On the Robust Control Design for a Class of Continuous-Time Dynamical Systems with a Sample-Data Output

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Abstract: This paper deals with a class of continuous nonlinear control systems in the presence of sampled outputs. The dynamical models under consideration are described by ordinary differential equations with additive bounded uncertainties. The linear feedback control design proposed in the manuscript is based on an extended version of the classical invariant ellipsoid method. The stability/robustness analysis of the resulting closed-loop system involves the celebrated "descriptor techniques" from the extended Lyapunov methodology. Finally, the implementability of the proposed control design scheme is illustrated by a computational example. A brief discussion on the principal numerical issues is also included.

Keywords: switched control systems, affine systems, optimal control

1. INTRODUCTION

Nonlinear dynamical systems with the sample-data outputs are mathematical models of various modern control systems consisting of a part governed by differential equations, and discrete (or stepwise) outputs. These models can represent an extremely wide range of systems of practical interest (see e.g., (2; 3; 4; 16; 17)). The sample-data outputs can also be interpreted in some practical systems as a result of application of a quantified procedure to an original continuous model. In this paper, we study a specific family of the dynamical systems with the quasi-Lipschitz uncertainties. Roughly speaking, the nonlinear uncertainty effects in the given dynamic models are modeled by the quasi-Lipschitz right-hand sides of the corresponding state equations. We are particularly interested in effective algorithms for an appropriate robust control design that guarantees a "practical stability" of the resulting closed-loop system. Recently, the robust control methodologies have attracted a lot of attention, both theoretical results and applications were developed, (see e.g., (1; 2; 3; 4; 16; 17)). The robust control approach to be discussed in our contribution is based on two fundamental ideas, namely, on the well-known invariant ellipsoid approach and some advanced Lyapunov techniques (the descriptor method). We refer to (4; 5; 6; 7; 9; 10; 14; 15; 17) for the corresponding theoretical and computational details.

Recall that a set in the state space is said to be positively invariant (for a given dynamical system) if any trajectory initiated in this set remains inside the set at all future time instants (13; 10). The theoretical questions related to the existence and possible constructive characterizations of an invariant set are very sophisticated mathematical questions. Under some structural assumptions, it is possible to apply the celebrated invariant ellipsoid method and to determine an invariant set constructively (see e.g., (14; 15)).

This set will be usually chosen in the form of an ellipsoid in the given state space of the system. Note that the main theoretical tool in the usual "practical stability" analysis is the celebrated Lyapunov and Lyapunov-Krasovski approaches (see e.g., (17)). The existence question and a constructive characterization of an invariant set for a given control system is ordinarily a sophisticated mathematical problem statement. From the point of view of practical applications this problem can be replaced by a relaxed concept of the above invariant property, namely, by the attractivity property. Under some weak assumptions related to the structure of the examined dynamics, it is possible to specify an attractive set constructively. This set will be also chosen in the form of an (attractive) ellipsoid. Our paper deals with a generalization of the conventional ellipsoid schemes in the sense of the above-mentioned attractivity set. We create an attractive ellipsoid that possesses some minimal properties and use it in the design of the stabilizing feedback strategy. From the numerical point of view the control synthesis problem is reduced to an auxiliary LMI-constrained optimization problem (14; 15). The last constitutes an analytical consequence of the advanced Lyapunov-based techniques used in our manuscript (see (5; 6)).

The remainder of the paper is organized as follows. In Section 2 we formulate our main problem and discuss some necessary mathematical concepts. Section 3 is devoted to the theoretical analysis of the extended version of the invariant ellipsoid method. We use a Luemberger-type estimator and construct a minimal attractive ellipsoid that guarantees stability of the system in a practical sense. Section 4 is devoted to the associated numerical techniques. We propose an implementable algorithm for the constructive treatment of the robust control design problem under consideration. Section 5 summarizes the paper.
2. PROBLEM FORMULATION AND SOME PRELIMINARIES

The basic inspiration for studying continuous-time control systems with sample-data outputs is the dynamical model of the form

\[ \dot{x}(t) = f(t, x(t)) + Bu(t) + v_x(t) \]
\[ y(t) = Cx(t) + \omega_y(t), \quad \bar{y}(t) = y(t) \chi_{[t_k, t_{k+1})}(t) \]  

