Dynamic Consensus and Formation: Fixed and Switching Topologies

Jun Xu ∗ Tao Li ∗∗ Lihua Xie ∗∗∗ Kai Yew Lum ∗

∗ Temasek Laboratories, National University of Singapore, Singapore 117411. Email: {tslxu,tslumky}@nus.edu.sg
∗∗ Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Email: litao@amss.ac.cn.
∗∗∗ School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798, Email: elhxie@ntu.edu.sg

Abstract: This paper addresses the dynamic consensus and formation problems of multi-agent systems for both leader-less and leader-follower cases under fixed and switched topologies. Necessary and/or sufficient conditions are presented in terms of graph topology, detectability, stabilizability, rank condition and Lyapunov inequality constraints. These conditions explicitly reveal how the intrinsic dynamics of the agents and the communication topology affect consensusability and formationability. In addition, several constructive procedures for protocol design are proposed to achieve consensus and formation. In particular, the so-called separation principle is established, which simplifies the design procedure greatly.

Keywords: Co-operative control, Convergence analysis, Agents

1. INTRODUCTION

In recent years, the interplay between control and communications has attracted increasing attention. It is well-known now that the communication constraints and information flow can have significant influence on the performance of networked control systems, especially for networked multi-agent systems. The consensus problem is one of the fundamental problems for networked multi-agent systems [Ren and Beard, 2008, Olfati-Saber et al., 2006, Li and Zhang, 2009]. In [Olfati-Saber et al., 2006, Olfati-Saber and Murray, 2004, Ren and Beard, 2008, Ma, 2009], static consensus protocols for continuous-time systems are presented using a static state feedback control. For discrete-time counterpart, You and Xie [2010a] give a necessary and sufficient condition for consensusability. However, when not all the states are available, the output feedback control has to be considered. A sufficient condition with mild assumptions for multi-agent systems with static output feedback control (SOFC) to reach consensus is presented in [Ma, 2009]. In [Wang et al., 2009], a sufficient condition for SISO structure is presented. Note that the applicability of the SOFC-based protocols is inherently limited due to their existence conditions and design procedures, e.g., the stabilizability and detectability generally cannot guarantee the existence of a SOFC (thus more difficult for simultaneous stabilization) and the design procedure is usually BMI (bilinear matrix inequality)-based. Hence, dynamic output feedback control (DOFC)-based protocols have been studied in the literature [Fax and Murray, 2004, Li et al., 2010, Seo et al., 2009]. In [Fax and Murray, 2004], a necessary and sufficient condition using simultaneous stabilization for a local controller to stabilize a certain formation dynamics is given. An observer-based controller is proposed in [Li et al., 2010], where a separation principle-like condition is presented. In [Seo et al., 2009, Wang et al., 2009, You and Xie, 2010b], the DOFC is also observer-based essentially. All these DOFC-based protocols are only for fixed topologies.

In this paper, we will further consider the necessary and sufficient conditions of consensus and formation using generalized dynamic output feedback control with different information levels. Our work contains all these existing situations as special cases. Meanwhile, protocols with different properties are designed using separation principle. In addition, time-varying topologies are also considered.

The notation in this paper is standard. \( \mathcal{R} \) and \( \mathcal{C} \) denote the real and complex space, respectively. \( A^T \) (\( A^* \)) is the (conjugate) transpose of \( A \). \( A > 0 \) (\( A \geq 0 \)) means that \( A \) is positive definite (semidefinite). \( A^\perp \) is an orthogonal complement of \( A \), i.e., the basis of the null space of \( A^T \). \( A^* \) is the empty set. \( j \) is the pure imaginary number such that \( j^2 = -1 \). \( 1 \) is a vector with ones as its entries. \( \odot \) is the symbol for the Kronecker product. \( \mathcal{L}_0 \in \mathcal{R}^{n\times n} \) is the Laplacian matrix of a digraph \( \mathcal{G} \) (0-1 digraph) with \( n \) vertexes. Denote the adjacency matrix of \( \mathcal{G} \) as \( \mathcal{A}_0 = [a_{ij}] \in \mathcal{R}^{n\times n} \) and the in-degree matrix as \( \mathcal{D} \). \( \lambda_i \) is the eigenvalue of \( \mathcal{L}_0 \). More properties of graphs are listed in Appendix and [Ma, 2009, Ren and Beard, 2008].

2. DYNAMIC CONSENSUS AND FORMATION

2.1 Problem Statement

Consider the following multi-agent system:

\[
\dot{x}_i = A x_i + B u_i \tag{1}
\]

\[
y_i = C x_i, \quad i = 1, 2, ..., n \tag{2}
\]

where \( x_i \in \mathcal{R}^n \), \( u_i \in \mathcal{R}^r \), \( y_i \in \mathcal{R}^m \) are the state, control input and output of the agent \( i \), respectively. The communication topology among agents is represented by a directed graph \( \mathcal{G} \), which consists of a node set \( \mathcal{V} = \{1, 2, ..., n\} \) and an edge set....
\( \mathcal{E} \subseteq V \times V \). For the agent \( i \), denote its neighbor set as \( N(i) \). The consensus and formation are defined as follows. For the system (1)-(2), if there exists a \( u(t) \in \mathcal{U} \), where \( \mathcal{U} \) is a set of admissible control inputs, such that

\[
\lim_{t \to \infty} \| x_j(t) - x_i(t) \| = 0, \quad i, j = 1, \ldots, n
\]

we then say that the system asymptotically reaches a consensus. Similarly, if there exists a \( u(t) \in \mathcal{U} \), such that

\[
\lim_{t \to \infty} \| x_j(t) - x_i(t) - (h_j - h_i) \| = 0, \quad i, j = 1, \ldots, n
\]

where \( \mathcal{H} = (h_1^T, \ldots, h_n^T)^T \) is a given formation, then we say that the system asymptotically reaches a formation \( \mathcal{H} \).

