On Rearrangements of Exponentially Stable Discrete Linear Systems

Adam Czornik Aleksander Nawrat

Department of Automatic Control, Silesian Technical University, Akademicka 16, 44-101 Gliwice, Poland, (e-mail: Adam.Czornik@polsl.pl)

Abstract: In this paper we investigate the influence of rearrangements of terms in the evolution operator of discrete system on the exponential stability. We present a necessary and a sufficient condition for a permutation of natural number to carry any exponentially stable system into exponentially stable system.

1. INTRODUCTION

Let \(\mathbb{X} = (\mathbb{X}, |||\cdot|||)\) be a complex or real Banach space. We consider time-varying discrete linear systems of the form

\[ x(n + 1) = A(n)x(n), \quad n \in \mathbb{N} \]  

(1)

where \(A = (A(n))_{n \in \mathbb{N}}\) is a bounded sequence of bounded linear operators on \(\mathbb{X}\). The evolution operator for (1) is defined as

\[ A(m, k) = A(m - 1)\ldots A(k) \]

for \(m > k\) and \(A(n, m) = I\), where \(I\) is the identity operator. System (1) is called uniformly exponentially stable (UES), if the evolution operator satisfies

\[ \|A(m, k)\| \leq cq^{m-k}, \]

(2)

for some constants \(c, q\), \(0 < q < 1\), \(c > 1\), where \(\|A(m, k)\|\) denote the induced operator norm.

Stability problems for time-varying discrete-time systems in Banach spaces have received some attention over the last few years, see Caughlan and Logemann [2009], Logemann [1992], Michel et al. [2005], Przyluski [1985], Przyluski [1988] and Przyluski [1984], and references therein. Several authors have noted the possibility of studying functional properties of these systems in this context. For these applications it is essential that discrete-time systems on infinite dimensional spaces are considered.

In this paper we will investigate the problem of describing permutations \(\sigma\) of natural numbers such that, if systems (1) is UES, then so is the following system

\[ x(n + 1) = A(\sigma(n))x(n), \quad n \in \mathbb{N}. \]  

(3)

Such a problem arises in natural way when we consider stability of switched linear systems, see Czornik and Jurgas [2008], Czornik [2005], Daafouz et al. [2002], Liberzon and Morse [1999], and references therein.

2. MAIN RESULTS

The UES of system (1) may be characterized by the discrete time version of the Bohl exponent Daleckii and Krein [1974] (named generalized spectral radius in Przyluski [1988]).

Definition 1. The Bohl exponent \(\beta(A)\) of system (1) is defined in the following way

\[ \beta(A) = \inf \{ \beta : \exists c_\beta \geq 1, m \geq k \geq 0 \Rightarrow \|A(m, k)\| \leq c_\beta \beta^{m-k} \}. \]

The role of the Bohl exponent for UES and it simplest properties are given in the next Theorem taken from Przyluski [1988].

Theorem 1. System (1) is UES if and only if \(\beta(A) < 1\). Moreover

\[ \beta(A) = \limsup_{k,m \to \infty} \|A(m, k)\|^{1/\tau} = \inf_{\tau \in \mathbb{R}} \sup_{m \in \mathbb{N}} \|A(t + m, m)\|^\tau \]  

(4)

Before we will present the main results of this paper we introduce some notation. For two nonempty subsets \(A\) and \(B\) of \(\mathbb{N}\) we write \(A < B\) when \(a < b\) for any \(a \in A\) and \(b \in B\). Denote by \(|A|\) cardinality of a set \(A\), and if \(n, m \in \mathbb{N}, n \leq m\), then

\[ I(n, m) = \{n, n + 1, \ldots, m\}, \]

\[ I'(n, m) = \{n, n + 1, \ldots\}. \]