(1)

where \( x(t) \in \mathbb{R}^n \), \( t \in \mathbb{R}_+ \) is the state vector and \( u(t) \in \mathbb{R}^m \) is the control input. The vector \( y(t) \in \mathbb{R}^q \) describes the system output for every \( t \in \mathbb{R}_+ \) and \( v_x(t) \in \mathbb{R}^n \). \( \omega_y(t) \in \mathbb{R}^p \) are bounded perturbations associated with the inputs \( x(t) \) and outputs \( y(t) \), respectively. Moreover, \( f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) is an unknown nonlinear function, \( B \in \mathbb{R}^{m \times n} \), \( C \in \mathbb{R}^{p \times n} \) are given system matrices. The variable \( \bar{y} \) describes the real available sample-data outputs of the system. Note that in difference to \( y(t) \) the stepwise values \( \bar{y} \) is given a "measurable" output of the system under consideration. By \( \chi_{[t_k, t_{k+1})} \) we denote here the characteristic function of the time interval \([t_k, t_{k+1})\). For a feasible control function \( u(\cdot) \) the initial dynamical system (1) is considered on the positive semi-line \( \mathbb{R}_+ \) and for any initial state vector \( x(0) = x_0 \in \mathbb{R}^n \). We also introduce the standing assumptions, namely, hypothesis (A):  

- \( ||v_x(t)||_{Q_2}^2 + ||\omega_y(t)||_{Q_2}^2 \leq 1 \) \( \forall \ t \in \mathbb{R}_+ \);  
- \( ||f(t, x) - Ax(t)||_{Q_2} \leq \delta + ||x||_{Q_2} \) \( \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \);  
- the pair \((A, B)\) is controllable and \((A, C)\) is observable.

Here \( \delta > 0 \), \( ||\cdot||_{Q_1} \), \( ||\cdot||_{Q_2} \), \( \delta \geq ||Q_1||_{Q_2} \), \( ||\cdot||_{Q_2} \) are weighted (by some given matrices \( Q^*_n, Q^*_o, Q_1, Q_2 \) Euclidean norms of the corresponding vectors. Using the quasi-Lipschitz property of \( f(t, x) \) (see (A)), we define the auxiliary function \( \omega_x(t) := v_x(t) + f(t, x(t)) - Ax(t) \) and rewrite (1) as follows:

\[ \dot{x}(t) = Ax(t) + Bu(t) + \omega_x(t) \]
\[ y(t) = Cx(t) + \omega_y(t), \quad \bar{y}(t) = y(t) \chi_{[t_k, t_{k+1})}(t) \]  

(2)

where \( ||v_x(t)||_{Q_2}^2 \leq \delta + ||x||_{Q_2}^2 \) \( \forall \ t \in \mathbb{R}_+ \). A control system of the type (1) is usually associated with a set \( \mathcal{U} \) of feasible control functions \( u(\cdot) \). In this paper we deal with a class of linear feedback control strategies. The presence of uncertainties in the state and output dynamics motivates an additional formal step in the design procedure of an appropriate control law. This necessary step is related to a suitable observation scheme for the given initial system (1). In this paper, we use the standard Luenberger observer for this purpose:

\[ \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(\bar{y}(t) - C\hat{x}(t)), \quad \hat{x}(0) = x_0 \]  

(3)

where \( L \) is a \( n \times q \)-matrix. Application of (3) to the initial system (1) gives a rise to the explicit definition of the set \( \mathcal{U} \) of all feedback control functions of the type \( u(t) = K\hat{x}(t) \). Evidently, a control this type is characterized by a \( (m \times n) \)-gain matrix \( K \). We now introduce some additional auxiliary variables, namely, \( \Delta y(t) := \bar{y}(t) - y(t) \) and also consider the error-vector \( e(t) := x(t) - \hat{x}(t) \). Evidently, the estimating error \( e(t) \) satisfies the differential equation

\[ \dot{e}(t) = (A - LC)(t) + \omega_x(t) - L(\omega_y(t) + \Delta y(t)) \]  

(4)

with the initial condition \( e(0) = e_0 \). Combining (3)-(4) we now can write the resulting closed-loop variant of system (2) in the compact form

\[ \dot{z}(t) = Az(t) + F\omega(t) + \psi(t), \quad z(0) = (x_0, e_0) \]  

