Optimality Conditions for the Guaranteed Result in Time-Delay Control Problems

Nikolay Yu. Lukoyanov

Abstract: Control under disturbances or counteraction is considered for time-delay dynamical systems with Lipschitz data. The value of the optimal guaranteed result of control is defined within the game-theoretical approach of N.N. Krasovskii and A.I. Subbotin as a nonanticipating functional of the motion realization. An appropriate Lipschitz continuity of this functional is proved, and its nonlocal and infinitesimal characterization including the Hamilton–Jacobi formalism is elaborated.

Keywords: control theory; differential games; dynamic programming; time-delay.

1. INTRODUCTION

In control theory, especially in problems of feedback control, the notion of the value function plays a key role. In optimal control (see Pontryagin et al., 1964, Bellman and Kalaba, 1965), it is the value of the optimal result. In differential games (see Isaacs, 1965, Krasovskii, 1970, Krasovskii and Subbotin, 1988), the problem of control under disturbances or counteraction is considered, it is the value of the optimal guaranteed result.

For control problems of dynamical systems described by ordinary differential equations, various types of conditions that characterize the value function are known. One can mention here such conditions as Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations (see Bellman and Kalaba, 1965, Isaacs, 1965), -u- and -v-stability properties (see Krasovskii, 1970, Krasovskii and Subbotin, 1988), inequalities for directional derivatives (Subbotin, 1980 and also Berkovitz, 1989, Vinter and Wolenski, 1990), invariance properties of the graph (Guseinov, 1985), inequalities for conjugate derivatives (see Subbotin and Tarasiev, 1985, Subbotin, 1995), viscosity properties and inequalities for sub- and supergradients (Lions and Souganidis, 1985).

In this paper, functional analogues of such conditions are considered for time-delay dynamical systems described by functional differential equations of retarded type.

Some results concerning characterization of the value of the optimal result in control problems of time-delay systems can be found in Soner (1988), Barron (1990), where a viscosity-type technique is applied to functional HJB equations with Frechet derivatives. In Wolenski (1994) an appropriate functional HJB equation is derived by evaluating certain Dini directional derivatives.

The present paper is devoted to characterization of the value of the optimal guaranteed result in time-delay problems of control under disturbances or counteraction; and, in particular, this paper deals with an appropriate functional HJBI equation. The problem is posed within the game-theoretical approach of Krasovskii and Subbotin (1988) on the basis of the functional interpretation proposed by Krasovskii (1959) for time-delay systems and developed by Osipov (1971) from the viewpoint of differential games. The value of the optimal guaranteed result is defined as a nonanticipating functional of the motion realization (the value functional). Properties of -u-stability and -v-stability of this functional can be obtained from the differential games theory (see, e.g., Osipov, 1971). In order to derive the adequate functional HJBI equation, and generalizing inequalities, as in Subbotin (1980), special co-invariant derivatives (ci-derivatives) (see, e.g., Kim, 1999) and the corresponding directional derivatives are evaluated. To characterize the value functional as a well posed generalized solution to the HJBI equation, constructions given by Krasovskii (1976) for unification of differential games, the minimax approach (see Subbotin, 1995), and the viscosity technique (see Crandall and Lions, 1983) are developed.

2. TIME-DELAY CONTROL PROBLEM

Many real processes have the “time-delay or retarded effect” when the future evolution depends not only on the current state but also on the history that has led to this state. It often happens that, together with the useful control, unpredictable disturbances or counteractions influence the process. Following the game-theoretical approach (see Krasovskii, 1970, Krasovskii and Subbotin, 1988, Osipov, 1971, see also Krasovskii, 1959 for related functional interpretation of the process), the problem of
optimization of such processes on a finite time interval can be formalized as follows.

