Multi-Stage Optimization Approach for Polymer Flooding Optimal Control Problems

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Abstract: Polymer flooding is an important tertiary recovery technology. The optimization of polymer injection can be formulated as an optimal control problem. Since the polymer injection has special slug structure, the size of slug and injection concentration must be optimized, simultaneously. Thus in this paper, a multi-stage optimization approach based on the control vector parameterization (CVP) is proposed for solving this optimal control problem. A piecewise-constant approximation is used to describe the injection structure in which each constant piece represents an injection slug, and the optimization problem can be solved as a nonlinear programming (NLP) problem. For illustration, an experiment example is solved by using the proposed approach.

Keywords: optimal control; nonlinear programming; optimization problems.

1. INTRODUCTION

Petroleum is an important energy, but is non-renewable. After many years of water flooding recovery, most old oil fields have been in high water cut stage in China, but a large amount of oil is still retained in the reservoir. So it is necessary to apply the tertiary recovery technologies to enhance the final oil recovery. Polymer flooding is an important tertiary recovery technology and is most wildly used in China. Since the polymer is expensive, it is necessary to plan a reasonable injection strategy, which can enhance the oil recovery and control the polymer cost. The common method for injection strategy planning is based on reservoir simulators. In this simulator based method, several feasible injection strategies must be selected first, then by running the simulator, the performance of these strategies can be evaluated and the best one is chosen as the "optimal" injection strategy. The performance of this simulator based method is heavily dependent on the original selected strategies and the optimized result is poor.

The use of optimal control to plan the oil extraction was first discussed by Lee and Aronofsky (1958). Then the optimal control theory was applied to solve the water flooding optimization problems (Kraaijevanger et al., 2007; Brouwer and Jansen, 2004; Masoud and Geir, 2009). Ramirez (1987) solved the chemical flooding problems using the optimal control method. The optimal control methods for reservoir recovery can be mainly classified into two groups: gradient based methods and iterative dynamic programming (IDP) methods. Effati et al. (2006) applied the IDP method to solve the water flooding optimization problems. Guo et al. (2009) realized the polymer flooding optimization using the parallel IDP method. Compared with the gradient based methods, IDP does not need the gradient calculation, but its computation is very huge. Thus the gradient based methods are the most common types used in reservoir recovery optimization (Masoud and Geir, 2009; Li et al., 2008; Zhang et al., 2010).

Limited to the polymer flooding technological conditions, in actual applications the polymer is injected in segmented forms. So optimization of the segment size and injection concentration simultaneously is necessary. Schlegel and Marquardt (2006) proposed a multi-stage optimization approach to solve the multi-stage dynamic optimization problems in the chemical process systems. In this multi-stage approach, each stage corresponded to a specific control arc, and the control parameters and switching times acted as decision variables.

In this paper, considering the reality of polymer segmented injection, we propose a multi-stage optimization approach for solving the polymer flooding optimal control problems. According to the given number of injection slugs, the time horizon is partitioned into several subintervals by using the control vector parameterization (CVP) method. And the original optimal control is reformulated as a multi-stage optimization problem. In order to avoid the influence of variations of the stage length on numerical precision, a time scale factor is introduced to do normalized treatment for the stage length. The numerical solutions are obtained by using the sequential quadratic programming (SQP) method.

2. PROBLEM FORMULATION

2.1 The Optimization Problem

The mathematical model of polymer flooding is a distributed parameter system. We use a conventional finite
difference approximation to describe the reservoir and obtain the continuous time form of the model (for more details, see Aziz and Settari, 1986). The polymer flooding optimal control problem can be described as the following ordinary form.

\[
\begin{aligned}
\min_u J &= \int_{t_0}^{t_f} \varphi(x, u, t) dt \\
\text{s.t.} & \quad \dot{x}(t) = f(x, u, t), x(0) = x_0, \\
& \quad g(x(t_f)) \leq 0, \\
& \quad LB \leq u(t) \leq UB,
\end{aligned}
\]

where \( x \) denotes the state vector which includes reservoir press, water saturation, reservoir polymer concentration, and polymer cost per unit time, \( u \) denotes the injection strategy needed to be optimized, \( LB, UB \) are the lower and upper bounds of control \( u \), \( g(x(t_f)) \) denotes the total polymer loading constraint. Since the polymer loading can be obtained by solving the system dynamic model, \( g(x(t_f)) \) is written here in a fully implicit form. \( J \) is the objective function of the optimization problem (1), i.e., minimum of the cost, maximum of the cumulative oil product (COP), or maximum of the net present value (NPV).

The aim of the polymer flooding optimal control problem is to find a feasible polymer injection to maximize (or minimize) the given performance objective under the conditions of obeying system dynamic and given constraints. Usually the optimized control of problem (1) may be continuous, but in actual oil recovery, polymer can only be injected in slug structures. The design of injection slug mainly includes the calculations of the slug sizes and injection concentrations.