(5)

where \( z(t) := (\hat{x}(t), e(t))^T, \quad \omega(t) := (\omega_x(t), \omega_y(t)) \) for all \( t \in \mathbb{R}_+ \) and

\[ A := \begin{bmatrix} A + BK & LC \\ 0 & A - LC \end{bmatrix}, \quad F := \begin{bmatrix} 0 & L \\ I & -L \end{bmatrix}, \quad \psi(t) := (LA\Delta y(t), -\Delta y(t))^T \]

Note that the bounded variable \( \omega(t) \) introduced above can also be interpreted as a bounded uncertainty parameter or an exogenous disturbance for the given system (5). From our basic assumptions we easily deduce that

\[ ||\omega(t)||_{Q_2}^2 := ||\omega_x(t)||_{Q_2}^2 + ||\omega_y(t)||_{Q_2}^2 \leq \delta + ||x||_{Q_2}^2 \]

(6)

for every \( t \in \mathbb{R}_+ \). Here \( Q := (Q_f, Q_o) \). Let us also introduce the transformation (block-) matrix \( M := (I, I) \), where \( I \) denotes a unit matrix of the dimensionality \( n \times n \), in order to obtain the relation \( x(t) = Mz(t) \). We use this representation of the vector \( x(t) \) in the next section.

For a long time various problems of optimal feedback-type control design for a IO system with arbitrary bounded disturbances have been recognized as a major challenge in the control engineering. For example, the idea of optimal rejection of bounded disturbances was examined in (2) and (3). Let us now consider the state space of system (5) and introduce the ellipsoid with the center at the origin \( E := \{ z \in \mathbb{R}^{2n} \mid z^TPz \leq 1 \} \), where \( P \) is a positive defined symmetrical \( 2n \)-matrix. Our aim is to generate a feedback control strategy such that \( E \) to be an attractive set of the closed-loop system (5). Evidently, the control strategy under consideration is also related to a suitable selection procedure for the matrix \( L \) from observer (3). Moreover, the parameter matrix \( P \) that determines the attractive ellipsoid \( E \) can be chosen in an optimal manner. In this paper we follow the robust or practical stability methodology (see e.g., (14; 15; 16; 17)) and define \( P \) such that the size of the invariant ellipsoid \( E \) will be minimal. This minimizing problem will be considered under some natural restrictions for the "free" parameters, namely, for matrix \( P \), for the observer matrix \( L \) and for the gain matrix \( K \). The above "minimality" property can be formalized by the following minimization problem

\[ \min_{P>0, \ P^T = P, \ K, \ L} \ tr(P^{-1}) \]  

(7)

subject to \( P \in \Gamma_1(z(0), z(\cdot))(K, L) \in \Gamma_2 \) where \( z(\cdot) \) is a solution of the closed-loop system (5), \( \Gamma_1(z(0), z(\cdot)) \) is a set of symmetric and positive defined \( 2n \)-matrices that guarantee the invariance property of the corresponding ellipsoid \( E \). Evidently, this set is characterized by the dynamics of the control process under consideration. In this sense we use the above notation \( \Gamma_1(z(0), z(\cdot)) \) (the dependence of \( z(0) \) and \( z(\cdot) \)). Moreover, \( \Gamma_2 \) is a subset of the space of \( (m \times n + n \times q) \times (m \times n + n \times q) \)-dimensional matrices. This set describe the admissible system/observer matrices \( K \) and \( L \). Clearly, problem (7) is equivalent to a maximization of \( tr(P) \) over the same constraints set \( \Gamma_1 \otimes \Gamma_2 \). This last maximization problem guarantees the minimal "size" of an invariant ellipsoid \( E \). Note that the set \( \Gamma_1 \otimes \Gamma_2 \) in (5) is a set of restrictions and determines a class of admissible matrices \( P \) and \( \Theta \) such that \( E \) has the property to be invariant for the corresponding closed-loop system (4). The main problem is now to give a constructive characterization of the above restrictions set \( \Gamma_1 \otimes \Gamma_2 \).
3. THE ATTRACTIVE ELLIPSOID METHOD

The conventional ellipsoid method is usually based on an adequate Lyapunov-analysis of the attracting sets. We refer to (7) for some classical attraction Lyapunov theorems. For aims of the extended ellipsoid techniques proposed in this section we firstly formulate a useful abstract result.