The purpose of this paper is to study consensus and formation using the following distributed DOFC-based consensus protocol for the agent \( i, i \in \mathcal{V} \):

\[
\dot{x}_i = M x_i + N y_i + V \sum_{j=1}^{n} a_{ij} (x_j - x_i) + H \sum_{j=1}^{n} a_{ij} (y_j - y_i)
\]

where \( M, N, L, K, S, R, V, H \) are gain matrices with proper dimensions to be determined and \( \xi_i \) is the internal state. When all the gain matrices are non-zero matrices, it means that the protocol of the agent \( i \) uses full local information from itself (\( y_i \) and \( \xi_i \)) and its neighbors (including the external output information \( y_j \) and the internal state information \( \xi_j, j \in N(i) \)). However, there are some cases that not all local information can be used for design. For example, suppose that \( y_j \) presents the location information of the agent \( i \) via GPS signals. When the GPS is not available, \( y_j \) may be unknown. In this case, we may only have the relative information \( y_i - y_j \) (e.g., distance) through some on-board sensors. Thus \( N \) and \( H \) are zero matrices. We will present some basic properties of the DOFC-based consensus protocols using full local information in Subsection 2.2 and consider partial local information cases in Subsection 2.3.

### 2.2 Dynamic Consensus with Full Local Information

**Theorem 1.** Suppose that \( G \) contains a spanning tree. There exists a protocol (3)-(4) such that a consensus is asymptotically reached for all initial states if and only if \( \mathbb{K}_d = \emptyset \), where

\[
\mathbb{K}_d = \{ K : \mathcal{A} + B \mathcal{K} \xi_i \text{ is Hurwitz}, \quad i = 2, \ldots, n \}
\]

where

\[
\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{K}_1 = \begin{pmatrix} R & S \\ N & M \end{pmatrix},
\]

\[
\mathcal{A}_2 = \begin{pmatrix} K & L \\ H & V \end{pmatrix}, \quad \hat{\mathcal{C}} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}
\]

\[
\mathcal{K} = [\mathcal{K}_1, \mathcal{K}_2], \quad \hat{\mathcal{C}} = \begin{pmatrix} \hat{C} \\ \mathcal{C}_i \end{pmatrix}, \quad \hat{\mathcal{C}}_i = \begin{pmatrix} \hat{C} \\ \lambda_i \mathcal{C}_i \end{pmatrix}
\]

**Proof:** Define

\[
\delta_i = x_1 - x_i, \quad \zeta_i = \xi_1 - \xi_i
\]

Then

\[
\dot{\zeta} = [I \otimes \mathcal{A} - (\mathcal{L} + 1 \cdot \alpha^T) \otimes \mathcal{B}] \zeta
\]

where

\[
\alpha = \begin{pmatrix} a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} \sum_{j=1}^{n} a_{j1} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{n2} \\ -a_{n3} & \cdots & \cdots & \sum_{j=1}^{n} a_{nj} \end{pmatrix}
\]

Since the Laplacian matrix is \( \mathcal{L}_g = \sum_{j=1}^{n} a_{ij} - \alpha^T \), we can easily deduce that 1) the remaining \( n-1 \) eigenvalues of \( \mathcal{L}_g \) (except 0) are determined by \( \mathcal{L}_1 + 1 \cdot \alpha^T \), 2) there exist a Jordan matrix \( J \) and an invertible matrix \( T \) such that \( T^{-1} [\mathcal{L}_1 + 1 \cdot \alpha^T] T = J \). Thus \( (T \otimes I)^{-1} [I \otimes \mathcal{A} - (\mathcal{L}_1 + 1 \cdot \alpha^T) \otimes \mathcal{B}] (T \otimes I) = I \otimes \mathcal{A} - J \otimes \mathcal{B} \). That means the eigenvalues of \( I \otimes \mathcal{A} - (\mathcal{L}_1 + 1 \cdot \alpha^T) \otimes \mathcal{B} \) is governed by \( \mathcal{A} + B \mathcal{K}_i \). Note that the choice of the difference between vector indices 1 and \( i \) is without loss of generality. Hence, the consensus is reached if and only if \( \mathcal{A} + B \mathcal{K}_i \) is Hurwitz for a certain \( K, i = 2, \ldots, n \).

**Theorem 2.** If \( G \) contains any spanning tree, then for a given \( K \in \mathbb{K}_d \), we have

\[
v(t) \to [v_1(t) \otimes 1] = [v_1(0) \vdots v_n(0)], \quad t \to \infty, \quad v_i = [x_i \xi_i]
\]

where \( r^T = [r_1, \ldots, r_n] \in \mathbb{R}^n \) is the left eigenvector of \( \mathcal{L}_g \) associated with the eigenvalue 0, satisfying \( r^T 1 = 1 \).

