About Controlled Impact of Two Bodies*

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Abstract: A model of mechanical system consisting of visco-elastic racket and rigid ball is introduced. A class of optimal control problems according to the different cases of the termination of interaction during the impact are formalized. One task from this class is solved as an example of the general approach. The paradox phenomenon in geometric condition is considered as an optimal control problem.

1. INTRODUCTION

The optimal control tasks for the systems with impacts during the impact phase provide a wide area of research because they are not fairly investigated. If in these tasks the time of the system control is found from the condition of the termination of interaction then the task’s solution is reduced to the known Pontryagin’s maximum principle. In the case when the control time is fixed the optimal control task is reduced to the one with the phase restrictions. The necessary conditions are obtained by Gammekideze and the maximum principle is proved by Dubovitskii and Milutin (Arutunov et al. [2006]). If the surface of the termination of interaction in the explicitly depends on time and control then it is the case of the mixed restrictions. There are too few results on optimal control here.

There are two approaches to the description of impact phase in impact mechanics, namely, geometric and physical (Kozlov, Treshev [1991]). In the considered model it is supposed that a rigid body interacts with a visco-elastic racket parameterized by its rigidity where both a control force acting on the body in a contact phase and a total impulse of this force are bounded during the impact period. Special technic based on so called space-time transformation and developed in Miller, Bentsman [2003] is used to research a problem of impact of a rigid body with constraint.

A formalized impact problem statement is entered through a contact set. It allows to reduce different known conditions of the termination of interaction to the one type. In frameworks of the considered model an optimal control task of a racket with the aim of minimizing an after impact velocity is solved. The paradox phenomenon in geometric condition is considered as an optimal control task. It should be noticed that private optimal control tasks in an impact phase are considered in Miller, Bentsman [2007], Miller, Bentsman [2006], Bentsman, Miller [2007] (geometric condition), Bentsman et al. [2009] (geometric condition and multiple impact), Galyaev et al. [2006], Bentsman et al. [2008] (physical condition).

2. MOTION EQUATIONS OF BALL-RACKET SYSTEM

2.1 Space-Time Transformation

The task of perpendicular collision of a perfect rigid ball of mass $\mathcal{M}$ with a visco-elastic racket of mass $\mathcal{M}$ is considered. Initially a ball moves left on a racket. It is supposed that control force is applied to a racket, bounded on its value and parameterized by the racket rigidity. For the ball-racket system the motion equations are valid:

$$\begin{align*}
\ddot{z}(t) &= F_c \mathbf{I}_c \{C\}, \\
M\ddot{y}(t) &= -F_c \mathbf{I}_c \{C\} + F_{\text{cont}},
\end{align*}$$

(1)

where $z, y$ are coordinates of a ball and a racket, respectively, $\mathbf{I}_c \{C\}$ is an indicator of the impact phase, $\{C\}$ is a contact set and $F_{\text{cont}} = \sqrt{\mathcal{M}}u(t\sqrt{\mathcal{M}})$ is a control force. The visco-elastic force which is acting on a ball is given by the expression

$$F_c = 2\sqrt{\mathcal{M}}k(y - \dot{z}) + \mu a(y - z),$$

here $k, a$ are coefficients of visco and elastic properties of a racket. It is assumed that the restriction $|u(t\sqrt{\mathcal{M}})| \leq \mathcal{v}_0$ is tied to the control force. The following time-space coordinates transformation $x_1 = \sqrt{\mathcal{M}}z, x_2 = \dot{z}, x_3 = \sqrt{\mathcal{M}}y, x_4 = \dot{y}, s = \sqrt{\mathcal{M}}t$ reduces the motion equations (1) to the form

$$\begin{align*}
\dot{x}_1(s) &= x_2, \\
\dot{x}_2(s) &= (a(x_3 - x_1) + 2k(x_1 - x_2))\mathbf{I}_c \{C\}, \\
\dot{x}_3(s) &= x_4, \\
\dot{x}_4(s) &= -\frac{(a(x_3 - x_1) + 2k(x_1 - x_2))\mathbf{I}_c \{C\} + u(s)}{M},
\end{align*}$$

(2)