Lemma 1. Let a function \( V : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies the following differential inequality \( V(t) \leq -aV(t) + \beta \), where \( \alpha > 0 \) and \( \beta > 0 \). Then \( \lim_{t \to \infty} V(t) \leq \beta/\alpha \).

This result can be proved by direct computations applied to a specially constructed "cutting function" and some standard properties of continuous functions, namely, the Weierstrass Theorem. We refer to (7) for the corresponding proof.

Let us now introduce the following Lyapunov-Krasovskii functional associated with trajectories \( z(\cdot) \) and the corresponding derivatives \( \dot{z}(\cdot) \) of (5) (see (7)):

\[
V(z(\cdot), \dot{z}(\cdot))(t) = z^T(t)P^{-1}z(t) + \int_{t-h}^t e^{(s-t)}z^T(s)Sz(s)ds + ah \int_0^h \int_{t-h}^t e^{(s-t)}z^T(s)R\dot{z}(s)dsd\theta
\]

where \( S \) and \( R \) are positive definite symmetric matrices, \( h := \max_4 |k_{t+1} - k_t| \) and \( a > 0 \). This analytical construction will be used in the further analysis of the attractive/invariance properties of \( E \). Let us firstly compute the derivative of the function \( V(z(\cdot), \dot{z}(\cdot))(t) \) introduced above.

Lemma 2. The derivative \( dV/dt \) of \( V(z(\cdot), \dot{z}(\cdot)) \) along the trajectory \( x(\cdot) \) and its derivative \( \dot{z}(\cdot) \) of (5) is given by the following relation

\[
\frac{dV}{dt}(z(\cdot), \dot{z}(\cdot))(t) = 2z^T(t)P^{-1}\dot{z}(t) - a \int_{t-h}^t e^{(s-t)}z^T(s)Sz(s)ds + ah \int_0^h \int_{t-h}^t e^{(s-t)}z^T(s)R\dot{z}(s)dsd\theta - e^{-ah}z^T(t)Sz(t - h) + \dot{z}(t)^2 - h^2z^T(t)R\dot{z}(t)
\]

Proof: Evidently, the derivative of the first term of \( V \) is equal to \( 2z^T(t)P^{-1}\dot{z}(t) \). The derivative of the second term of \( V \) can be evaluated by a direct computation. The application of the well-known differentiating formula for integrals with variable limits (see e.g., (17)) to parts of \( V \) implies the resulting relation for \( dV/dt(z(\cdot), \dot{z}(\cdot))(t) \). □

The next result is usually called the "A-matrix inequality" (see (17) for details).

Lemma 3. For any matrices \( X, Y \in \mathbb{R}^{n \times n} \) and any symmetric positive definite matrix \( \Delta \in \mathbb{R}^{n \times n} \) the following is true \( X^T Y + Y^T X \leq X^T (I + \Delta) X + Y^T (I + \Delta^{-1}) Y \). Moreover, we have \( (X + Y)^T (X + Y) \leq X^T (I + \Delta) X + Y^T (I + \Delta^{-1}) Y \).

We are now able to formulate our first significant result.

Theorem 4. Let all conditions from Section II are fulfilled. Let also \( \alpha := a, \beta := b(1 + \delta) + (\lambda_1 + \lambda_2)|||L|||_Q \), where \( \lambda_1 \) and \( \lambda_2 \) are the maximal eigenvalues of some positive defined \((2n \times 2n)\)-dimensional matrices \( A_1 \) and \( A_2 \), respectively. Assume that the following nonlinear minimization problem

\[
\min_{P > 0, \ P = \ P^T, \ K, \ L} \tr(P^{-1})/\beta
\]

subject to

\[
\begin{align*}
& |z^T(t)Pz(t)| \leq 1 & \text{if } z(0) \in E \\
& \frac{d}{dt} (z(\cdot), \dot{z}(\cdot))(t) \leq -\alpha V(z(t)) + \beta & \text{if } z(0) \in \mathbb{R}^{2n} \setminus E
\end{align*}
\]

where \( z(\cdot) \) is a solution of the closed-loop system (5), has an optimal solution \((P, K, \dot{L})\). Then the ellipsoid determined by the matrix \( P = \alpha P^{-1}/\beta \) is a minimal attractive ellipsoid for system (5).