**Proof:** Define

\[
v = \begin{pmatrix} v_1 \\ v_n \end{pmatrix}, \quad \bar{T} = [I] \in \mathbb{R}^{n \times n}, \quad T^{-1} = \begin{pmatrix} r^T \\ \bar{T} \end{pmatrix},
\]

\[
\mathcal{L}_g \bar{T} = \bar{J} = \begin{pmatrix} 0 & 0 \\ 0 & J_d \end{pmatrix}
\]

where \( \bar{T} \) denotes the part we are not interested in and \( \bar{J}_d \) is a Jordan matrix with the non-zero eigenvalues of \( \mathcal{L}_g \) as its diagonal entries. It is easy to observe that \( \dot{v} = (I \otimes \mathcal{A} + \mathcal{L}_g \otimes \mathcal{B}) v \), which gives the solution

\[
v(t) = e^{(I \otimes \mathcal{A} + \mathcal{L}_g \otimes \mathcal{B}) t} v(0)
\]

\[
= (\bar{T} \otimes I) e^{(I \otimes \mathcal{A} + \bar{J} \otimes \mathcal{B}) t} (\bar{T}^{-1} \otimes I) v(0)
\]

By Theorem 1, \( (I \otimes \mathcal{A} + \bar{J} \otimes \mathcal{B}) \) is Hurwitz,

\[
v(t) \to (1 \otimes I) e^{A t} v(0), \quad t \to \infty
\]

Thus we complete the proof.
fluenced by $\bar{K}_2$ from (15). Note that here $\bar{K}$ is a full variable matrix, thus we have the following result.

**Theorem 3.** $\mathcal{K}_d \neq \emptyset$ if and only if $\bar{K}_{d1} = \{\bar{K}_1 : \bar{A} + \bar{B}\bar{K}_1\bar{C} \text{ is Hurwitz}\} \neq \emptyset$.

**Proof:** For necessity, note that $\bar{A} + \bar{B}\bar{K}_1\bar{C} = (\bar{A} + \bar{B}(K_1 + \alpha_1K_2)\bar{C}) + \beta_2\bar{K}_2\bar{C}$, where $\lambda_1 = \alpha_1 + \beta_2$. Based on Lemma 14 in Appendix, $\bar{A} + \bar{B}\bar{K}_1\bar{C}$ is Hurwitz only if $\{\bar{A} + \bar{B}(K_1 + \alpha_1K_2)\bar{C}\}$ is Hurwitz. Since $\bar{K}_1$ is a full variable matrix, $\mathcal{K}_{d1} \neq \emptyset$ implies that $\mathcal{K}_{d2} = \{\bar{K} : (\bar{A} + \bar{B}(K_1 + \alpha_1K_2)\bar{C}) \text{ is Hurwitz}\} \neq \emptyset$. For sufficiency, since $\mathcal{K}_{d1} \neq \emptyset$, we simply choose $\bar{K}_1 = K_1$ and let $\bar{K}_2$ such that $\bar{B}\bar{K}_2\bar{C}$ sufficiently small (an extremely case is that $\bar{K}_2$ is the null of $\bar{B}$ or $\bar{C}^T$), so that $\mathcal{K}_d \neq \emptyset$. Thus we complete the proof. ∎

2.3 Partial Local Information and Separation Principle

Note that $\Psi_1 = \bar{A} + \bar{B}\bar{K}_1\bar{C}$ is similar to the matrices $\Phi_1$ via

$$\Pi = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \text{ i.e., } \Theta_1 = \Pi \Psi_1 \Pi^{-1}.$$ 

Let $S = S_1 - R\xi, L = L_1 - K\xi, -A + NC + M - BS_1 = 0, H\xi - BL_1 + V = 0$ (16)

Then $\Theta_1$ can be simplified as:

$$\begin{pmatrix} A + BS_1 + \lambda_1BL_1 & \lambda_1BL + BS \\ 0 & A - NC + BR\xi + \lambda_1(-HC + BK\xi) \end{pmatrix}.$$ 

**Corollary 4.** Assume that the controller gain $K$ satisfies (16) and one of the following additional constraints

$$N = N_1 + BR, H = BK, S_1 = 0$$ (17)

$$H = H_1 + BK, N = BR, S_1 = 0$$ (18)

$$H = H_1 + BK, N = BR, L_1 = 0$$ (19)

Then there exists a protocol (3)-(4) such that a consensus is asymptotically reached for all initial states if and only if 1) $(A, B)$ is stabilizable; 2) $(A, C)$ is detectable; and 3) if $A$ is not Hurwitz, then $G$ contains any spanning tree.

**Proof:** With the constraints (16) and one of (17)-(19), $\bar{A} + \bar{B}\bar{K}_1\bar{C}$ is similar to the matrices

$$\begin{pmatrix} A + \lambda_1BL_1 & \lambda_1BL + BS \\ 0 & A - N_1C \end{pmatrix}.$$ 

Then there are two situations. When $A$ is Hurwitz, the problem is trivial. Now consider the case $A$ is not Hurwitz. In fact, according to Lemmas 12 and 15 in Appendix, we can conclude that $\mathcal{K}_d \neq \emptyset$ if and only if 1)-3) hold. Now we only need to prove that $G$ contains a spanning tree if there exists at least an eigenvalue $Re(\lambda) \geq 0$. If $G$ has no spanning tree when there exist $Re(\lambda(A)) > 0$, then there are at least two eigenvalues with $Re(\lambda_1) > 0$. For such a $\lambda_1$, $i \geq 2$, $A - \lambda_1H_1C$ (or $A + \lambda_1BL_1$) is Hurwitz, which contradicts the assumption. Thus we can draw the conclusion. ∎

