Analysis of Networked Event-Based Control with a Shared Communication Medium: Part II - Slotted ALOHA *

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Abstract: We analyze the performance of event-based control, where the loop is closed with an ALOHA communication system. Therefore, we first derive the packet loss probability of slotted ALOHA with an arbitrary arrival process. Then, we use this result to get the packet loss probability of event-based control with slotted ALOHA. Finally, we take the packet loss probability as well as the packet delay into account to compare the performance of event-based control with slotted and unslotted ALOHA.

Keywords: Networked Control System, event-based control, ALOHA communication system.

1. INTRODUCTION

In Åström and Bernharsson (2002) the time-triggered and event-based approaches were compared from a control point of view to scalar systems without packet losses. For systems with integrator dynamics, it was shown that time-triggered control needs three times more samples than event-based control to achieve the same performance. Hence, it is often suggested to use event-based control for NCS because it reduces the utilization of the shared resources, see, e.g., Otanez et al. (2002); Heemels et al. (2008); Henningsson et al. (2008); Wang and Lemmon (2002); Lunze and Lehmann (2010).

However, from a communication point of view it is not that simple because loss and delay also depend on the traffic pattern. Rabi and Johansson (2009) started to consider some details of the bus arbitration. Therefore, a loss model, where the loss probability depends on the parameters of the control and communication system is introduced. Based on this model, time-triggered and event-based control for an integrator system is compared with the assumption that the packet loss probability is equal.

In Blind and Allgöwer (2011), we continued to analyze the interaction between control and communication. In that work, we compared time-triggered and event-based control of NCS with a shared communication system with the assumption that the bandwidth is equal. Therefore, we assumed that the agents send whenever an event occurs, i.e., unslotted ALOHA was used to transmit the event-based traffic. Under these assumptions, the performance of event-based control is drastically reduced because of packet losses and time-triggered control turns out to be superior. Obviously, there exist many other communication systems, which are able to transmit the event-based traffic with a smaller packet loss probability than unslotted ALOHA. In most cases, this is achieved by buffering packets, i.e., loss is traded against delay. In this work, we use the slotted ALOHA communication system, where packet loss is reduced by the usage of timeslots, for a first analysis of this tradeoff.

The remainder of this work is outlined as follows. We first give the problem setup in Sec. 2.1 and describe slotted ALOHA in Sec. 2.2. We use these results in Sec. 3 to derive the loss probability of slotted ALOHA, when the traffic is generated by event-based control. Finally, we compare the performance of event-based control with the two versions of ALOHA in Sec. 4 and conclude in Sec. 5.

2. PRELIMINARIES

2.1 Control World

The problem setup is similar to the one of Åström and Bernharsson (2002); Rabi and Johansson (2009); Blind and Allgöwer (2011). There are $N$ agents, each has to control a system with integrator dynamic:

$$dx_i = u_i dt + dv_i,$$  \hspace{1cm} (1)

where $x_i \in \mathbb{R}$ is the state of system $i$, the disturbance $v_i(t) \in \mathbb{R}$ is a Wiener process with unit incremental variance and $u_i(t) \in \mathbb{R}$ the control signal. We assume that the control signal $u_i(t)$ is a sequence of impulses, which reset the state to the origin if applied. These $N$ agents transmit the control information over a shared communication medium.

In event-based control without packet losses, the samples are generated whenever the state $x_i$ exceeds the bound $\Delta_i$, i.e., when $|x_i| \geq \Delta_i$. If there are packet losses, a slightly different scheme must be used to prevent infinitesimally small interarrival times. If a packet is lost, then both bounds are shifted by the boundary increment $\Delta_i$. Consequently, after a lost packet, the state is exactly between the two bounds. Hence, the distribution of the period between two events,

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the so called \emph{interevent} or \emph{interarrival time}, is the same as
the one of the lossless case.

Fig. 1 shows an example of event-based control with packet loss. In this example, the state of the system starts at the
origin and the bounds are \( \pm \Delta_i \). At time \( t_1 \), the state reaches the bound \( -\Delta_i \), an event is generated and the state is reset to the origin. At time \( t_2 \), the state reaches the bound \( \Delta_i \) and an event is generated. We assume that the corresponding packet is lost. Hence, the bounds are shifted to 0 and \( 2\Delta_i \). At time \( t_3 \), the state reaches the bound \( 2\Delta_i \), an event is generated and the state is reset to the origin. Since the packet arrives, the bounds are set to \( \pm \Delta_i \).