Proof: Let us consider the both two families of constraints in the minimization problem, namely, the cases \( z(0) \in E \) and \( z(0) \in \mathbb{R}^{2n} \setminus E \). Our aim is to show that the composed function \( V(t) = V(z(\cdot), \dot{z}(\cdot))(t) \) satisfies the inequality from Lemma 1. Let us estimate the sixth term of the derivative of \( V(z(\cdot), \dot{z}(\cdot))(t) \)

\[
\int_{t-h}^t e^{(s-t)}z^T(s)R\dot{z}(s)ds \leq e^{-ah} \int_{t-h}^t z^T(s)R\dot{z}(s)ds.
\]

Applying the Jensen’s Inequality to the above integrals we get

\[
\int_{t-h}^t z^T(s)R\dot{z}(s)ds \geq \int_{t-h}^t z^T(s)dsR \int_{t-h}^t z^T(t)ds.
\]

Now we have

\[
\frac{dV}{dt}(z(\cdot), \dot{z}(\cdot))(t) \leq 2z^T(t)P^{-1}\dot{z}(t) - a \int_{t-h}^t e^{(s-t)}z^T(s)Sz(s)ds + z^T(t)Sz(t) - e^{-ah}z^T(t)Sz(t - h) - ah \int_0^h \int_{t-h}^t e^{(s-t)}z^T(s)R\dot{z}(s)dsd\theta - he^{-ah} \int_{t-h}^t z^T(s)dsR \int_{t-h}^t z^T(s)ds + h^2z^T(t)R\dot{z}(t)
\]

Adding \( aV((z(\cdot), \dot{z}(\cdot))(t) - b||w(t)||_Q^2 \), where \( b > 0 \), to the both sides of the last inequality we now deduce

\[
\frac{dV}{dt}(z(\cdot), \dot{z}(\cdot))(t) + aV(z(\cdot), \dot{z}(\cdot))(t) - b||w(t)||_Q^2 \leq 2z^T(t)P^{-1}\dot{z}(t) - a \int_{t-h}^t e^{(s-t)}z^T(s)Sz(s)ds + z^T(t)Sz(t) - e^{-ah}z^T(t)Sz(t - h) - ah \int_0^h \int_{t-h}^t e^{(s-t)}z^T(s)R\dot{z}(s)dsd\theta - he^{-ah} \int_{t-h}^t z^T(s)dsR \int_{t-h}^t z^T(s)ds + h^2z^T(t)R\dot{z}(t) + a \int_{t-h}^t e^{(s-t)}z^T(s)Sz(s)ds + ah \int_0^h \int_{t-h}^t e^{(s-t)}z^T(s)R\dot{z}(s)dsd\theta - b||w(t)||_Q^2
\]