**Remark 1.** The results stated in Corollary 4 together with Lemma 15 in Appendix are interesting. First, it makes the design satisfy the so-called separation principle. For example, under the constraint (16) and (17), we can design the gains $L_1$ and $N_1$ separately. $L_1$ can be designed using the procedure introduced in the proof of Lemma 15. For $N_1$, simply choose one such that $A - N_1C$ is stable. $R$ and $K$ can be arbitrarily assigned. And then $L, N, S, H, M$ and $V$ can be obtained accordingly. Second, using the consensus result in Corollary 4 we can easily derive a robust protocol for slowly switched connected topologies, as long as the switched topologies are known a priori, or at least the minimum no-zero eigenvalue of all Laplacian matrices is known a priori. Note that we may obtain some other conditions than (17)-(19) using different transformations.

Now we consider some cases that only partial local information is used for design. First consider the case that $N$ and $R$ are zero, i.e., $y_i$ is unknown for the agent $i$.

**Corollary 5.** The protocols (20) and (21) satisfy the condition of Corollary 4.

$$\dot{x}_i = A\xi_i + BS\xi_i + H n \sum_{j=1}^{n} a_{ij} \xi_j$$ (20)

$$u_i = L \sum_{j=1}^{n} a_{ij} (\xi_j - \xi_i)$$ (21)

$$\xi_i = A\xi_i + BS\xi_i + H n \sum_{j=1}^{n} a_{ij} \xi_j$$ (21)

$$u_i = S\xi_i$$ (21)

where $\chi_{ji} = y_j - C\xi_j - y_i + C\xi_i$.

**Proof:** The proof is similar to that of Corollary 4. Thus we omit the details. ∎

Note that the generalized DOFC protocol is reduced to observer-based protocol here, i.e., either (20), where $\xi_i$ in (20) estimates $x_j - x_i$, or (21), where $\xi_i$ in (21) estimates $x_i$. In fact, (21) is suggested in Li et al. [2010] and [Wang et al., 2009] and (20) is proposed for discrete-time cases in [You and Xie, 2010a].

Second, we consider the case that besides $N = 0$, $R = 0$, the gains $V$ and $L$ are also zero matrices, i.e., both $y_i$ and $\xi_i - \xi_j$ are unknown. Note that this can reduce the communication load. If (20) or (21) is used, then the consensusability is achieved only if $A$ is stable. When $A$ is not stable, a stronger condition may be enforced. For example, if further $S = 0$ (or $H = 0$), then $\mathcal{K} = \{L : A + \lambda_1BLC \text{ is Hurwitz}, i = 2, ..., n \} \neq \emptyset$ is the necessary and sufficient condition for $\mathcal{K}_d \neq \emptyset$ (in this case, $M$ must be stable).

There are some other interesting cases. For example, in the case that $y_i - y_i$ is unknown, if $K = 0, R = 0, H = 0, S = 0, M = A - NC, V = BL$, then the separation principle holds. If one of the following conditions holds: 1) $R = 0, N = 0, H = 0, S = 0, M = A - NC, V = BL$; 2) $R = 0, N = 0, H = 0, L = 0, M = A - BS = 0, V = BKC$, then $\mathcal{K}_d \neq \emptyset$ if and only if $L \neq \emptyset$. For the case of $V = 0, L = 0$, Fax and Murray [2004] provides some similar results. When $\bar{K}_1$ is zero matrix, the problem is similar to static output protocols and can be partially solved using Lemma 16.

**Remark 2.** Now a critical question is that what the benefits of using more information for the design are. First, we can see that the existence condition for consensusability and formationability with less information may be stronger than that with more information, e.g., the existence condition of the aforementioned protocols with $N = 0, R = 0, L = 0$ and $V = 0$ is stronger than that of (20) and (21). Second, the extra variables in the formulation may bring more freedom in pole/eigenstructure assignment, thus the system gains a fast convergent rate, which can be easily observed by (14). Third, the consensus value may be adjusted. For example, using the protocols (21) such that $\mathcal{K}_d \neq \emptyset, x_i \rightarrow (e^t \odot c^t)X(0)$ when $t \rightarrow \infty$, where $X(0) = [x_1(0)^T, ..., x_n(0)^T]^T$. If we add a $\bar{R} \neq 0$ to the
protocols (21) in (3)-(4), then \( x_i \to (rT \otimes e^{A+BRC})X(0) \) when \( t \to \infty \).

2.4 Dynamic Formation

When the information exchange of internal state is allowed, we design the formation protocol as follows:

\[
\dot{x}_i = M(x_i - h_i) + V \sum_{j=1}^{n} a_{ij} \chi_{ij} + N(y_i - Ch_i) + H \sum_{j=1}^{n} a_{ij} \chi_{ij} \tag{22}
\]

\[
u_i = S(x_i - h_i) + L \sum_{j=1}^{n} a_{ij} \chi_{ij} + R(y_i - Ch_i) + K \sum_{j=1}^{n} a_{ij} \chi_{ij} \tag{23}
\]

where \( \chi_{ij} = x_j - x_i - h_i + h_j \) and \( \chi_{ij} \) is defined in (20).

