alpha) \cdot g_i(s_j^* + s_j^*).
\]

Being a superposition of the continuous function, the function (6) is continuous in all its arguments. According to the respective theorem from Owen (1968), there exist a pure strategy Nash equilibrium at the game (5).

Note that the game (5) formally resembles to the Cournot competition (Moulin (1981)). For the search of the equilibrium at the game (5) we construct the system of equations for the search of the best answers of each agent \( \sum_{l \in M_i} \left( \sum_{j \in M_i} \gamma_{ij} \cdot q_{ij}^o \cdot f_i(0) - \gamma_{ij} \cdot q_{ij}^o \cdot \gamma_{ij} \cdot q_{ij}^o \cdot f_i(0) - \gamma_{ij} \cdot q_{ij}^o \cdot f_i(0) \right) \cdot p_i(0) = 
\]
\[
\forall l \in M_i, \forall i \in N \quad \text{by analogy with solving the problem of Cournot competition. After the following transformation we get the system of equations at the explicit form:}
\]
\[
\sum_{l \in M_i} \left( \sum_{j \in M_i} \gamma_{ij} \cdot q_{ij}^o \cdot f_i(0) - \gamma_{ij} \cdot q_{ij}^o \cdot \gamma_{ij} \cdot q_{ij}^o \cdot f_i(0) - \gamma_{ij} \cdot q_{ij}^o \cdot f_i(0) \right) \cdot p_i(0) = 
\]
\[
\forall l \in M_i, \forall i \in N .
\]

As noted above at (7), there are no control actions for uncontrolled concepts \( p_k(0) = 0 \), \( \forall k \notin \cup_{j \in M_i} M_j \). It is argued that if there exist the solution of the equation system (7) and the solution belong to the hypercube \( S_1 \times \cdots \times S_n \), then it is the pure strategy Nash equilibrium for the game (5). We can also argue that if the point of the pure strategy Nash equilibrium for the game (5) belong to the interior part of the hypercube \( S_1 \times \cdots \times S_n \), then it is the solution of the equation system (7).

From the previous two statements and the existence of the pure strategy Nash equilibrium for the game (5) it follows that if there are no solution of the equation system (7) which belong to the hypercube \( S_1 \times \cdots \times S_n \), then the point of the pure Nash equilibrium for the game (5) belong to the border of the hypercube \( S_1 \times \cdots \times S_n \).

5. EXAMPLE

Let us consider the example of two-agent interaction; cognitive maps of agents include three concepts «A», «B» and «C» (see the Fig. 2).

![Fig. 2. Cognitive maps of two agents](image-url)

Initial values of each concept are stated at graph nodes: \( A = 1, B = -7, C = 11 \). Let «A» be the target concept for both agents, «B» be the controlled concept of the first agent, «C» be the controlled concept of the second agent. Let \( T = 3 \) be the time point of the control effect. Control actions on the concept “B” and “C” are limited by the intervals [-30, 30] and [-20, 20] respectively. These intervals are sets of the first and second agent strategies respectively: \( S_1 = [-30, 30], S_2 = [-20, 20] \). Then we consider the two-agent game of the form (5). Let the first agent want the value of the concept «A» to unrestricted approximate to 0, and the second agent want the value of concept «A» to approximate to 2. In this case the agent target functions are:
\[
f_1 = -(x_i(3) - 0)^2 = (-1 - 0.04 p_d(0) + 0.02 p_k(4))^2, \quad f_2 = -(x_i(3) - 2)^2 = (-0.08 p_d(0) - 0.35 p_k(0) - 1)^2 .
\]

The vector (26.61, 3.23) \( \in S_1 \times S_2 \) is the
solution of the equation system (7). This solution is the unique Pareto optimal solution at the game, since only this solution provides the maximum possible payoff for both the first and second agents. This example illustrates the fact that agents with different initial targets can make an “action” coalition due to the difference in beliefs.

Let us change the influence restrictions. Let \( S_1 = [-10, 10], S_2 = [-15, 15] \). Then the solution of the equation system (7) is \((26.61, 3.23) \notin S_1 \times S_2\). In this case the bound point of \( S_1 \times S_2\): \((10, -0.57)\) is the unique Nash equilibrium. The values of agent utility functions are \( f_1(10, -0.57) = -0.35, f_2(10, -0.57) = 0\). However, the first agent can improve his payoff, if he tells misleading information about his cognitive map to the second agent; this alerts the volume of the following actions: \( B \rightarrow C \) from \(+1.2\) to \(+0.5\), \( C \rightarrow B \) from \(+0.9\) to \(+0.3\). Notice that the alert case doesn’t fit to our hypotheses stated at the point 3 that each agent knows true cognitive maps of other agents. In this case the Nash equilibrium of the second agent is shifted to the point \((-6.71, -4.39)\). The first agent can make it use keeping his true optimal strategy. The informational equilibrium \((10, -4.39)\) (Novikov, Chkhartishvili (2005), (2008a)) is the game effect, the values of utility functions are \( f_1(10, -4.39) = -0.26, f_2(10, -4.39) = -1.79\). Comparing payoffs of the first agent in two cases we can see that he improves the payoff from \(-0.35\) to \(-0.26\) through the disinformation of the second agent.

6. CONCLUSION

In this paper we considered the game of agents with different beliefs in the form of cognitive maps. We obtained the solution at the form of the pure strategy Nash equilibrium. Let us emphasis two main points. Firstly, the internal agent belief of the situation dynamics was formally described. In this case agents predict results of the action in different ways, but each agent believes in the adequacy of his result only. As a consequence, the agent payoff is calculated according to his belief only; during the decision making process each agent realizes that other agents have other targets and they perceive the environment in different ways. So, we can consider reflexive games (Novikov, Chkhartishvili (2005), (2008a)) at cognitive maps. Secondly, agent beliefs regarding the situation are represented by different cognitive maps, and there is no any information about the true cognitive map if such map exists. In other words, our problem of an agent action prediction doesn’t depend on the agent (or nobody), who has the adequate belief regarding the situation in general.

In the context of examples under the point 5, we demonstrated some effects appeared during the modeling of agent interactions with different beliefs. For example, agents with different initial targets and different awareness can make an “action” coalition. The example also shows that there is a possibility of improving the agent payoff through his disinformation of his opponents regarding his cognitive map. This allows setting and solving problems of informational control at cognitive maps, where agents as well as third parties can maintain control of their targets.

The results of this study can be used to develop decision support systems for conflict in semistructured situations.

REFERENCES