The sets \(I(n, m)\), \(I'(n, m)\) will be called interval and ordered interval, respectively. We say that a set \(A \subset \mathbb{N}\) (a sequence \((a_1, a_2, \ldots, a_l)\)) is a union of \(k\) mutually separated intervals (ordered intervals) if there exist \(k\) intervals \(I_1, \ldots, I_k\) (ordered intervals \(I'_1, \ldots, I'_k\)) such that \(A = I_1 \cup \ldots \cup I_k = (a_1, a_2, \ldots, a_l) = (I'_1, \ldots, I'_k)\) and \(\text{dist}(I_i, I_j) \geq 2\) (\(\text{dist}(I'_i, I'_j) \geq 2\)) for any distinct \(i, j \leq k\). If \(\sigma\) is a permutation of natural numbers and \(n, m \in \mathbb{N}, n \leq m\), then \(\kappa_n(n, m) = \kappa'_n(n, m)\) is such a number \(k\) that set \(\sigma(I(n, m))\) (sequence \((\sigma(n), \ldots, \sigma(m))\)) is a union of \(k\) mutually separated intervals (ordered intervals). For linear operators \(A(n), A(n+1), \ldots, A(m), n \leq m\) on \(\mathbb{X}\) symbol

\[ \prod_{i=n}^{m} A(i) \]

denotes \(A(m)A(m-1)\ldots A(n)\). We say that a permutation \(\sigma\) preserves UES if for any UES system (1), system (3) is UES. Finally, we say that a permutation...
\[ \beta(A) = \beta_\sigma(A), \]
where \( \beta_\sigma \) is the Bohl exponent of (3). Because for all complex \( z \) we have \( \beta(zA) = |z| \beta(A) \), then from Theorem 1 we see that \( \sigma \) preserves UES.

The next two theorems contain the main results of the paper.