Let times $t^* < t_0 < T$ be fixed. By symbol $C$ denote the set of continuous functions $x(t) = \{ x(t) \in \mathbb{R}^n, t_0 \leq t \leq T \}$ from $[t_0, T]$ to $\mathbb{R}^n$. Let the process state at the current time $t \in [t_0, T]$ be described by the vector $y(t) \in \mathbb{R}^n$, and the motion $y(t) = \{ y(t) \in \mathbb{R}^n, t_0 \leq t \leq T \}$ be described by the functional differential equation

$$
\dot{y}(t) = f(\tau, y(\cdot), u(\cdot), v(\cdot)), \quad t \leq \tau \leq T,
$$

$$
y(\tau) = x(t^*) \quad \text{for} \quad \tau \in [t^*, t],
$$

with the initial condition

$$
y(\tau) = x(t^*) \quad \text{for} \quad \tau \in [t^*, t],
$$

where $\dot{y}(t) = dy(t)/dt$, $u(\cdot)$ is the current control action, $v(\cdot)$ is the current disturbance action or counteraction, $P$ and $Q$ are known compact finite-dimensional sets.

Denote by symbols $u(\cdot) = \{ u(t) \in P, t \leq \tau < T \}$ and $v(\cdot) = \{ v(t) \in Q, t \leq \tau < T \}$ the control and disturbance (or counteraction) realizations.

Let the process quality be estimated by the cost functional

$$
\gamma(y(\cdot), u(\cdot), v(\cdot)) = \sigma(y(\cdot)) - \int_{t}^{T} h(\tau, y(\cdot), u(\cdot), v(\cdot)) d\tau.
$$

It is assumed in (1) and (3) that, for any fixed $u \in P$ and $v \in Q$, mappings

$$
[t_0, T] \times C \ni (\tau, y(\cdot)) \mapsto f(\tau, y(\cdot), u, v) \in \mathbb{R}^n,
$$

$$
[t_0, T] \times C \ni (\tau, y(\cdot)) \mapsto h(\tau, y(\cdot), u, v) \in \mathbb{R}
$$

are nonanticipating, i.e., for every $t \in [t_0, T]$ and any $x(\cdot), y(\cdot) \in C$ satisfying (2), the following equalities hold:

$$
f(t, x(\cdot), u, v) = f(t, y(\cdot), u, v),
$$

$$
h(t, x(\cdot), u, v) = h(t, y(\cdot), u, v).
$$

This property of nonanticipation determines the retarded structure of the considered dynamical system.

The goal of the control is to minimize functional (3). Note that, since disturbance actions (counteractions) are unknown, the worst-case might occur when disturbances (counteractions) maximize (3).

### 3. ASSUMPTIONS

Throughout this paper, $\| \cdot \|$ refers to Euclidian norm of a vector, and $(\cdot, \cdot)$ stands for the scalar product of vectors.

Let $t \in [t_0, T]$, $w(\cdot) \in C$, $\vartheta_j \in (0, t_0 - t_m]$, $j = 1, \ldots, J$ and $t_m \in [t_0, T]$, $t_{m-1} < t_m$, $m = 1, \ldots, M$, $t_M = T$.

Denote

$$
\rho(\tau, w(\cdot)) = 1 + \max_{t^* \leq \xi \leq \tau} \| w(\xi) \|,
$$

$$
\mu(\tau, w(\cdot)) = \| w(\tau) \| + \sum_{j=1}^{J} \| w(\tau - \vartheta_j) \| + \sqrt{\int_{t^*}^{\tau} \| w(\xi) \|^2 d\xi},
$$

$$
\mu_\sigma(w(\cdot)) = \sum_{m=1}^{M} \| w(t_m) \| + \sqrt{T} \| w(\xi) \|^2 d\xi.
$$

The basic assumptions are given next.

(A1) The following mappings are continuous:

$$
f : [t_0, T] \times C \times P \times Q \mapsto \mathbb{R}^n,
$$

$$
h : [t_0, T] \times C \times P \times Q \mapsto \mathbb{R}.
$$

(A2) There exists a constant $\alpha > 1$ such that

$$
\| f(\tau, y(\cdot), u, v) \| \leq \alpha \rho(\tau, y(\cdot)),
$$

$$
\| h(\tau, y(\cdot), u, v) \| \leq \alpha \rho(\tau, y(\cdot))
$$

for all $(\tau, y(\cdot), u, v) \in [t_0, T] \times C \times P \times Q$.