As in Åström and Bernhardsson (2002); Rabi and Johansson (2009), we use the variance of the state as costs:

\[
J_i = \lim_{M \to \infty} \frac{1}{M} \int_0^M E[x_i(t)^2]dt. \tag{2}
\]

Note that the variance is identical to the average power and second moment here.

In Åström and Bernhardsson (2002), it was shown that the costs of event-based control without packet loss are:

\[
J_{eb,i} = \frac{T_{eb,i}}{6}, \tag{3}
\]

where \( T_{eb,i} = \Delta_i^2 \) is the average interarrival time.

In Rabi and Johansson (2009), the costs of event-based control with a packet loss probability \( p_i \) have been derived. They are:

\[
J_{eb,loss,i} = \frac{\Delta_i^2(5p_i + 1)}{6(1 - p_i)} = \frac{\Delta_i^2}{6} + \frac{\Delta_i^2 p_i}{1 - p_i}, \tag{4}
\]

where \( J_{eb,loss,i} \) are the additional costs due to packet losses.

In the same work, also the Probability Density Function (PDF) of the interarrival times of event-based control is given as:

\[
f(t|\Delta_i) = \frac{1}{\Delta_i} \sqrt{\frac{2}{\pi t}} \sum_{k=-\infty}^{\infty} \frac{(4k + 1)e^{-\frac{(4k+1)^2\Delta_i^2}{8t}}}{\Delta_i}, \tag{5}
\]

To simplify notation, we can write (5) as function of a normalized PDF \( f(t|\Delta_i) \):

\[
f(t|\Delta_i) = \frac{1}{\Delta_i} f(\frac{t}{\Delta_i}|1). \tag{6}
\]

As already noted, the event-based approach generates events at arbitrary times. Hence, a suitable communication system must be used and analyzed. Such a communication system is the well known ALOHA system, which was introduced and analyzed by Abramson (1970). There are two versions of this system: unslotted and slotted ALOHA. In the unslotted ALOHA system, which is also called pure ALOHA, the sources are allowed to transmit at arbitrary times, without checking whether or not the shared communication medium is currently used. In the slotted ALOHA system, the time is divided into \( s \)lots and a source waits for the start of the next slot to begin sending. In order to analyze these systems, it is assumed, that all packets have the same length, called \emph{packet duration} \( \tau \). Moreover, they are only lost due to collisions, which occur if two or more sources are sending at the same time.

Fig. 2 shows an example of packet transmissions in a slotted ALOHA system. First, during slot 1, source 2 gets a new packet and waits until the start of slot 2 to begin sending. Now, during slot 2, source 1 gets a new packet and waits until the start of slot 3 to begin sending. Note that this prevents a collision. During slot 4 both sources get a new packet and wait until the next slot for sending. Thus, both sources are sending in slot 5, i.e., they collide and both packets are lost.

From the previous example, we see that a collision occurs if two or more sources get a new packet during the same slot. We assume that the slot length is identical to the packet duration and thus, the \emph{vulnerable period} is equal to the packet duration for slotted ALOHA. Note that the vulnerable period is twice the packet duration for unslotted ALOHA. Hence, we expect to have fewer collisions in slotted ALOHA. For the classical analysis of the ALOHA system in Abramson (1970); Tanenbaum (2003); Rom and Sidi (1990) it is assumed that packets are generated by a Poisson distribution with a mean of \( \lambda \) packets per packet duration. Obviously, the PDF of the interarrival times for event-based control, as given by (5), are not Poisson. Therefore, we cannot use these results for our analysis of event-based control with an ALOHA communication system. However, the unslotted ALOHA system with arbitrary interarrival times has been analyzed in Sant (1980). The probability of a packet loss for such a system is summarized in the following theorem:

\begin{theorem} (Sant (1980)). Let there be \( N \) users in the system and let each transmit packets with the same packet duration \( \tau \). For user \( j \), let \( f_j(x) \) and \( F_j(x) \) denote the density and distribution functions, respectively, of the packet transmisssions in a slotted ALOHA system.