Theorem 6. Suppose that \( \mathcal{G} \) contains a spanning tree. Consider the system (1)-(2). There exists a protocol (22)-(23) such that the prescribed formation is asymptotically reached for all initial states if and only if \( \mathcal{K}_d \neq \emptyset \) and

\[
A(h_i - h_j) = 0, \quad i, j = 1, ..., n \tag{24}
\]

Proof: Define

\[
\tilde{\xi}_i = x_i - x_1 - h_i + h_1 \tag{25}
\]

\[
\tilde{\chi}_{1i} = x_1 - x_i - h_i + h_1 \tag{26}
\]

Then

\[
\tilde{\xi} = [I_{n-1} \otimes A - (\mathcal{L} + \mathbf{1} \cdot \alpha^T) \otimes BK] \tilde{\xi} + \left(I_{n-1} \otimes \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} h_2 - h_1 \\ h_n - h_1 \end{pmatrix} \tag{27}
\]

where \( \tilde{\xi}_i = (\tilde{\xi}_i^T \tilde{\chi}_{1i}^T)^T \), \( \tilde{\xi} = (\tilde{\xi}_1^T \tilde{\chi}_{11}^T \cdots \tilde{\chi}_{nn}^T)^T \). From (27), we know that besides the conditions for the consensus in Theorem 1, \( A(h_i - h_j), i, j = 1, 2, ..., n \) must be zero by considering the particularity of (25)-(26). The remaining is similar to that of Theorem 1. Thus we omit the details. \( \diamond \)

Remark 3. The additional equation constraint (24) seems to be strange. However, it is automatically satisfied for the double-integrator dynamics discussed in [Ren and Beard, 2008]. We may also consider the choice of the controller gains for different formability. Note that besides the choice of the controller gains, whether the multiplication of the gain matrix and the state include the item \( h_i \) also affects the formability. In fact, if we slightly change (23) as

\[
u_i = S\tilde{\xi}_i + R(y_i - Ch_i) + L \sum_{j=1}^{n} a_{ij} \chi_{ij} + K \sum_{j=1}^{n} a_{ij} \chi_{ij},
\]

then the formability is equivalent to \( \mathcal{K}_d \neq \emptyset \) and \( (A + BS)(h_i - h_j) = 0, i, j = 1, ..., n \). Thus it is no doubt under certain circumstance, we may relax the constraint by importing the gains into the equation constraints via changing the protocols.

Remark 4. When there exists a leader with the following dynamics:

\[
\begin{align*}
\dot{x}_0 &= A\tilde{x}_0 \\
y_0 &= C\tilde{x}_0
\end{align*}
\]

and the other agents are followers with dynamics (1)-(2), the consensus and formation problem can be defined similarly.

Note that in this case, the topology is slightly different from previous one. Define a diagonal matrix \( D \in \mathbb{R}^{n \times n} \) with its diagonal entries \( d_{ii} = 1 \) or 0, i.e., if the arc \((0, i)\) exists, then \( d_{ii} = 1 \), otherwise \( d_{ii} = 0 \). Denote the new graph as \( \mathcal{G} \) which contains the leader agent 0 and following agents 1, ..., \( n \). Denote the Laplacian matrix of \( \mathcal{G} \) as \( L_g \). Then

\[
\mathcal{L}_g = \begin{pmatrix} \mathcal{L}_g + D & -D1 \\ 0 & 0 \end{pmatrix} \tag{28}
\]

Consider the DOFC-based consensus protocol using the relative information as follows:

\[
\begin{align*}
\dot{x}_i &= Mx_i - Vd_i x_i + \sum_{j=1}^{n} a_{ij} (\xi_j - \xi_i) + H \sum_{j=1}^{n} a_{ij} (y_j - y_i) + 1 + a_{ij} (y_j - y_i) \\
 & + K \sum_{j=1}^{n} a_{ij} (y_j - y_i) + Kd_i (y_j - y_i),
\end{align*}
\]

(29)

Without giving proofs, we present some results here.

Theorem 7. Suppose that \( \mathcal{G} \) contains a spanning tree rooted at the node 0. There exists a protocol (29)-(30) such that a leader-follower consensus is asymptotically reached for initial condition if and only if \( \mathcal{K}_d \neq \emptyset \), where \( \mathcal{K}_d = \{ K : A + BS, V \} \). \( \mathcal{K}_d \) is Hurwitz, \( i = 1, ..., n \), \( \hat{K} = [K_1, K_2], K_1 = \begin{pmatrix} 0 & -S \\ 0 & M \end{pmatrix} \), \( \hat{K}_1 = \begin{pmatrix} -K & L \\ H & -V \end{pmatrix} \). \( \lambda_{ij}, i = 1, ..., n \) are the eigenvalues of \( \mathcal{H} = \mathcal{L}_g + D \).

The separation principle also holds under some circumstances.

Corollary 8. Suppose that \( \mathcal{G} \) contains a spanning tree rooted at the node 0 and the controller gain \( K \) satisfies one of the following conditions:

\[
K = 0, S = 0, M = A, V = BL - HC \tag{31}
\]

\[
K = 0, L = 0, M = A - BS, V = HC \tag{32}
\]

Then there exists a protocol (29)-(30) such that a leader-follower consensus is asymptotically reached for all initial states if and only if \( (A, B) \) is stabilizable, and \( (A, C) \) is detectable.