(A3) There exist time-delays $\vartheta_j \in (0, t_0 - t_m]$, $j = 1, \ldots, J$, such that, for any compact subset $D \subset C$, one can find a constant $\lambda = \lambda(D) > 0$ for which the following inequalities are valid

$$
\| f(\tau, y(\cdot), u, v) - f(\tau, x(\cdot), u, v) \| \leq \lambda \mu(\tau, y(\cdot) - x(\cdot)),
$$

$$
\| h(\tau, y(\cdot), u, v) - h(\tau, x(\cdot), u, v) \| \leq \lambda \mu(\tau, y(\cdot) - x(\cdot))
$$

for all $(\tau, u, v) \in [t_0, T] \times P \times Q$ and $x(\cdot), y(\cdot) \in D$.

(A4) There exist the estimation times $t_m \in [t_0, T]$, $t_{m-1} < t_m$, $m = 1, \ldots, M$, $t_M = T$, such that, for any compact set $D \subset C$, one can find a constant $\lambda = \lambda(D) > 0$ for which the following relation takes place

$$
\| \sigma(x(\cdot)) - \sigma(y(\cdot)) \| \leq \lambda \mu_\sigma(y(\cdot) - x(\cdot))
$$

for all $x(\cdot), y(\cdot) \in D$.

(A5) For all $(\tau, y(\cdot), s) \in [t_0, T] \times C \times \mathbb{R}^n$, the following equality holds:

$$
\min_{w \in P} \max_{v \in Q} \left[ \langle s, f(\tau, y(\cdot), u, v) \rangle - h(\tau, y(\cdot), u, v) \right]
$$

$$
= \max \min_{v \in Q} \left[ \langle s, f(\tau, y(\cdot), u, v) \rangle - h(\tau, y(\cdot), u, v) \right].
$$

In particular, due to assumptions (A1)–(A3) the initial value problem (1), (2) has a uniquely absolutely continuous on $[t_0, T]$ solution $y(t) \in C$ for every possible initial position $(t, x(\cdot)) \in [t_0, T] \times C$ and any Borel measurable realizations $u(\cdot) : [t, T] \mapsto P$ and $v(\cdot) : [t, T] \mapsto Q$.

Lipschitz continuity assumptions (A3) and (A4) provide in the sequel an appropriate Lipschitz continuity of the value of the optimal guaranteed result. These assumptions
are typical for dynamical systems described by differential equations with discrete and distributed time-delays.

In differential games (A5) is called “Isaacs condition” (see, e.g., Isaacs [1965]) or “the saddle point condition in a small game” (see, e.g., Krasovskii and Subbotin [1988]). This assumption does not play a crucial role in the analysis until Section 7.

4. VALUE FUNCTIONAL

In order to define the value of the optimal guaranteed result in the time-delay control problem (1)–(3), consider control strategies with memory. Such a strategy is treated as a nonanticipating mapping

\[ [t_0, T] \times C \ni (\tau, y(\cdot)) \mapsto U = U(\tau, y(\cdot)) \in P. \]

This strategy acts on the system in the discrete time scheme on the basis of some partition of the interval \([t, T]\):

\[ \Delta = \{ \tau_i : \tau_i = t, \tau_i < \tau_{i+1}, i = 1, \ldots, N, \tau_{N+1} = T \}. \]

The pair \((U, \Delta)\) constitutes the control law that forms step-by-step the piecewise constant control realization \(u(\cdot)\):

\[ u(\tau) = U(\tau_i, y(\cdot)), \tau_i \leq \tau < \tau_{i+1}, i = 1, \ldots, N. \]

Denote by \(S(t,x(\cdot),U,\Delta)\) the set of all possible triples \((y(\cdot), u(\cdot), v(\cdot))\), such that

- \(v(\cdot) : [t,T] \mapsto Q\) is a Borel measurable function,
- \(u(\cdot) : [t,T] \mapsto P\) is the piecewise constant realization defined according to (4),
- \(y(\cdot) : [t,T] \mapsto \mathbb{R}^n\) is a continuous function which is absolutely continuous on \([t,T]\) and satisfies (2),
- the triple \((y(\cdot), u(\cdot), v(\cdot))\) satisfies (1) for almost all \(\tau \in [t,T]\).