\begin{enumerate}
\item Source 1
\begin{itemize}
\item packet 1.1
\item packet 1.2
\end{itemize}
\item Source 2
\begin{itemize}
\item packet 2.1
\item packet 2.2
\end{itemize}
\end{enumerate}
interarrival times. Let $\frac{1}{\lambda_j}$ be the average packet interarrival time for user $j$. Then the steady-state probability $p_i^u$ of a clear transmission for user $i$ (assuming that he can possibly interfere with himself) is given by

$$p_i^u = \left(1 - \int_0^\tau f_i(x)dx\right)^2 \times \prod_{j=1, j\neq i}^N \left(1 - \lambda_j \int_0^{2\tau} [1 - F_j(x)]dx\right). \quad (7)$$

If user self-interference is precluded, i.e., if for each $j$, $f_j(x)$ and $F_j(x)$ are zero for $x < \tau$, then the steady-state probability $\tilde{p}_i$ of a clear transmission for user $i$ reduces to

$$\tilde{p}_i = \prod_{j=1, j\neq i}^N \left(1 - \lambda_j \int_0^{2\tau} [1 - F_j(x)]dx\right). \quad (8)$$

Remark 2. Note that in the previous theorem it is assumed that all packets involved in a self-interference are lost. However, this does not have to be the case in slotted ALOHA. Here, a packet can get lost due to self-interference only while it is waiting for the begin of the next slot. Thus, the agent can decide to drop only the waiting packet but keep the new one.

Now, we adapt the previous theorem to slotted ALOHA:

**Theorem 3.** Let there be $N$ users in the system and let each transmit packets with the same packet duration $\tau$. Let the slot length be identical to the packet duration. For user $i$, let $f_i(x)$ and $F_i(x)$ denote the density and distribution functions, respectively, of the packet interarrival times. Let $\frac{1}{\lambda_i}$ be the average packet interarrival time for user $j$. Then the steady-state probability of a packet collision for user $i$ is given by

$$p_i^c = 1 - \frac{1}{\tau} \int_0^\tau \left(1 - \int_0^x f_i(t)dt\right) \left(1 - \int_0^{\tau-x} f_i(t)dt\right)dx \times \prod_{j=1, j\neq i}^N \left(1 - \lambda_j \int_0^{\tau} [1 - F_j(x)]dx\right) \quad (9)$$

if all packets involved in a self-interference are lost.

$$p_i^l = 1 - \frac{1}{\tau} \int_0^\tau \left(1 - \int_0^{\tau-x} f_i(t)dt\right)dx \times \prod_{j=1, j\neq i}^N \left(1 - \lambda_j \int_0^{\tau} [1 - F_j(x)]dx\right) \quad (10)$$

if the newest packet survives a self-interference.

$$\tilde{p}_i = 1 - \prod_{j=1, j\neq i}^N \left(1 - \lambda_j \int_0^{\tau} [1 - F_j(x)]dx\right) \quad (11)$$

if self-interference is precluded.

**Proof.** First, we proof the case without self-interference. Eq. (11) follows from (8) due to the fact that the vulnerable period is $\tau$ for slotted ALOHA, instead of $2\tau$ for unslotted ALOHA.

Now, we analyze the losses due to self-interference. Fig. 3 shows the time intervals between the event generation and the begin and end of the slot. The time interval between the event generation and the end of the slot is called forward recurrence time $\tau_F$. Similarly, the time interval between the event generation and the begin of the slot is denoted backward recurrence time $\tau_B$. Note that $\tau_F = \tau - \tau_B$. Packets are lost due to self-interference if the same user generates another event while a packet is waiting for the start of the next slot. Now, we have to distinguish the two cases.

If the newest packet survives a self-interference, then a packet is lost due to self-interference if and only if an event occurs during $\tau_F$. Consequently, the steady-state probability that a packet is not lost due to self-interference is

$$\frac{1}{\tau} \int_0^\tau \left(1 - \int_0^{\tau-x} f_i(t)dt\right)dx. \quad (12)$$

If all packets involved in a self-interference are lost, then a packet is lost due to self-interference, if and only if an event occurs during $\tau_B$ or $\tau_F$. Consequently, the steady-state probability that a packet is not lost due to self-interference is

$$\frac{1}{\tau} \int_0^\tau \left(1 - \int_0^{\tau-x} f_i(t)dt\right) \left(1 - \int_0^{\tau-x} f_i(t)dt\right)dx. \quad (13)$$

Finally, (9) and (10) follows from the fact that a clear transmission for user $i$ occurs if and only if he neither interferes with any other user nor himself.