**Remark 1.** Since the injection velocity of polymer solution is fixed in practical operation and the reservoir pore volume is a constant, the ratio of injection solution volume and reservoir pore volume can actually indicate the injection time, and its unit is PV.

So in this paper, we use PV as the unit of time. A typical three-stage slug injection strategy is shown in Fig. 1.

![Fig. 1. Typical polymer three-stage slug injection strategy](image)

2.2 Necessary Conditions of Optimality

According to the Pontryagin maximal principle (Pontryagin et al., 1962), the first order necessary optimality conditions for the optimal control problem (1) can be obtained. The Hamiltonian is defined as

\[
H(x, \lambda, u, t) = \varphi(x, u, t) + \lambda^T f(x, u) \tag{2}
\]

where \( \lambda \) is called adjoint vector which has the same dimension as the state vector \( x \).

Assume \( u^* \) is the feasible optimal control with system response \( x^* \). Then there exists an adjoint state \( \lambda^* \) such that in time horizon \([t_0, t_f]\) it holds

\[
\dot{x}^*(t) = f(x^*, u^*), \quad x^*(0) = x_0, 
\]

\[
\dot{\lambda}^*(t) = -\frac{\partial \varphi^*}{\partial x} - \lambda^* \frac{\partial f^*}{\partial x}, \quad \lambda^*(t_f) = \frac{\partial \mu^* g(x^*(t_f))}{\partial x}, \tag{3b}
\]

\[
H(x^*, \lambda^*, u^*, t) \leq H(x^*, \lambda^*, u, t), \tag{3c}
\]

\[
\mu^* g(x^*(t_f)) = 0, \tag{3d}
\]

\[
LB \leq u^*(t) \leq UB, \tag{3e}
\]

where \( \mu^* \) is a Lagrange multiplier associated with the final time constraint and satisfies

\[
\mu^* \begin{cases} = 0, & g^*(x(t_f)) < 0, \\ > 0, & g^*(x(t_f)) = 0. \end{cases} \tag{3f}
\]

If the final time \( t_f \) is free, an additional optimality condition also must be satisfied,

\[
H^*|_{t_f} = 0. \tag{3f}
\]

Equations (3a)-(3f) define a two-point boundary value problem which is very difficult to solve.

3. SOLUTION APPROACH

Direct optimization methods have been proved to be powerful tools for solving optimal control problems and they have been studied extensively in the last 20 years. Control vector parameterization (CVP) (Goh and Teo, 1988) is the common direct optimization method for solving the optimal control problems. The basic idea of CVP method is to convert the optimal control problem into a nonlinear programming (NLP) problem by replacing the control vector with a set of piecewise polynomials.

In this paper, the B-spline functions are used to approximate the control vector. The approximate control \( u_p \) acting on the time interval \([t_0, t_f]\) can be formulated as

\[
u_p(t) = \sum_{k=1}^{N} \hat{u}_k v_k^{(m)}(t), \tag{4}
\]

where \( \hat{u}_k \) is a scalar value and denotes the corresponding decision parameter, \( N \) denotes the number of decision parameter, \( m = 1, 2, \ldots \).
ψ_k(m)(t) := \begin{cases} 1 & \text{if } t_k \leq t < t_{k+1} \\ 0 & \text{else} \end{cases},
ψ_k(m−1)(t)
\sum_{k=m}^{m−1} ψ_k(m−1)(t), \quad m > 1,
where \( m = 1 \) denotes the piecewise constant approximation and \( m = 2 \) denotes the piecewise linear approximation.

3.1 Multi-stage Optimization Approach

Considering the slug structure of polymer flooding, based on the CVP method a multi-stage optimization approach is proposed to optimize the injection concentration and slug size, simultaneously.

For convenience, we introduce a new state variable \( z \) and convert the problem (1) into its Mayer type. The new state \( z \) satisfies
\[
\dot{z} = \varphi(x, u, t), \quad z(0) = 0.
\]
Define \( y = [x, z]^T \) and \( F = [f, \varphi]^T \), the Mayer type optimal control problem can be formulated as
\[
\begin{align*}
\min J &= z_f \\
\text{s.t. } \dot{y}(t) &= F(y, u, t), y(0) = y_0, \\
g(y(t_f)) &\leq 0, \\
LB &\leq u(t) \leq UB.
\end{align*}
\]
According to the polymer injection strategy (i.e., single-stage slug, two-stage slug, and three-stage slug), the polymer injection process is partitioned in time domain. If the injection strategy is selected as three-stage slug, the injection process should be partitioned into four pieces. Piecewise constant approximate method is used to realize the injection strategy.