**Theorem 2.** If
\[
\lim_{m,n\to\infty} k'_{\sigma}(n,m) = 0, \tag{5}
\]
then \( \sigma \) preserves UES.

**Proof.** Suppose that (1) is UES and let \( c, q \) be such that (2) holds. Let us select \( \varepsilon > 0 \) such that
\[
eq 1 \varepsilon \tag{6}
\]
According to (5) there exists \( n_0 \) such that
\[
k'_{\sigma}(n,m) < \varepsilon \tag{7}
\]
for all \( n, m \in \mathbb{N}, m - n > n_0 \). Let fix natural \( n, m, m - n > n_0 \) and denote by \( I_1, \ldots, I'_{k(n,m)} \) mutually separated ordered intervals, such that
\[
(\sigma(n), \ldots, \sigma(m)) = (I_1, \ldots, I'_{k(n,m)}).
\]
According to definition of \( c, q \) we have
\[
\|A_\sigma(m,n)\| = \left\| \prod_{i=1}^{k'} A(j) \right\| \leq c \prod_{i=1}^{k'} \prod_{j \in I'_i} A(j) \leq c \prod_{i=1}^{k'} \left( cq \right)^{m-n} = c^{k'_{\sigma}(n,m)} q^{m-n},
\]
where \( A_\sigma \) is evolution operator for (3). Applying (7) to the last inequality, we obtain
\[
\|A_\sigma(m,n)\| \leq c^{k'_{\sigma}(n,m)} q^{m-n} = (c q)^{m-n} \
\]
Taking into account formulas (4) and (6) we obtain
\[
\|A_\sigma(m,n)\| \leq c^{k'_{\sigma}(n,m)} q^{m-n} = (c q)^{m-n}.
\]
We will show that there exists a one dimensional (a.e \( \mathbb{X} = \mathbb{R} \)) system (1) which is UES, but system (3) is not. Fix numbers \( c \) and \( q \), \( 0 < q < 1, c > 1 \) such that
\[
eq 1 \varepsilon \tag{9}
\]
Now we define a sequence of intervals \( I(n, m), l = 1, 2, \ldots \) in the following way. For \( l = 1 \) let \( n_1, m_1 \in \mathbb{N}, n_1 - m_1 > 1 \) be such that
\[
k(n_1, m_1) > \varepsilon_0 (n_1 - m_1).
\]
Next define mutually separated intervals
\[
I_i^{(1)} = \left( a_i^{(1)}, b_i^{(1)} \right),
\]
\[
i = 1, \ldots, k_\sigma(n_1, m_1) \text{ such that } \sigma(I(n_1, m_1)) = \bigcup_{i=1}^{k_\sigma(n_1, m_1)} I_i^{(1)}.
\]
If \( I(n_1, m_1), \ldots, I(n_l, m_l) \) are defined, then for each
\[
I(n_p, m_p), p = 1, \ldots, l \text{ consider mutually separated intervals}
\]
\[
I_i^{(p)} = \left( a_i^{(p)}, b_i^{(p)} \right),
\]
\[
i = 1, \ldots, k_\sigma(n_p, m_p) \text{ such that } \sigma(I(n_p, m_p)) = \bigcup_{i=1}^{k_\sigma(n_p, m_p)} I_i^{(p)}.
\]
Let us choose \( n_{l+1}, m_{l+1} \in \mathbb{N}, n_{l+1} - m_{l+1} > l \), such that
\[
\bigcup_{p=1}^{l} k_\sigma(n_p, m_p) \geq \varepsilon_0 (m_{l+1} - n_{l+1}).
\]
Finally we define a sequence \( (A(n))_{n \in \mathbb{N}} \) of real numbers as follows
\[
A(i) = \begin{cases} c^{|J^{(p)}_i|} q & \text{if } i \in J^{(p)}_i \\ 0 & \text{otherwise} \end{cases}
\]
We will show that (1) is UES, but (3) is not. We have
\[
\prod_{i=1}^{k(n_1, m_1)} A_\sigma(i) = \prod_{i=1}^{k(n_1, m_1)} A_j = \prod_{i=1}^{k(n_1, m_1)} \left( c q |J^{(p)}_i| \right) = c^{k(n_1, m_1)} q^{m_1-n_1} = \left( c q^{m_1-n_1} \right).
\]
It implies together with (9) that \( \beta_\sigma(A) > 1 \) and by Theorem 1 system (3) is not UES. By the definition of \( (A(n))_{n \in \mathbb{N}} \) it is clear that
\[
\prod_{i=1}^{m_i} A(i) \neq 0
\]
if and only if \( I(n, m) = J^{(i)}_t \) for certain \( t \in \mathbb{N} \) and \( i = 1, \ldots, k_\sigma(n_t, m_t) \) but then
\[
\prod_{s \in J^{(i)}_t} A(s) = c^{|J^{(p)}_t|} q^{|J^{(p)}_t|}.
\]
It implies that (1) is UES.

**Theorem 3.** If \( \sigma \) preserves UES, then
\[
\lim_{m,n\to\infty} k_\sigma(n,m) = 0. \tag{8}
\]

**Proof.** Suppose that \( \sigma \) preserves UES and
\[
\lim_{m,n\to\infty} k_\sigma(n,m) = \varepsilon_0 > 0.
\]
Then for each \( l \in \mathbb{N} \) there are
\[
p, q \in \mathbb{N}, q - p > l, p > l
\]
such that
\[
k_\sigma(p, q) \geq \varepsilon_0 (q - p).
\]
with ordered interval replaced by intervals and therefore the following statement is true.

Corollary 4. A permutation $\sigma$ preserves UES of all systems (1) with commuting operators $(A(n))_{n \in \mathbb{N}}$ if and only if it satisfies the condition (8).

The last Corollary implies, in particular, that a permutation $\sigma$ preserves UES of all scalar systems (1) if and only if it satisfies the condition (8). Unfortunately, in general case, neither condition (8) is sufficient nor condition (7) is necessary for permutation $\sigma$ to preserve USE of (1).

3. CONCLUSIONS

In this paper we considered the problem of describing the permutations of terms in the evolution operator of discrete system which preserve exponential stability. We present a necessary and a sufficient condition for such permutation. The problem of finding a necessary and sufficient condition remains open.

REFERENCES