3. TIME-VARYING TOPOLOGIES

When the topologies are time-varying, consider an infinite sequence of nonempty, bounded and contiguous time intervals \([t_k, t_{k+1}), k = 0, 1, ..., \) with \( t_0 = 0, t_{k+1} - t_k \leq T \) for some constant \( T > 0 \). Assume that for each \([t_k, t_{k+1}), t_k \in [t_k(t_{k+1}), \cdots, [t_k(t_k + 1)]^t, \cdots, [t_k(t_{k+1}) + 1], t_k = t_k(t_{k+1}) + t_k(t_k + 1) \) satisfying \( t_k(t_k + 1) \geq T, 0 \leq j \leq m_k - 1 \) for some integer \( m_k > 0 \) and a given constant \( \tau \) such that \( G(t) \) is not changed for \( t \in [t_k(t_k + 1)]^t \) and changed at time \( t_k(t_k + 1) + 1 \). Index all \( G(t) \) in time sequence. Let \( \mathcal{S} \) denote an index set for all graphs defined on the vertices \( \{1, 2, ..., n\} \) and \( \mathcal{S} \subset \mathcal{B} \) be the index set of all switched topologies of system dynamics. There is no doubt that \( \mathcal{S} \) is a finite set.

It is well-known that a switched system is asymptotically stable if all individual subsystems are asymptotically stable and the switching is sufficiently slow, i.e., \( \tau \) is large enough, so as to allow the transient effects to dissipate after each switch [Liberzon, 2003]. If we assume that the dwell time \( \tau \) is larger
than the lower-bound/critical dwell time as stated in Liberzon [2003]. Then there exists a protocol (20)-(21) such that a leader-less consensus is asymptotically reached for all initial states if \((A, B)\) is stabilizable, and \((A, C)\) is detectable, and the finite switching topology \(\mathcal{G}(t)\) always contains a spanning tree. We can also obtain a similar result for leader-follower cases. However, this strong connection condition and slow switching condition are conservative. In the following, we relax the requirement by using the concept of the so-called joint spanning tree.

We call that the graph \(\mathcal{G}(t)\) for \(t \in [t_k, t_{k+1}]\) has a joint spanning tree if the union of the graphs \(\mathcal{G}(t)\) for \(t \in [t_k, t_{k+1}]\) has a spanning tree. Suppose for \(t \in [t_k, t_{k+1}], \sigma(t) \in S\). The associated graph is \(\mathcal{G}_{\sigma(t)}\), whose components are \(S_{\sigma}^{(1)}, \ldots, S_{\sigma}^{(n)}\). Without loss of generality, we assume that there exist some components \(S_{\sigma}^{(1)}, \ldots, S_{\sigma}^{(n)}\), \(n_{\sigma(t)} \leq n_{\sigma(t)}\), whose \(L^j + 10 \alpha^T, j = 1, \ldots, n_{\sigma(t)}\), have positive real-parts eigenvalues, where \(L^j\) and \(\alpha^T\) are defined similar to (12). Define \(\ell(t) = \left\{v : v \in \bigcup_{m=1}^{n_{\sigma(t)}} \mathcal{V}(S_{\sigma}^{(m)})\right\}\), where \(\mathcal{V}(S_{\sigma}^{(m)})\) is the index set of all vertexes of the corresponding components \(S_{\sigma}^{(m)}\). The following lemma is straightforward, as if not, it will contradict the fact that \(\mathcal{G}(t)\) contains a joint spanning tree.

Lemma 9. If \(\mathcal{G}(t)\) contains a joint spanning tree for each interval \([t_k, t_{k+1}]\), then \(\bigcup_{t_k \leq t \leq t_{k+1}} \ell(t) = \{1, \ldots, n\}\).

Before presenting the main result, a hypothesis is proposed: for any undirected (0-1) graph, the matrix \(L + 1 - \alpha T\) is similar to a diagonal matrix. As the choice of elements of the Laplacian matrix \(L_{\sigma(t)}\) for a given dimension is finite (1 for connection and 0 for disconnection), it is possible to use greedy method to check all the \(2^{n_{\sigma(t)}}\) combinations. Simulation shows that for \(n_{\sigma(t)} = 8\), there is no exception.

Theorem 10. Assume that \(\mathcal{G}(t)\) is undirected and its corresponding \(L(t) + 1 - \alpha T\) is diagonal. There exists a protocol (20) (or (21)) such that a leader-less consensus is asymptotically reached for all initial states if \((A, B)\) is stabilizable, and \((A, C)\) is detectable, and the switching topology \(\mathcal{G}(t)\) contains a joint spanning tree for each interval \([t_k, t_{k+1}]\).

Proof: We only consider the case (21). A constructive method similar to Lemma 15 is applied. Let \(\lambda^\sigma = \alpha^\sigma + j\beta^\sigma\) be the eigenvalues of all Laplacian matrices \(L_{\sigma(t)}\) of system dynamics, \(\sigma(t) \in S\). Arrange all the non-zero \(\lambda^\sigma\) for all \(\sigma(t) \in S\) into a set \(A\).