As already noted, slotted ALOHA was developed to reduce the loss probability and it is a very well known fact that this is the case for Poisson traffic. The next lemma shows that this is also the case for an arbitrary interarrival time distribution.

**Lemma 4.** Suppose that $f_i(x) > 0 \forall x \neq 0$ and $F_j(x) > 0 \forall x \neq 0$. Then the loss probability of slotted ALOHA is smaller than the loss probability of unslotted ALOHA, i.e.,

$$p_i^c < p_i^s < p_i^u, \quad \tilde{p}_i < \tilde{p}_i^u. \quad (2)$$

**Proof.** Part (2) of the lemma follows from the fact that $0 < F_j(x) < 1 \forall x \neq 0$, and thus $\int_0^{2\tau} [1 - F_j(x)]dx > \int_0^{\tau} [1 - F_j(x)]dx$.

Part (1) of the lemma follows from part (2), the fact that $\int_0^\tau f_i(t)dt < \int_0^\tau f_i(t)dt \forall x < \tau$ and $f_i(x) > 0$.

**3. LOSS PROBABILITY OF EVENT-BASED CONTROL WITH SLOTTED ALOHA**

In this section, we combine the two parts of the previous section to get the loss probability of event-based control
with slotted ALOHA. Therefore, we assume that the newest packet survives a self-interference.

Our first theorem gives the loss probability of event-based control with a slotted ALOHA communication system for the most general case.

**Theorem 5.** Suppose there are \( N \) agents, each agent controls a system as given by (1) via the event-based approach. These agents use a slotted ALOHA communication system with a packet duration \( \tau \) and agent \( i \) uses the boundary increment \( \Delta_i \). Then the loss probability \( p_i^s \) for the packets of agent \( i \) is:

\[
p_i^s(\Delta_1, \ldots, \Delta_N, \tau) = 1 - \frac{1}{\tau} \int_0^\tau \left( 1 - \int_0^{\tau-x} f(t|\Delta_i)dt \right) dx \\
\times \prod_{j=1, j\neq i}^{N} \left( 1 - \frac{1}{\Delta_j} \int_0^\tau [1 - \int_0^x f(t|\Delta_j)dt]dx \right),
\]

where \( f(t|\Delta_i) \) is the PDF of the interarrival times, given by (5).

**Proof.** We get (14) by plugging (5) into (10), using the definition of the Cumulative Distribution Function (CDF): \( F_j(x) = \int_0^x f_j(t)dt \) and taking \( 1/\lambda_i = \Delta_i^2 \) into account. \( \square \)

If we assume that all agents use the same boundary increment \( \Delta \), then all sources have the same PDF for the interarrival times and Thm. 5 simplifies to the following corollary.

**Corollary 6.** Suppose, all \( N \) agents use the same boundary increment \( \Delta \). Then the loss probability \( p^s \) for each agent is:

\[
p^s(\Delta, \tau, N) = 1 - \frac{1}{\tau} \int_0^\tau \left( 1 - \int_0^{\tau-x} f(t|\Delta)dt \right) dx \\
\times \left( 1 - \frac{1}{\Delta} \int_0^\tau [1 - \int_0^x f(t|\Delta)dt]dx \right)^{N-1},
\]

where \( f(t|\Delta) \) is the PDF of the interarrival times, given by (5).