2.2 Space Transformation

Linear system (2) can be compactly rewritten (hereinafter the tensor designation is used and the summation executes on recurring index) as follows

$$\dot{x}_i = \hat{\mathbf{A}}_{ij} x_j + \hat{\mathbf{B}}_i u_i, \quad i, j = 1, 4,$$

(3)

$$\hat{\mathbf{A}}_{ij} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-a & -2k & a & 2k \\
0 & 0 & 0 & 1 \\
\frac{a}{M} & \frac{2k}{M} & \frac{a}{M} & \frac{2k}{M}
\end{pmatrix}, \quad \hat{\mathbf{B}}_i = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}.$$
with the initial condition \( x_i(0) = (0, V_0, 0, 0)^T \). The moment of the termination of interaction is defined by the expression:

- for physical condition \( \tau = \inf\{ s > 0 : a(x_3 - x_1) + 2k(x_4 - x_2) = 0 \} \);
- for geometric condition \( \tau = \inf\{ s > 0 : x_3 - x_1 = 0 \} \).

Claim 1. The contact set is \( C = \{ s : s \leq \tau \} \). The general type of the moment of the termination of interaction is

\[
\tau = \inf\{ s > 0 : \tilde{D}_i x_i = 0 \}.
\] (4)

So the controlled impact in the transformed time \( s \) lasts on the time interval \( s \in [0, \tau] \).

It is assumed the initial velocity of the ball is great enough to supply the ball reflection off a racket. The optimal control problem is concluded to minimize after impact velocity of the ball. The optimal control criteria takes the form

\[
J = \min_{x(\tau)} (-x_2(\tau) + V_0) = \min_u \int_0^\tau (a(x_1 - x_3) + 2k(x_2 - x_4))ds = \min_u \int_0^\tau \tilde{C}_i x_i ds,
\] (5)

where \( \tilde{C}_i = (a, 2k, -a, -2k)^T \). The variables substitution

\[
\xi_i = R_{ij} x_j, \quad R_{ij} = \begin{pmatrix}
\frac{M}{M+1} & 0 & -\frac{M}{M+1} & 0 \\
0 & \frac{M}{M+1} & 0 & -\frac{M}{M+1} \\
1 & 0 & \frac{M}{M+1} & 0 \\
0 & 1 & 0 & \frac{M}{M+1}
\end{pmatrix}
\]

brings equations (3) to the form

\[
\dot{\xi}_i = \tilde{A}_{ij} \xi_j + \tilde{B}_i u; \quad i, j = \overline{1, T},
\] (6)

where

\[
\tilde{A}_{ij} = R_{il} \tilde{A}_{lm} R_{mj}^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\frac{M+1}{M} & -2k\frac{M+1}{M} & 0 & 0 \\
0 & 0 & \frac{M}{M+1} & 0 \\
0 & 0 & 0 & \frac{M}{M+1}
\end{pmatrix},
\]

\[
\tilde{B}_i = R_{ij} \tilde{B}_j = \left(0, -\frac{1}{M+1}, 0, \frac{1}{M+1}\right)^T, \quad \text{and initial conditions are set by the vector } \xi_i(0) = \left(0, V_0 \frac{M}{M+1}, 0, V_0 \frac{M}{M+1}\right)^T.
\]

In new notations the optimal control criteria (5) is given by the expression

\[
J = \min_u \int_0^\tau \tilde{C}_i \xi_i ds,
\] (7)

where \( \tilde{C}_i = \tilde{C}_j R_{ji}^{-1}, \quad \tilde{C}_i = \left(a \frac{M+1}{M}, 2k \frac{M+1}{M}, 0, 0\right)^T \).

The moment of the termination of interaction now is defined as follows

\[
\tau = \inf\{ s > 0 : \tilde{D}_i \xi_i = 0 \}, \quad \tilde{D}_i = \tilde{D}_j R_{ji}^{-1},
\] (8)

where for geometric condition \( \tilde{D}_i = \left(-\frac{M+1}{M}, 0, 0, 0\right)^T \), for physical condition \( \tilde{D}_i = \left(-a \frac{M+1}{M}, -2k \frac{M+1}{M}, 0, 0\right)^T \).