Analogous to Corollary 5, we can see that \(\Psi_1 = A + B\mathcal{K}\mathcal{C}\), \(l \in S\), is similar to \(\Theta_l = \begin{pmatrix} A + BS & BS \\ 0 & A - \lambda_l HC \end{pmatrix}\). Construct the \(H\) using the method in Lemma 15, \(H = \begin{pmatrix} 2P - C^T C & 0 \\ 0 & A - \lambda_l HC \end{pmatrix}\), where \(P\) is the solution of \(AT^T P + P A - \tau C^T C < 0\), \(P > 0\), \(\tau > 0\), \(\tau = \frac{1}{\delta}\) and \(\alpha = \min A\). Thus, for certain \(W_l > 0\), \(A - \lambda_l HC)^T P + P(A - \lambda_l HC) = -W_l < 0\). Construct \(S = -B^T Q^{-1}\), where \(Q\) is the solution of \(AQ + QA^T - B^T B < 0\), \(Q > 0\). Now the task is to find a common Lyapunov matrix \(P\) for \(\Theta_l\) such that the Lyapunov inequality holds, i.e., \(\Theta_l P + Q^T P < 0\), where \(Q = Q^{-1}\). That is possible since \(W_l = W_c - \frac{1}{Q} Q B S W_l^{-1} S^T B^T Q < 0\) where \(W_c = \phi(A + BS)^T Q + Q(A + BS))\). By adjusting \(\phi\), i.e., making \(\phi\) large enough, \(W_l < 0\) always holds for all \(l \in S\) as \(S\) is a finite set. Thus, \(\Theta_l P + Q^T P < 0\) holds for all \(l \in S\). Define \(\hat{P} = \Pi^T \Pi\), \(\tilde{P} = \Pi^T (\Theta_l P + Q^T P) \Pi < 0\). For all \(l \in S\), there exists a \(\varphi > 0\) such that \(\tilde{P} \Psi_1 + \Psi_1^T \tilde{P} + \varphi I < 0\).

There exist invertible matrices \(T_l\) such that \((T_l \odot I)^{-1} [I \odot A - \left(C^j + 1 \alpha^T \right) \odot B(T_l I)] = I \odot \Theta_l, T_l^T (C^j + 1 \alpha^T) T_l = \lambda_l = diag\{\lambda_l^{(1)}, \ldots, \lambda_l^{(n_l)}\}\), where \(\lambda_l^{(i)}\) are nonnegative real scalars. Thus we define \(\varsigma = (T_l \odot I)^{-1}\). Since \(T_l\) is invertible, instead of proving the stability of \(\varsigma\), we can prove the stability of \(\varsigma\). Now we can define the Lyapunov function for the switching system \(\varsigma\) similar to (11). \(V(t) = \varsigma^T (I \odot P) \varsigma\)

Clearly, \(V(t) > 0\) for \(\varsigma \neq 0\).

\(V(t) = c^T (I \odot A + \varsigma) (I \odot P) + (I \odot P) (I \odot A - \varsigma) (I \odot B) \varsigma\)

\(= c^T (I \odot \Psi_1 + \Psi_1^T) \varsigma < -\varphi \sum_{j \in \ell(t)} c_j^T c_j\)

Now we can use a similar technique introduced in [Ni and Cheng, 2010]. The main tools are Cauchy’s convergence criteria and Barbalat’s lemma [Khalil, 2002]. The extended version is available upon request.

Remark 5. For the leader-follower case, suppose for \(t \in [t_k, t_{k+1}], \sigma(t) \in S\) (\(S\) is similarly defined as \(S\)) and the associated graph is \(\mathcal{G}_{\sigma(t)}\). Assume that there are some components \(S_{\sigma(t)}^{(1)}, \ldots, S_{\sigma(t)}^{(n_{\sigma(t)})}\) of \(\mathcal{G}_{\sigma(t)}\) satisfying 1) the leader agent 0 is included; and 2) \(L_{\sigma(t)} + D^T, j = 1, \ldots, n_{\sigma(t)}\), have positive real-parts eigenvalues, where \(L_{\sigma(t)}^{(1)}\) and \(D^T\) are defined similar to (28). Define \(\ell(t) = \left\{v : v \in \bigcup_{m=1}^{n_{\sigma(t)}} \mathcal{V}(S_{\sigma(t)}^{(m)})\right\}\). Thus, if \(\mathcal{G}(t)\) contains a joint spanning tree rooted at the node 0 for each interval \([t_k, t_{k+1}]\), then \(\bigcup_{t_k \leq t \leq t_{k+1}} \ell(t) = \{0, 1, \ldots, n\}\).

Theorem 11. Assume that \(\mathcal{G}\) is undirected. There exists a protocol (29)-(30) under the equation constraint (31) or (32) such that a leader-follower consensus is asymptotically reached for all initial states if \((A, B)\) is stabilizable, and \((A, C)\) is detectable, and the switching topology \(\mathcal{G}(t)\) contains a joint spanning tree rooted at 0 for each interval \([t_k, t_{k+1}]\).

4. CONCLUSION

In this paper, some necessary and sufficient conditions for dynamic consensusability and formability have been presented. Most of these conditions were proved by constructive methods. We show that by the constructive methods, the design protocol can be used for not only the time-invariant communication topologies but also the time-varying communication topologies as long as these the minimum non-zero eigenvalue of the Laplacian matrices of these topologies are known a priori. Our future work will continue to study the properties of these protocols, such as the specified convergence region. We will also address the corresponding problems for the constrained input situation and partial consensus and formation situations.