Let \( \alpha := a \). Since \( b||w(t)||_Q^2 \leq b(\delta + 1) + b|Mz(t)||_Q^2 \), we modify the right hand side of the last inequality and obtain
the next estimation given by the relation
\[ 2z^T(t)P^{-1} \dot{z}(t) + z^T(t)Sz(t) - e^{-ah} z^T(t-h)Sz(t-h) - \]
\[ h e^{ah} \int_{t-h}^t \dot{z}(s) ds R \int_{t-h}^t \dot{z}(s) ds + h^2 z^T(t) R z(t) + \]
\[ a z^T(t)P^{-1} z(t) + z^T(s) S z(s) ds - b z^T(t) Q \omega(t) + \]
\[ b z^T(t) M^T M z(t) \]
Rewrite \( \Delta \theta(t) = C M \Delta z(t) + \Delta \omega_y(t) \) and compute the quotient \( \Delta \omega(t) = \frac{1}{P^{-1}} \dot{z}(s) ds \). Let us introduce the new notation \( \Delta \omega_y(t) := \dot{\omega}(t) - \omega(t) \). Let \( \Pi_b \) be some matrices of the suitable dimension. Following the idea of the "descriptor method" developed in (6) and (7) for systems with time-delays, we now consider the term
\[ 2(z^T(t)P_b + z^T(t)P_c) \times (\dot{A}z(t) + F \omega(t) + \]
\[ (0, \ LCM \int_{t-h}^t \dot{z}(s) ds) + (0, \ L \Delta \omega_y(t)^T - \dot{\omega}(t)) \]
It is evident that this relation defined on admissible trajectories (solutions) \( z(\cdot) \) of the main system (5) is equal to zero. Using Lemma 3, we also compute the upper bounds for the terms involving \( \Delta \omega_y(t) \)
\[ 2z^T(t)P_b(0, \ L \Delta \omega_y(t)) + \leq z^T(t) P_b A^{-1} P_b z(t) + (L \Delta \omega_y(t))^T A_b \Delta \omega_y(t), \]
\[ 2z^T(t)P_c(0, \ L \Delta \omega_y(t)) + \leq z^T(t) P_c A_2^{-1} P_c z(t) + (L \Delta \omega_y(t))^T A_2 \Delta \omega_y(t), \]
and finally obtain
\[ (L \Delta \omega_y(t))^T (\lambda_1 + \lambda_2) L \Delta \omega_y(t) \leq (\lambda_1 + \lambda_2) ||L||^2_{\omega_y} \]
Consider the extended vector
\[ \eta(t) := (z(t), \dot{z}(t), \int_{t-h}^t \dot{z}(s) ds, (t-h), \omega(t))^T \]
and compute the estimation
\[ \frac{dV}{dt}(z(\cdot), \dot{z}(\cdot)) + \alpha V(z(\cdot), \dot{z}(\cdot)) - \beta ||\omega(t)||_Q^2 \leq \]
\[ \frac{dV}{dt}(z(\cdot), \dot{z}(\cdot)) + \alpha V(z(\cdot), \dot{z}(\cdot)) - \beta + (\lambda_1 + \lambda_2) ||L||^2_{\omega_y} \leq \eta^T W \eta \]
where \( \beta := b(1 + \delta) \) and
\[ W := \begin{bmatrix} w_{11} & w_{12} & w_{13} & 0 & w_{15} \\
 w_{21} & w_{22} & w_{23} & 0 & w_{25} \\
 w_{31} & w_{32} & w_{33} & 0 & 0 \\
 0 & 0 & 0 & w_{44} \\
 w_{51} & w_{52} & 0 & 0 & w_{55} \end{bmatrix} \]
The elements of \( W \) are given as follows
\[ w_{11} := ((2 + a) I + M^T Q \eta M) P^{-1} + S + \]
\[ 2 \Pi_b \tilde{A} + \Pi_b A_1^{-1} \Pi_b \]
\[ w_{12} := P^{-1} - \Pi_b + \Pi_b \tilde{A} w_{13} := \Pi_b \tilde{A} \]
\[ w_{15} := \Pi_b F w_{22} := h^2 R - 2I_b + \Pi_b A_2^{-1} \Pi_b \]
\[ w_{23} := \Pi_b M \]
\[ w_{32} := \Pi_b M^T \]
\[ w_{33} := -Re^{-ah} w_{44} := -Se^{-ah} \]
\[ w_{51} := F^T \Pi_b w_{52} := F^T \Pi_b w_{55} := -b Q \]
\[ M := (0, \ LCM)^T \]
where \( M := (0, \ LCM)^T \) and select some matrices \( \Pi_b \), \( \Pi_c \) and the optimal matrices \( P, \ K, \ L \) such that \( W \leq 0 \). Then we conclude that the function \( V(t) \) defined as \( V(z(\cdot), \dot{z}(\cdot))(t) \) satisfies the conditions of Lemma 1. From this fact it follows that the ellipsoid given by \( P = \alpha P^{-1}/\beta \), where \( \beta \) has the minimality property in the sense of the above optimization problem, is a minimal attractive ellipsoid of the closed-loop system (5). □

Evidently, a set of admissible matrices \( P, \ K, \ L \) such that \( W \leq 0 \) is a non-specified set in the corresponding finite-dimensional Euclidean space. The initial minimization problem from Theorem 1 is a strongly nonlinear problem of mathematical programming with constraints given in the form of nonlinear matrix inequalities. The general solution procedures to these optimization problems are usually given by very sophisticated algorithms. The same is also true with respect to the possible effective numerical approximations. Therefore, we replace this initial nonlinear optimization problem by an adequate linear problem. Our aim is to relax the given nonlinear matrix-constrain \( W \leq 0 \) by a suitable system of LMI's. Moreover, the cost functional will be also relaxed to a linear functional that approximates the initial costs.