3. STATEMENT OF OPTIMAL CONTROL PROBLEM

The transformations from section 2 allow to state optimal control problem. Now the motion equations are given by the system of differential equations

\[
\dot{\xi}_i = \tilde{A}_{ij} \xi_j + \tilde{B}_i u; \quad i, j = \overline{1, T}; \quad \xi_i \in \mathbb{R}^4
\] (9)

with matrix \( \tilde{A}_{ij} \) and vector \( \tilde{B}_i \) from equation (6), the initial conditions \( \xi_i(0) \) and the restriction on the control force \( u \in [-u_0, u_0] \subset \mathbb{R} \). The full time of the system control \( \tau \) is defined from the condition

\[
\tau = \inf\{ s > 0 : \tilde{D}_i \xi_i = 0 \}.
\] (10)

It is needed to find the minimum of the next criteria

\[
J = \min_u \int_0^\tau \tilde{C}_i \xi_i ds.
\] (11)

The Lagrangian for the optimal control problem is written in the form

\[
\mathcal{L} = \int_0^\tau \left( p_i (\dot{\xi}_i - \tilde{A}_{ij} \xi_j - \tilde{B}_i u) + \lambda \tilde{C}_i \xi_i \right) ds + \lambda_0 \xi_i(0) + \lambda_5 \tilde{D}_i \xi_i(\tau),
\] (12)

where \( \lambda_0 \geq 0, \lambda_i \) at \( i = \overline{1, T} \), \( \lambda_5 \) are some constant values, \( p_i \) are some functions of the time.

Pontryagin’s function is equal to

\[
H(p, \xi, u) = p_i (\tilde{A}_{ij} \xi_j + \tilde{B}_i u) - \lambda_0 \tilde{C}_i \xi_i.
\] (13)

Theorem 2. For the optimal control problem (9)–(11) with Lagrangian (12) the Pontryagin’s maximum principle is valid. If \( (\xi_{opt}(\cdot), u_{opt}(\cdot), \tau) \) supplies a minimum to Lagrangian (12), then necessarily

- Euler’s equations on \( \xi \), which give the conjugate system:

\[
\dot{p}_i + p_i \tilde{A}_{ij} - \lambda_0 \tilde{C}_j = 0.
\] (14)

- the transversality conditions at the moments 0 and \( \tau \):

\[
p_i(0) = \lambda_i, \quad \lambda_5 \tilde{D}_i + p_i(\tau) = 0;
\] (15)

- the minimum condition on \( u \):

\[
\min_{u \in [-u_0, u_0]} -p_i \tilde{B}_i u;
\] (16)

- the stationarity condition at the moment \( \tau \):

\[
(\lambda_0 \tilde{C}_j + \lambda_5 \tilde{D}_i \tilde{A}_{ij} \xi_j(\tau) + \lambda_5 \tilde{D}_i \tilde{B}_i u(\tau) = 0
\] (17)

are fulfilled.

See Arutunov et al. [2006] for the proof of the theorem.
4. SOLUTION OF OPTIMAL CONTROL PROBLEM

It is found from the conditions (15) and (17), that \( H(p(\tau), \zeta(\tau), u(\tau)) = 0 \). As the Pontryagin’s function derivative on an optimal trajectory equals to zero, the equality \( H = 0 \) is fulfilled identically on it. Due to the solution of conjugate system (14), the conditions (15) and the fact \( C_3 = C_4 = D_3 = D_4 = 0 \), it is obtained \( p_2(s) = p_4(s) = 0 \) at \( s \in [0, \tau] \). Therefore only first two equations of direct (9) and conjugate (14) systems should be taken into account while solving of the optimal control problem. So in the formulas (9)-(17) it is assumed that \( i, j = 1, 2 \).

Claim 3. It is supposed than in the solution of equations (9) and (17) oscillating components are presented.

Hence the new designations are entered \( \lambda = k \frac{M + 1}{M} \), \( \omega^2 + \lambda^2 = a \frac{M + 1}{M} \).

4.1 Solution of Optimal Control Problem with Physical Condition of Termination of Interaction

There are two cases \( \lambda_5 = 0 \) and \( \lambda_5 \neq 0 \).