Appendix A. SOME USEFUL LEMMAS

Lemma 12.olfati-Saber and Murray [2004], Ma [2009] The Laplacian matrix \(L_{\mathcal{G}}\) of a directed graph \(\mathcal{G}\) has no eigenvalues with negative real part and at least one zero eigenvalue. \(L_{\mathcal{G}}\) has only one zero eigenvalue if and only if \(\mathcal{G}\) contains a spanning tree. In the latter case, assume that \(\lambda_i, i = 2, \ldots, n\) are the non-zero eigenvalues of \(L_{\mathcal{G}}\) and \(0 \leq \lambda_2 \leq \cdots \leq \lambda_n\).
Lemma 13. The matrix $H = Lg + D$ is positive stable (i.e., its all eigenvalues have positive real-parts) if and only if $G$ contains a spanning tree rooting at node 0. If $G$ is undirected, the matrix $H$ is positive stable if and only if $G$ is connected.

**Proof:** By applying Lemma 12, we can conclude that $G$ contains a spanning tree if and only if the Laplacian matrix $\bar{L}_g$ of $G$ has only one zero eigenvalue. Thus the eigenvalues of $L_g + D$ have positive real-parts. Reversely, if $H$ have eigenvalues of positive real-parts, then based on $\bar{L}_g \begin{pmatrix} x \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} x \\ 0 \end{pmatrix}$, we can easily obtain the eigenvalue of $\bar{L}_g$. Meanwhile, node 0 should be the root. Otherwise, there exist more than one zero eigenvalues. $
abla$

**Remark 6.** A similar result with different notation has been stated in Hu and Hong [2007], where the sufficient and necessary condition is that the node 0 is the global reachable in $G$.

**Lemma 14.** Boyd et al. [1994] For a complex matrix $\Phi$, $\Phi < 0$ if and only if

$$\begin{pmatrix} Re(\Phi) & Im(\Phi) \\ -Im(\Phi) & Re(\Phi) \end{pmatrix} < 0 \quad (A.1)$$

**Lemma 15.** Suppose that $G$ contains a spanning tree. $(A, B)$ is stabilizable if and only if $\mathbb{K}_c = \{ F : A + \lambda_i B F \text{ is Hurwitz}, i = 2, ..., n \} \neq \emptyset$. $(A, C)$ is detectable if and only if $\mathbb{K}_o = \{ L : A + \lambda_i LC \text{ is Hurwitz}, i = 2, ..., n \} \neq \emptyset$.

**Proof:** The sufficiency is obvious. Now we prove the necessity. If $\alpha_i, \beta_i \in \mathbb{R}$. If all $\lambda_i$ are real, since $\text{Rank}([A - sI, B]) = \text{Rank}([A - sI, -\alpha_i B])$, $\alpha_i \neq 0$, the necessity is proved. Meanwhile, there exist $F$ and $P > 0$

$$(A - BF)^T P + P (A - BF) < 0 \quad (A.2)$$

(A.2) can be restated as

$$(A - BF)^T Q (A - BF) < 0 \quad (A.3)$$

where $Q = P^{-1} > 0$. Note that (A.3) can be rewritten as

$$QA^T + AQ - (Y^T BY + BY) < 0 \quad (A.4)$$

by letting $FQ = Y$. By Finsler’s lemma Boyd et al. [1994], there exists a matrix $Y$ satisfying (A.4) if and only if there exists a scalar $\tau > 0$ such that

$$QA^T + AQ - \tau (BB)^T < 0 \quad (A.5)$$

Now we choose $\bar{\tau} = \frac{\tau}{2}$, where $\alpha = \text{min} \{\alpha_i\}$, then $\bar{\tau} \alpha_i \leq \tau$. As $BB^T \geq 0$, (A.5) implies that

$$QA^T + AQ - \bar{\tau} (BB)^T < 0$$

Thus we can choose $Y = \frac{\tau}{2} B^T$, i.e., $F = \frac{\tau}{2} B^T Q^{-1}$.

Now if $\beta_i \neq 0$, we have

$$0 > (A - \alpha_i B F)^T (A - \alpha_i B F) Q = (A - (\alpha_i + \beta_i) B F)^T (A - (\alpha_i + \beta_i) B F) Q$$

i.e., $\frac{(A - (\alpha_i + \beta_i) B F)^T P + P (A - (\alpha_i + \beta_i) B F)}{2} < 0$. Then we can conclude that $(A, -\lambda_i B)$ is stabilizable. $
abla$

**Lemma 16.** Suppose that $G$ contains a spanning tree and there exist $P > 0$ and $W$ such that one of the following conditions holds.

$$A^T P - C^T W^T + PA - WC < 0, \quad B^T P (C^T)^{-1} = 0$$

Then $K = \{ L : A + BLC \text{ is Hurwitz} \} \neq \emptyset$ if and only if $K = \{ L : A + \lambda_i BLC \text{ is Hurwitz}, i \in \{2, ..., n\} \}$.

**Proof:** The proof is based on Corollary 1 in Geromel et al. [1996]. We omit the details here.

**Remark 7.** With the aid of Lemmas 15 and 16, we can derive the consensability condition for static state/output feedback protocols. In fact, for static state feedback protocol, the consensability is equivalent to the stabilizability of $(A, B)$. Under the precondition of Lemma 16, the consensability of the output feedback protocols is equivalent to $\mathbb{K} \neq \emptyset$. Moreover, all conditions are obtained by constructive methods.

**REFERENCES**