We use the additional notation \( P^{-1} := \text{diag}(P_1, P_2) \) and \( R := \text{diag}(R_1, R_2) \). Moreover, we also introduce the matrices \( S := \text{diag}(S_1, S_2), A_1 := A_2 := \text{diag}(P_1^{-1}, P_2^{-1}) \) and \( Q_y := \text{diag}(Q_y^1, Q_y^2) \). Define the block-components of matrix \( MTQM \) by \( M_{ij}, i, j \in \{1, 2\} \). Let us now introduce the following auxiliary block-matrices \( W_i := \{W_{11}\}_{i,j=1,2} \), where \( W_{11} \) is a symmetrical matrix, and
\[ G_1 \begin{bmatrix} Y_1^T C & 0 & 0 & 0 & 0 \\
 w & C^2 Y_2 & A_2 X_2 - C^2 Y_2 & -2Y_2^T C & -2Y_2^T C \\
 & . & G_3 & 0 & 0 \\
 . & . & . & h^2 R_2 - 2Y_2^T C & -2Y_2^T C \\
 . & . & . & -R_1 e^{-ah} & 0 \\
 w := X_2 A - Y_2 C + A_2^T X_2 - C^2 Y_2^2 + M_2 X_2 + (3 + a) X_2 + S_2 \\
 W_{22} := \text{diag}(-S_1 e^{-ah}, -S_2 e^{-ah}, -b Q_y^1, -b Q_y^2, \]
\[ G_4, G_5) \]
\[ W_{12} := \begin{bmatrix} 0 & 0 & 0 & Y_2^T & 0 \\
 0 & 0 & 0 & Y_2^T & 0 \\
 0 & 0 & 0 & Y_2^T & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Moreover, let \( G_1 := \{G_1^i\}_{i,j=1,2} \) and
\[ g^2 := \begin{bmatrix} -G_2 - 2X_2 & I & X_1 A^T + Y_1 B^T \\
 -2X_2 & I & X_1 + A_4 \\
 -G_4 - S_1 & X_1 - A_3 \\
 -G_5 - h^2 R_1 & X_1 - A_4 \end{bmatrix} \]
\[ g_5 := \begin{bmatrix} H & I & L X_2 \end{bmatrix} \]
with \( G_1^i := -G_1 - 2X_2, G_2^1 := G_1^1 := I \) and
\[ G_4 \triangleq : X_1 A^T + AX_1 + Y_1 B^T + BY_1^T + X_1 M_1 + (3 + a)X_1 + \Lambda_3 \]

We also consider the family of the associated matrix parameters \( \Upsilon := (X_1, X_2, Y_1, Y_2, R, S, G_1, G_2, G_3, G_4, G_5, H) \) of the corresponding dimension.

We are now able to formulate an over-relaxation to the initial minimization problem from Theorem 1. The solutions set to this relaxed auxiliary problem contains the solution set of the above-mentioned initial auxiliary problem.

**Theorem 5.** Assume that the following auxiliary optimization problem

\[
\min_{X_1} \text{tr}(X_1^*) + \text{tr}(H) \\
\text{subject to } W \leq 0 \quad G^1 \leq \hat{G}^2 \leq 0 \quad G^3 \leq 0 \quad (8) \\
G^4 \leq 0 \quad G^5 \leq 0 \quad X_1 \geq 0 \quad X_2 \geq 0 \quad H \geq 0
\]

has a solution \( \hat{X} \). Then the solution set of (8) contains the solution set of the initial minimization problem from Theorem 4. The ellipsoid \( E \) defined by the matrix \( P = \text{diag}(\hat{X}_1, \hat{X}_2^{-1}) \) approximates a minimal attractive ellipsoid for the system (5). Moreover, the corresponding gain and observer matrices are given as follows:

\[ K = \hat{X}_1^{-1} \hat{Y}_1, \quad L = \hat{Y}_2 \hat{X}_2^{-1} \quad (10) \]

The proof of this result is based on linear over-approximation of the set given by the matrix inequality. Evidently, the resulting optimization problem (8) is a linear minimization problem with the constraints in the form of LMI. This over-relaxation of the initial nonlinear problem (discussed in Theorem 4) provides a basis for numerical approaches to the practically stable feedback control design. This design procedure is defined by the optimal control/observer matrices (10). The corresponding approximation of the minimal attractive ellipsoid for system (5) is given by (9).