The optimal guaranteed result \(\varphi\) of the control is defined as follows:

\[ \varphi = \varphi(t,x(\cdot)) = \inf_{U,\Delta} \sup_{\delta} \gamma \left( S(t,x(\cdot),U,\Delta) \right). \] (5)

By this definition and due to (A1)–(A3), for any \(\zeta > 0\), on the one hand, there exists such a control law \(\{U, \Delta\}\) that provides the inequality

\[ \gamma \leq \varphi(t,x(\cdot)) + \zeta \]

for all Borel measurable disturbance realizations \(v(\cdot)\); and, on the other hand, every possible control law \(\{U, \Delta\}\) can be counteracted by an appropriate Borel measurable realization \(v^\circ(\cdot)\) so that

\[ \gamma \geq \varphi(t,x(\cdot)) - \zeta. \]

At the terminal time \(t = T\), it is natural to set

\[ \varphi(T,x(\cdot)) = \sigma(x(\cdot)). \] (6)

Relations (5) and (6) define a nonanticipating functional:

\[ [t_0, T] \times C \ni (t,x(\cdot)) \mapsto \varphi = \varphi(t,x(\cdot)) \in \mathbb{R}. \]

According to the terminology of differential games it is called the functional of the optimal guaranteed result or the value functional in control problem (1)–(3). The paper studies conditions which unambiguously characterize this functional.

5. HJBI EQUATION WITH CO-INARIANT DERIVATIVES

Let \((t,x(\cdot)) \in [t_0,T] \times C\) and \(\text{Lip}(t,x(\cdot))\) be the set of functions \(y(\cdot) \in C\) that satisfy (2) and are Lipschitz continuous on \([t,T]\).

Suppose that there exist \(\partial_t \varphi = \partial_t \varphi(t,x(\cdot)) \in \mathbb{R}\) and \(\nabla_x \varphi = \nabla_x \varphi(t,x(\cdot)) \in \mathbb{R}^n\) such that, for any \(y(\cdot) \in \text{Lip}(t,x(\cdot))\), the following relation is valid:

\[ \varphi(\tau,y(\cdot)) - \varphi(t,x(\cdot)) = (\tau - t)\partial_t \varphi + \langle y(\tau) - x(t), \nabla_x \varphi \rangle + o_\varphi(\tau - t), \]

\[ \tau \in (t,T), \] (7)

where \(o_\varphi\) may depend on the choice of \(y(\cdot) \in \text{Lip}(t,x(\cdot))\), \(o_\varphi(\delta) / \delta \to 0\) as \(\delta \downarrow 0\).

The differentiability property defined by (7) is typical for the Lyapunov functionals in time-delay systems. According to the terminology given in Kim [1999] values \(\partial_t \varphi\) and \(\nabla_x \varphi\) are called co-invariant derivatives (ci-derivatives). In Aubin and Haddad [2002] similar derivatives are called Cio-derivatives by the name of “the muse of history”. In the present paper the term “ci-derivatives” is used, and continuous nonanticipating functionals possessing the differentiability properties like (7) are called ci-smooth.

So, if the value functional (5),(6) is ci-smooth then it can be described by the following HJBI equation with ci-derivatives (see, e.g., Lukoyanov [2003] for details):

\[ \partial_t \varphi(t,x(\cdot)) + H \left( t, x(\cdot), \nabla_x \varphi(t,x(\cdot)) \right) = 0, \] (8)

where

\[ H(t,x(\cdot),s) = \min_{u \in P} \max_{v \in Q} \left[ \langle s, f(t,x(\cdot),u,v) \rangle - h(t,x(\cdot),u,v) \right]. \] (9)

However, as a rule, the value functional does not possess sufficient smoothness properties to be considered as the classical solution of HJBI equation, and can only be treated as its appropriate weak solution. Thus, nonsmooth techniques should be elaborated to characterize the value functional.

6. LIPSCHITZ CONTINUITY OF THE VALUE FUNCTIONAL

Let \(t_m, m = 0,1,\ldots,M\), be estimation times from (A4) and \(\tau \in [t_0,T]\), \(w(\cdot) \in C\).
Denote
\[ M(\tau) = \max \left\{ m = 0, 1, \ldots, M \mid t_m \leq \tau \right\}, \]
\[ \mu_\varphi(\tau, w(\cdot)) = M(\tau) \sum_{m=0}^{\left\lfloor \frac{\tau}{\lambda} \right\rfloor} \|w(t_m)\| + \ldots \]
that\[ \varphi(\tau, y^*(\cdot)) - \varphi(t, x(\cdot)) \leq \tau \int_{t}^{\tau} \left[ \langle \dot{y}^*(\xi), s \rangle - H(\xi, y^*(\cdot), s) \right] d\xi \]
for all \( \tau \in [t, T] \).