Let \( \lambda_5 = 0 \). From the transversality condition (15) at the moment \( \tau \) follows, that \( p_1(\tau) = p_2(\tau) = 0 \). The stationarity condition (17) is fulfilled automatically. The solution of the conjugate system, satisfying these conditions, is

\[
\begin{align*}
p_1(s) &= \lambda e^{(s - \lambda) \sin(\omega(\xi + \nu)) + \cos(\omega(s - \nu)) - 1}.
\end{align*}
\]

The constant \( \lambda_0 \) is defined by the equality \( \lambda = \omega \). The moment of control switching \( s_p \) is found from the condition \( p_2(s_p) = 0 \) in the equations (18). For this purpose it is necessary to solve the equation

\[
e^{\lambda(s_p - \tau)} \left( \frac{\lambda}{\omega} \sin(\omega(s_p - \tau)) + \cos(\omega(s_p - \tau)) \right) = 1 = 0.
\]

Rewrite it in the form

\[
e^{\lambda(s_p - \tau)} \sin(\omega(s_p - \tau) + \psi) = \sin(\psi), \quad \psi = \arctan \frac{\omega}{\lambda}.
\]

So \( s_p - \tau < 0 \), the solution of the last equation can exist, when \( (s_p - \tau + \psi) \in [-2\pi, -\pi] \) or \( (s_p - \tau) \in [\pi + \psi, 2\pi + \psi] \). Therefore the control switching does not occur on time interval \( s \in [0, \tau] \). The sign \( p_2(s) \) is calculated on this interval and it is found that \( u_{opt}(s) = -\text{sign}p_2(s)u_0 = u_0 \) at \( s \in [0, \tau] \).

If the inequality \( \lambda_5 \neq 0 \) is true, then \( p_1(\tau) = \omega^2 + \lambda^2 \).

It follows from conditions \( H = 0 \) and \( \xi_i(\xi) = 0 \), that \( p_1(\tau)\xi_2(\tau) + p_2(\bar{D}_2)\xi_2(\tau) = 0 \). So at the moment \( \tau \) next coordinate equalities are fulfilled

\[
\xi_1(\tau) = -\frac{4\lambda^2}{(\omega^2 + \lambda^2)^2} \frac{u(\tau)}{M + 1},
\]

\[
\xi_2(\tau) = \frac{2\lambda}{\omega^2 + \lambda^2} \frac{u(\tau)}{M + 1}.
\]

As stated above \( \xi_2(\tau) < 0 \), \( u(s) = -\text{sign}p_2(s)u_0 \) and hence \( u(\tau) = -u_0 \). The expressions (19) take the form

\[
\xi_1(\tau) = \frac{4\lambda^2}{(\omega^2 + \lambda^2)^2} \frac{u_0}{M + 1},
\]

\[
\xi_2(\tau) = -\frac{2\lambda}{\omega^2 + \lambda^2} \frac{u_0}{M + 1}.
\]

Corollary 4. The coordinates equalities (20) are the conditions of the ball stop.

4.2 Main Results

During the solving of the optimal control problem with physical condition of the termination of interaction the following results are obtained.

- The optimal control law is \( u_{opt}(s) = u_0 \) at \( s \in [0, \tau] \).
- There is no control switching here. The direction of the control force must always coincide with the direction of the initial velocity of a ball.
- The expressions (20) cause the conditions of the ball stop. If the initial velocity of a ball allows it to be stopped then the optimal control problem with another optimal criteria can be stated, for example, with the minimal time criteria of the ball stop.

The solving of the optimal control problem with physical condition of the termination of interaction shows how to solve similar problem with another vector \( D_1 \). Also it seems interesting to compare the values \( \tau \) and \( x_i(\tau) \) for different vectors \( D_1 \).

5. PARADOX PHENOMENON IN GEOMETRIC CONDITION AS OPTIMAL CONTROL PROBLEM

In this section the ability of ball moving in the right direction under the zero initial conditions is considered. The geometric condition of the termination of interaction is chosen.