An interesting question is how the loss probability scales with the packet duration \( \tau \) and the boundary increments \( \Delta_i \). To answer this question, we define \( \rho_i := \tau/\Delta_i, = \tau/\Delta_i \) as the offered load of agent \( i \). Using this definition, we get the following theorem for the loss probability:

**Theorem 7.** Suppose there are \( N \) agents, each agent controls a system as given by (1) via the event-based approach. These agents use a slotted ALOHA communication system and agent \( i \) offers the load \( \rho_i \). Then the loss probability \( p_i^s \) for the packets of agent \( i \) is:

\[
p_i^s(\rho_1, \ldots, \rho_N) = 1 - \frac{1}{\rho_i} \int_0^{\rho_i} \left( 1 - \int_0^{\rho_i-x} f(t|1)dt \right) dx \\
\times \prod_{j=1, j\neq i}^{N} \left( 1 - \int_0^{\rho_j} [1 - \int_0^x f(t|1)dt]dx \right),
\]

where \( f(t|1) \) is the PDF of the interarrival times, given by (5).

**Proof.** Eq. (16) follows by using (6) in (14) and a change of the integration variables. \( \square \)

Again, we can simplify this theorem if all agents use the same boundary increment \( \Delta \) and thus offer the same load.

**Corollary 8.** Suppose, the offered load \( \rho \) of all \( N \) agents is equal. Then, the loss probability \( p^s \) for each agent is:

\[
p^s(\rho, N) = 1 - \frac{1}{\rho} \int_0^{\rho} \left( 1 - \int_0^{\rho-x} f(t|1)dt \right) dx \\
\times \left( 1 - \int_0^{\rho} [1 - \int_0^x f(t|1)dt]dx \right)^{N-1},
\]

where \( f(t|1) \) is the PDF of the interarrival times, given by (5).

**Remark 9.** From Thm. 7 and Cor. 8, we clearly see, that the loss probability depends on the offered load \( \rho \) of each agent and not the granularity given by \( \tau \) and \( \Delta \). Moreover, note that the loss probability is increasing with \( \rho \) increasing. Both observations are well known in communication theory.

Now, that we derived the loss probability of event-based control with a slotted ALOHA communication system, we will compare it with results from previous literature.

In Rabi and Johansson (2009), the loss probability of event-based control with a shared communication medium is given as:

\[
\bar{p}(\Delta, \tau, N) = 1 - (1 - \gamma/\Delta)^N, \tag{18}
\]

which can be written as

\[
\bar{p}(\rho, N) = 1 - (1 - \rho)^N, \tag{19}
\]

Note that \( (1 - \rho) \) is the fraction of time an user is not sending. Hence, \( \bar{p}(\rho, N) \) is in fact the probability that the communication medium is utilized, which is not equivalent to the loss probability.

In Blind and Allgöwer (2011), the loss probability of event-based control with an unslotted ALOHA communication system is given as:

\[
p^\mu(\rho, N) = 1 - \left( 1 - \int_0^\rho f(x|1)dx \right)^2 \\
\times \left( 1 - \int_0^{2\rho} [1 - \int_0^x f(t|1)dt]dx \right)^{N-1}, \tag{20}
\]

where \( f(t|1) \) is the PDF of the interarrival times, given by (5).

Finally, we compare with the help of Fig. 4 the different loss probabilities for different numbers of agents. Fig. 4 shows the analytically calculated loss probabilities \( p^\mu(\rho, N) \) and \( p^s(\rho, N) \) of unslotted and slotted ALOHA and the approximation \( \bar{p}(\rho, N) \) of Rabi and Johansson (2009) as well as the loss probability obtained by simulations in a linear and logarithmic scale. During one simulation each source generated 100.000 packets. We clearly see that the loss probability of unslotted ALOHA is much larger. Moreover, the loss probability \( p^s \) fits the loss probability obtained by simulations very well. Finally, for
In general, there are two reasons for performance degradation due to a shared communication system: delay and loss of packets. From Eq. (4), we see that increased costs due to packet losses are additive. Moreover, in Blind and Allgöwer (2011) we showed that the increased costs due to delay are also additive. Thus, we can divide the costs of networked event-based control into three parts:

\[ J_{eb,p,d} = J_{eb} + J_{eb,p} + J_{eb,d}, \]  

(21)

where

- \( J_{eb} \) are the costs of event-based control without loss and delay,
- \( J_{eb,p} \) are the additional costs due to packet losses,
- \( J_{eb,d} \) are the additional costs due to a delay with mean \( d \).