As is shown in Lukoyanov [2009] under (A1)–(A4) the following Lipschitz continuity property holds for the value functional (5),(6).

(L) For each compact set \( D_* \subset C \), there exists a constant \( \lambda_* = \lambda_*(D_*) > 0 \) such that
\[ \| \varphi(\tau, x(\cdot)) - \varphi(\tau, y(\cdot)) \| \leq \lambda_* \mu_\varphi(\tau, x(\cdot) - y(\cdot)) \]
for all \( \tau \in [t_0, T] \) and \( x(\cdot), y(\cdot) \in D_* \).

Due to this property of the value functional it can be characterized by evaluating certain Dini derivatives with respect to finite-dimensional directions only.

7. CHARACTERIZATION OF THE VALUE FUNCTIONAL

Define the lower and upper Dini derivatives \( \partial_{(1,1)}^- \varphi(t, x(\cdot)) \) and \( \partial_{(1,1)}^+ \varphi(t, x(\cdot)) \) of the nonanticipating functional \( \varphi : [t_0, T] \times C \rightarrow \mathbb{R} \) at the point \( (t, x(\cdot)) \in [t_0, T] \times C \) with respect to the direction \((1,1) \in \mathbb{R} \times \mathbb{R}^n \) as follows:
\[ \partial_{(1,1)}^- \varphi(t, x(\cdot)) = \liminf_{\delta \downarrow 0} \left[ \varphi(t + \delta, y(\cdot)) - \varphi(t, x(\cdot)) \right] \delta^{-1}, \]
\[ \partial_{(1,1)}^+ \varphi(t, x(\cdot)) = \limsup_{\delta \downarrow 0} \left[ \varphi(t + \delta, y(\cdot)) - \varphi(t, x(\cdot)) \right] \delta^{-1}, \]
where
\[ y(\tau) = \begin{cases} x(\tau) & \text{for } \tau \in [t_+, t], \\ x(t) + (\tau - t) \lambda & \text{for } \tau \in [t, T]. \end{cases} \]

Denote
\[ E^*(t, x(\cdot), v) = \co \{ (f(t, x(\cdot), u, v), h(t, x(\cdot), u, v)) \mid u \in P \}, \]
\[ E_* \{ t, x(\cdot), u \} = \co \{ (f(t, x(\cdot), u, v), h(t, x(\cdot), u, v)) \mid v \in Q \}, \]
where the symbol \( \co \) stands for the convex hull of the corresponding set.

Basing on (A2), define the set
\[ B(t, x(\cdot)) = \left\{ t \in \mathbb{R}^n : \|t\| \leq a \rho(t, x(\cdot)) \right\}, \]
and denote by \( \Omega(t, x(\cdot)) \) the set of such functions \( y(\cdot) \in \text{Lip}(t, x(\cdot)) \) that\[ \dot{y}(\tau) \in B(\tau, y(\cdot)) \]
for almost all \( \tau \in [t, T] \).
(C3.2) For any \( t \in [t_0, T) \), \( x(\cdot) \in C \), and any \( s \in \mathbb{R}^n \), there exists \( y(\cdot) \in \Omega(t, x(\cdot)) \) such that

\[
\varphi(t, y(\cdot)) - \varphi(t, x(\cdot)) \geq \int_t^\tau \left[ \langle \dot{y}(\xi), s \rangle - H(\xi, y(\cdot), s) \right] d\xi
\]

for all \( \tau \in [t, T] \).

(C4) For any \( t \in [t_0, T) \), \( x(\cdot) \in C \), and any \( s \in \mathbb{R}^n \), there exists \( y(\cdot) \in \Omega(t, x(\cdot)) \) such that

\[
\varphi(t, y(\cdot)) - \varphi(t, x(\cdot)) = \int_t^\tau \left[ \langle \dot{y}(\xi), s \rangle - H(\xi, y(\cdot), s) \right] d\xi
\]

for all \( \tau \in [t, T] \).