5.1 Problem Statement and Coordinates Transformation

Again the motion equations are given by

\[
\xi_i = \bar{A}_{ij}\xi_j + \bar{B}_{iu}, \quad i, j = 1, 4, \quad \xi_i \in \mathbb{R}_4, \quad u \in [-u_0, u_0],
\]

with the initial conditions \( \xi_i(0) = (0; 0; 0; 0) \). Now the control time \( \tau \) is fixed. It is necessary to find the minimum of the next criteria

\[
J = \min_u \int_0^\tau \bar{C}_i\xi_i ds \quad \text{under the restriction} \quad \bar{D}_i\xi_i \leq 0,
\]

where \( \bar{D}_i^T = \left( -\frac{M + 1}{M}; 0; 0; 0 \right)^T \). Next statement is formulated to reach the required effect of a ball motion.

Claim 5. It is supposed that in the solution of equations (21) oscillating components are absent.

The new designations \( \lambda = k \frac{M + 1}{M} \). \( \omega^2 + \lambda^2 = a \frac{M + 1}{M} \) are entered. At first the coordinates transformation \( \eta_i = Q_{ij}\xi_j \) is fulfilled, where
The motion equations (21) is rewritten as
\[ \eta_i = A_{ij}\eta_j + B_{iu}, \quad i, j = 1, 4, \]  
(23)
where
\[ A_{ij} = Q_{ij1}Q^{-1}_{m1}, \quad B_{i} = \frac{1}{M+1} \left( \frac{1}{1+\lambda}\eta_i, 0, -1 \right)^T. \]

The initial conditions aren’t changed and are defined by the vector \( \eta_i(0) = (0, 0, 0, 0) \). At the new coordinates the optimal control criteria takes the form
\[ J = \min_u \int_0^\tau \overline{C}_i\xi ds, \quad \overline{C}_i = \frac{\lambda^2 - \omega^2}{2\omega} (\lambda + \omega, \omega - \lambda, 0, 0)^T. \]  
(24)

The restriction is \( \overline{D}_i \eta_i \leq 0, \overline{D}_i = \bar{D}_i J_{ij1}, \) where \( \overline{D}_i = -\frac{M+1}{2M\omega} (\omega - \lambda, \lambda + \omega, 0, 0)^T \). As two variables \( \eta_1, \eta_4 \) don’t enter the criteria and the restriction, two last equations in the system (23) don’t depend on variables \( \eta_1, \eta_2 \) and initial conditions are equal to zero, then the variables \( \eta_1, \eta_4 \) can not be taken into account during the problem solving. The problem is reformulated as follows.

### 5.2 Optimal Control Problem

Now the motion equations are
\[ \eta_i = A_{ij}\eta_j + B_{iu}, \quad i, j = 1, 2, \]  
(25)
where
\[ A_{ij} = \left( \begin{array}{cccc} -\lambda - \omega & 0 & 0 & 0 \\ 0 & \omega - \lambda & 0 & 0 \end{array} \right), \quad B_{i} = -\frac{1}{M+1} \left( \frac{1}{\lambda - \omega}, \frac{1}{\lambda + \omega} \right)^T \]
with the initial conditions \( \eta_i(0) = (0, 0) \). The control time \( \tau \) is fixed. It is necessary to find the minimum of the next criteria
\[ J = \min_u \int_0^\tau C_i\eta_i ds \]  
under the restriction \( D_i \eta_i \leq 0, \]  
(26)
where vectors \( C_i = \frac{\lambda^2 - \omega^2}{2\omega} (\lambda + \omega, \omega - \lambda)^T, \) \( D_i = -\frac{M+1}{2M\omega} (\omega - \lambda, \lambda + \omega)^T \).