Obviously, these terms depend on the communication protocol, i.e., how and when the sources access the shared medium. In the unslotted ALOHA system, the sources are allowed to send immediately, without checking the medium. Obviously, this results in a high packet loss probability and there exist many different approaches to reduce the packet loss probability, e.g., the usage of time slots or carrier sensing. All these approaches have the buffering of packets in common, i.e., packet loss is traded against packet delay. In this section, we compare event-based control with a slotted and unslotted ALOHA communication system for a first analysis of this tradeoff.

Based on Lemma 4, we can compare the additional costs due to packet losses of event-based control with the two versions of ALOHA. Since \( J_{eb,p} \) as given in (4) is increasing with \( p \) and the loss probability of unslotted ALOHA is larger, we have:

\[ J_{eb,p}^u < J_{eb,p}^s, \]  

(22)

i.e., the additional costs due to packet losses are smaller for slotted ALOHA than for unslotted ALOHA.

In Blind and Allgöwer (2011) we showed that the additional costs due to a delay with mean \( d \) are:

\[ J_{eb,d} = d. \]  

(23)

In unslotted ALOHA, the delay \( d \) is just the packet duration \( \tau \). However, in slotted ALOHA, the mean delay is increased because the agents are not allowed to start sending at arbitrary times but have to wait for the start of the next slot. Since all slots have the same length, the mean residual waiting time is half the slot length. Thus we have for the mean delay of slotted ALOHA

\[ E[d] = \tau + \tau/2. \]

Obviously, we thus have

\[ J_{eb,d}^u > J_{eb,d}^s. \]  

(24)

Note that the costs of event-based control with packet losses and the additional costs due to delay scale with the packet duration \( \tau \). Thus, the costs of event-based control with unslotted ALOHA can be written as:

\[ J_{eb}^u = \tau \left( \frac{1}{6p} + \frac{p^u(p,N)}{(1-p^u(p,N))p} + 1 \right). \]  

(25)

The costs of event-based control with slotted ALOHA can be written as:

\[ J_{eb}^s = \tau \left( \frac{1}{6p} + \frac{p^s(p,n)}{(1-p^s(p,N))p} + 1 + 1/2 \right). \]  

(26)

Note that for the normalized costs, i.e., \( J_{eb}/\tau \) and \( J_{eb,u}/\tau \) the additional costs due to packet loss depends only on the utilization \( p \) and the additional costs due to the delay \( d \) is constant (1 for unslotted ALOHA and 1.5 for slotted ALOHA). Hence, for some values of \( p \), event-based control with unslotted ALOHA might be better than event-based control with slotted ALOHA. However, the additional costs due to the packet duration are twice the additional costs due to the waiting time. Consequently, unslotted ALOHA will never be much better.

Finally, Fig. 5 shows the normalized costs \( J_{eb}/\tau \) and \( J_{eb,u}/\tau \) of event-based control with slotted and unslotted ALOHA for different numbers of agents. Moreover, this figure also shows the normalized costs if the packet loss probability is approximated by \( \bar{p} \) and the normalized costs of time-triggered control \( J_{tt}/\tau \) with an optimal sampling time.

First, we see that there is a significant difference between the costs of event-based control with slotted and unslotted ALOHA for larger values of the utilization \( p \). The usage of slotted ALOHA significantly reduces the costs. Consequently, it is worthwhile to trade packet losses against delay. Nevertheless, the costs of time-triggered control are
still smaller. Finally, we see that the costs calculated with the approximation $\bar{p}$ of the loss probability fit the costs of event-based control with slotted ALOHA very well, except for $N = 2$.

5. CONCLUSION

In order to compare event-based control with the two versions of ALOHA, we first ignored the packet delay and focused our discussion on packet loss. Therefore, we derived the loss probability of slotted ALOHA and showed that it is indeed less than the loss probability of unslotted ALOHA. Thus, the additional costs due to packet losses are smaller for slotted ALOHA. While comparing the additional costs due to delay, we saw that the delay of slotted ALOHA is slightly larger than the delay of unslotted ALOHA. Thus, the additional costs due to delay are slightly larger for slotted ALOHA. However, these costs are relatively small when compared with the costs due to packet losses. Consequently, for event-based control it is worthwhile to trade loss against delay. Nevertheless, in our numerical calculations the costs of time-triggered control are still smaller. Thus it is important to analyze other communication protocols, like the family of CSMA protocols, which might result in a better performance than time-triggered control.

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