4. THE NUMERICAL ASPECTS

The example presented here corresponds to a separately excited DC motor with the following dynamics

\[ \dot{\Omega} = c \Phi_s I_r - B \Omega - \eta \quad (11) \]

where \( \Omega \) denotes the angular velocity of the shaft; \( I_r \) and \( I_s \) are the currents of the rotor circuit, and \( R_r \) and \( R_s \) the corresponding resistances. The rotor and stator voltages are expressed by \( U_r \) and \( U_s \). The rotor inductance is denoted here by \( L_r \) and \( \Phi_r \) is the stator flux. The parameters \( J \) and \( B \) in the above model express the moment of inertia of the rotor and the viscous friction coefficient, respectively. Finally, \( \eta \) denotes a parametrical uncertainty (see the general model (11)) and \( c \) represents a constant parameter that depends on the spatial architecture of the drive. The initial conditions for (11) are selected as follows \((\Omega^0, I_r^0, \Phi_s^0)^T = (1, 1, 1)^T\).

Let us apply the conventional linearization procedure to the original model (11) (see e.g., (11)). We consider this linearization of (11) around a given reference point \((\Omega^{ref}, I_r^{ref}, \Phi_s^{ref})\). The linearized model has the following form:

\[
\begin{align*}
\dot{x}_1 &= -\frac{B}{J} x_1 + \frac{c}{J} \Phi^{ref} x_2 + \frac{c I^{ref}}{J} x_3 - \eta \\
\dot{x}_2 &= -\frac{c}{L_r} \Phi_r x_1 - \frac{R}{L_r} x_2 - \frac{c}{L_r} \Omega^{ref} + \frac{1}{L_r} U_r \\
\dot{x}_3 &= -R_s + U_s
\end{align*}
\]

where \( x := (x_1, x_2, x_3)^T \) is a state vector of the linearized model. The initial conditions for the linearized system (12) are given by \((x_1^0, x_2^0, x_3^0)^T = (0,0,0)^T\). The corresponding linear state space representation is characterized by the following matrices:

\[
A = \begin{bmatrix}
B & c \Phi^{ref} & c I^{ref} \\
-\frac{c}{L_r} & -\frac{R}{L_r} & -\frac{c}{L_r} \\
0 & 0 & -R_s
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

and \( Q_r = I_{3 \times 3} \). As we can see the obtained model (12) has the form of the state equation from our basic system (2).

We now apply the solution procedure proposed in Section 5 (see Theorem 5) to the resulting problem (8) associated with the linearized system (12). The matrix \( P \) of a minimal ellipsoid \( E \) and the corresponding controller/observer matrices \( K, L \) are determined from (9)-(10). Recall that the linear system (12) is used in the control design procedure as an auxiliary model. This representation corresponds to the model (2) containing matrix \( A \). For the concrete simulation of the original nonlinear system (11) with the selected controller/observer matrices \( K \) and \( L \) we have chosen the following model parameters: The results of our computational experiment are presented in Fig. 1 and Fig. 2. Fig. 1 contains the projection of the obtained attractive ellipsoid and the system trajectory) of the three-dimensional \( z \)-state space on the two-dimensional subspace \((x_1, x_2)\). On the Fig. 2 we present the projection of the obtained ellipsoid and of the resulting three-dimensional dynamics on the two-dimensional subspace \((x_2, x_3)\).

### Table 1. Parameters for the DC motor

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
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<tr>
<td>c</td>
<td>0.03</td>
<td>Wb/rad</td>
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<tr>
<td>J</td>
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<td>kg/m²</td>
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<td>Rₙ</td>
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<td>Ohms</td>
</tr>
<tr>
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<td>Ohms</td>
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<td>mH</td>
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<td>rad/s</td>
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<tr>
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<td>Wb</td>
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</table>

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5. CONCLUDING REMARKS

In this contribution we deal with some new implementable algorithm for a robust control design for a class of systems with uncertainties and sample-data outputs. The design procedure is based on an extension of the conventional invariant ellipsoid method and incorporates an auxiliary nonlinear minimization problem with matrix-constraints. We next considered an over-relaxation of this initial optimization problem obtain an attractive ellipsoid with some minimal properties (a minimal "size") that can be interpreted as a maximal robustness or stability in the practical sense. Our paper also proposes a computational algorithm that generates the corresponding feedback control strategy. The effectiveness of the control design method is demonstrated by an illustrative numerical example.

REFERENCES