The Lagrangian for the optimal control problem is written in the form
\[ \mathcal{L} = \int_0^\tau (p_i(\eta_i - A_{ij}\xi_j - B_iu) + \lambda_0 C_i \eta_i) ds + \int_0^\tau D_i \eta_i dp(s) + \lambda_i \eta_i(0), \]  
(27)
where \( \lambda_i \geq 0, \lambda_i \) at \( i = 1, 2 \) are some constant values, \( p_i(s) \) are some functions of the time, \( \mu(s) \) is non negative Borel measure, concentrated on the set \( \{ s \in [0, \tau] | D_i \eta_i(s) = 0 \} \), all of them are not equal to zero simultaneously. Pontryagin’s function is
\[ H(p, \eta, u) = p_i(A_{ij}\xi_j + B_iu) - \lambda_0 C_i \eta_i. \]  
(28)

### Theorem 6

For the optimal control problem (25)–(26) with Lagrangian (27) the Pontryagin’s maximum principle is valid. If \( (\eta_{opt}(\cdot), u_{opt}(\cdot)) \) supplies a minimum to the Lagrangian (27), then necessarily

- the Euler’s equations:
  \[ p_i(s) = \int_0^\tau (p_i(s)A_{ij}) ds - \int_0^\tau D_j dp(s); \]  
  (29)
- the transversality conditions:
  \[ p_i(0) = \lambda_1; \]  
  (30)
- the minimum condition on \( u \):
  \[ \min_{u \in [-u_0, u_0]} -p_iB_iu_w \]  
  (31)
- the condition of the Pontryagin’s function constancy:
  \[ H(p(s), \eta_{opt}(s), u_{opt}(s)) = \text{const}, \quad s \in [0, \tau]. \]  
(32)

The proof of the theorem is arranged in Arutunov et al. [2006].

### 5.3 Double Impulse Control on Restriction Border

Further it is shown that the opportunity of reaching the value \( J(\tau) < 0 \) exists as a result of control by the system (25). General solution of these equations at zero initial conditions is defined by the formulas:
\[ \eta_1(s) = -\frac{\exp(-(\lambda + \omega)s)}{(M+1)(\lambda - \omega)} \int_0^s u(\zeta) \exp((\lambda + \omega)\zeta) d\zeta, \]  
(33)
\[ \eta_2(s) = -\frac{\exp(-(\lambda - \omega)s)}{(M+1)(\lambda + \omega)} \int_0^s u(\zeta) \exp((\lambda - \omega)\zeta) d\zeta. \]

It is supposed that the control has a time structure
\[ u(s) = \begin{cases} -u_1, & s \in [0, s_1], \\ u_2, & s \in [s_1, s_2], \end{cases} \]  
(34)
where \( s_1, s_2 \) are some moment of the time. So it is assumed that \( \Delta s_1 = s_1 - s_0, \Delta s_2 = s_2 - s_0 \) are sufficient small time intervals. The equality \( \overline{D}_i \eta_i(s_2) = 0 \) is fulfilled at the moment \( s_2 \), but on the interval \( s \in (0, s_2) the strict inequality \( \overline{D}_i \eta_i(s) < 0 \) is true. The equation bounding time intervals \( \Delta s_1, \Delta s_2 \) and controls \( u_1, u_2 \) is gained by integrating the expressions (33) with control law (34) and using the above assumptions:
\[ u_1 \Delta s_1^2 = u_2 \Delta s_2^2. \]  
(35)

The value of the optimization criteria, calculated from definition (26) by using (35), is equal approximately
\[ J \approx -\frac{u_1 s_1^2}{6(M+1)} (s_1(\omega^2 + 3\lambda^2) - 12\lambda \sqrt{\frac{u_1}{u_2}}). \]  
(36)
Therefore if in the formula (36) takes place
\[ s_1 > \frac{12\lambda}{\omega^2 + 3\lambda^2} \sqrt{\frac{u_1}{u_2}}, \] then \( J < 0. \) (37)

The inequality \( J < 0 \) means, that a ball begins to move to the right being carried away by a racket placed right from it.

6. CONCLUSION

This article is revealing several tasks solving of which is in the progress. From the author’s point of view the most important problems which should be mentioned are:

- a proof the sufficient conditions for the optimal control problem (9)–(12) for all conditions of the termination of interaction;
- a solution of optimal control problem (25)–(27), that concerns to the class of tasks with phase restrictions (It is needed to find a measure \( \mu(s) \) here);
- a calculation of a maximum effect of the application of the double impulse control on a restriction border and propagation this result to the repeating multiple action of double impulse control.

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REFERENCES


V. Kozlov, D. Treshev. *Billiards. (Genetic Introduction into Dynamics of Systems with Impacts).* Moscow University, Moscow, 1991.