(C5) The following inequalities hold:

\[
\min_{l \in \mathcal{B}(t, x(\cdot))} \left[ \partial_{l(1,l)}^+ \varphi(t, x(\cdot)) - \langle s, l \rangle \right] \leq -H(t, x(\cdot), s), \quad (13)
\]

\[
\max_{l \in \mathcal{B}(t, x(\cdot))} \left[ \partial_{l(1,l)}^- \varphi(t, x(\cdot)) - \langle s, l \rangle \right] \geq -H(t, x(\cdot), s), \quad (14)
\]

\( t \in [t_0, T) \), \( x(\cdot) \in C \), \( s \in \mathbb{R}^n \).

(C6) The functional \( \varphi \) is such that:

(C6.1) For any nonanticipating \( ci \)-smooth test-functional \( \psi : [t_0, T] \times C \rightarrow \mathbb{R} \),

\[
\partial_\psi \varphi(t, x(\cdot)) + H\left(t, x(\cdot), \nabla_\psi \varphi(t, x(\cdot))\right) \leq 0
\]

whenever \( t < T \) and

\[
\varphi(t, x(\cdot)) - \psi(t, x(\cdot)) = \min_{\tau \in [t_0, T]} \left[ \varphi(\tau, y(\cdot)) - \psi(\tau, y(\cdot)) \right]
\]

for some \( k = 1, 2, \ldots \).

(C6.2) For any nonanticipating \( ci \)-smooth test-functional \( \psi : [t_0, T] \times C \rightarrow \mathbb{R} \),

\[
\partial_\psi \varphi(t, x(\cdot)) + H\left(t, x(\cdot), \nabla_\psi \varphi(t, x(\cdot))\right) \geq 0
\]

whenever \( t < T \) and

\[
\varphi(t, x(\cdot)) - \psi(t, x(\cdot)) = \max_{\tau \in [t_0, T]} \left[ \varphi(\tau, y(\cdot)) - \psi(\tau, y(\cdot)) \right]
\]

for some \( k = 1, 2, \ldots \).

Remark 1. Assumption (A5) is crucial only for validity of (C1) or (C2) and can be omitted when (C3)–(C6) are considered.

Theorem 1 represents different forms of characterization of optimality properties of the value functional in control problems (1)–(3).

Conditions (C1.1) and (C1.2) follow from the differential games theory (see, e.g., Krasovskii [1970], Krasovskii and Subbotin [1988, 1971]) and describe respectively the so-called properties of \( u \)-stability and \( v \)-stability of the value functional. Inequalities (11) and (12) describe these properties in the infinitesimal form: (11) and (12) are obtained by passing to the limit in (C1.1) and (C1.2) when \( \tau \rightarrow t + 0 \). Note also that (11) and (12) are the direct generalization of the inequalities given in Subbotin [1980] for the value function in control problems of dynamical systems described by ordinary differential equations.

Conditions (C3.1) and (C3.2) express \( u \)-stability and \( v \)-stability of the value functional in terms of the Hamiltonian (9). These conditions are related to constructions of uniformization of differential games (see Krasovskii [1976]) and define functional \( \varphi \) as the value functional in an appropriate unified differential game. Conditions (C3.1) and (C3.2) can be presented in the form of (C4), and condition (C4) determines functional \( \varphi \) as a minimax-type solution to equation (8) (see, e.g., Subbotin [1995], and also Lukoyanov [2003, 2009]).

Inequalities (13) and (14) represent an infinitesimal form of conditions (C3.1) and (C3.2), respectively. For \( ci \)-smooth nonanticipating functionals, (13) and (14) together are equivalent to (8). So, these inequalities can be considered as a generalization of the functional HJ equation with \( ci \)-derivatives for the nonsmooth case. Note also that (13) and (14) as well as (11) and (12) can be used effectively to check the optimality of piecewise \( ci \)-smooth nonanticipating functionals and of envelopes of \( ci \)-smooth nonanticipating functionals (see, e.g., Lukoyanov [2001] for related technique). These properties are typical for the value functional in control problems (1)–(3).

Finally, (C6) characterizes the value functional \( \varphi \) as a viscosity-type solution to (8) (see Crandall and Lions [1983], Soner [1988], and also Lukoyanov [2009]).

REFERENCES