From (4), we can know that the control parameter \( \hat{u}_k \) is independent with time. We partition the optimal control problem (7) into a following N-stage approximate problem with each stage corresponding to a particular injection slug.

\[
\begin{align*}
\min J &= z_f \\
\text{s.t. } \dot{y}_k(t) &= F(y_k(t), \hat{u}_k(t_i)), t \in [t_{k-1}, t_k], \quad \forall k \in \mathbb{Z}, \\
y_k(t_0) &= y_0, \quad y_k(t_{k-1}) = y_{k-1}(t_{k-1}), \quad \forall k \in \mathbb{Z}\setminus\{1\}, \\
g(y_k(t_f)) &\leq 0, \\
LB &\leq \hat{u}_k \leq UB, \quad \forall k \in \mathbb{Z}, \\
0 &\leq t_0 < t_1 < \cdots < t_N = t_f,
\end{align*}
\]
where set \( \mathbb{Z} = \{1, 2, \ldots, N\} \) includes the index of all stage, \( N \) denotes the number of stage. \( \hat{u}_k, t_k \) are the decision variables of stage \( k \).

Remark 2. If \((\hat{u}_k^*, t_k^*)\) is a solution of problem (8), then \( u_p(\hat{u}_k^*, t_k^*) \) is a suboptimal control of original problem (1).

For the approximate optimal control problem (8), the gradient-based optimization algorithms are not effective because of the difficulty for optimizing the stage lengths. Thus the time scaling transformation is applied to map these variable lengths into a set of fixed lengths on the normalized time horizon.

Now we introduce the normalized time variable \( \tau \) and do normalized treatment for the stage lengths. At stage \( k \), \( t \in [t_{k-1}, t_k] \). We define a new time variable \( \tau_k \) and it satisfies \( \tau_k = \frac{t-t_{k-1}}{t_k-t_{k-1}} \) (i.e. \( \tau_k \in [0, 1] \)). Then we can obtain the derivative of state \( y_k \) w.r.t. \( \tau_k \) described as follows
\[
\dot{y}_k(\tau_k) = (t_k-t_{k-1})F(y_k(\tau_k), \hat{u}_k, (t_k-t_{k-1})\tau_k + t_{k-1}).
\]
The final time objective function can be rewrite as
\[
J = z_N(1).
\]
For convenience of inferring, we use the stage length \( \Delta t_k \) as optimization variable instead of the switching time \( t_k \), and it satisfies \( \Delta t_k = t_k - t_{k-1} \).

Using(8), (9), and (10), we obtain the normalized multi-stage optimization problem.
\[
\begin{align*}
\min & \quad J = z_N(1) \\
\text{s.t. } & \quad \dot{y}_k(\tau_k) = \Delta t_k F(y_k(\tau_k), \hat{u}_k, \Delta t_k \tau_k + \epsilon), \quad \tau_k \in [0, 1], \\
& \quad y_k(0) = y_0, \\
& \quad \sum_{k=1}^N \Delta t_k = t_f - t_0,
\end{align*}
\]
where \( \epsilon = \{0, k = 1, \sum_{i=2}^{k-1} \Delta t_i, k \in \mathbb{Z}\setminus\{1\} \} \).

3.2 Gradient Formulas

Based on the CVP method, we convert the optimal control problem into a NLP problem. Besides the objective and constraint function values, it also needs their gradients w.r.t. the decision variables to efficiently solve the NLP problem. In this section based on the Pontryagin maximal principle, we construct multi-stage adjoint systems to calculate the gradients required by the NLP solver.

First, we do not consider the influence of final time constraint to optimization problem and derive the gradients of objective function w.r.t decision variables.

Define the Hamiltonian function for objective function as
\[
H = \sum_{k=1}^N \lambda_k \Delta t_k F(y_k(\tau_k), \hat{u}_k, \Delta t_k \tau_k + \epsilon).
\]
The adjoint equation for objective function in stage $k$ is
\[
\dot{\lambda}_k = -\lambda^T_k \Delta t_k \frac{\partial F}{\partial y_k}, \quad \forall k \in \Xi.
\] (13)

The transversality condition for objective function in the last stage $N$ is
\[
\lambda_N(1) = \frac{\partial z_N(1)}{\partial y_N}.
\] (14)

The junction condition between stage $k$ and its neighboring stage $k+1$ can be expressed as
\[
\lambda_k(1) = \lambda_{k+1}(0), \quad \forall k \in \Xi \setminus N.
\] (15)

The gradients of the objective function for the approximate problem (11) are furnished in the following theorem.

**Theorem 1.** Under the conditions of (13), (14), and (15), the gradient of objective function w.r.t $\hat{u}_k$ is expressed as
\[
\frac{dJ}{d \hat{u}_k} = \int_0^1 \left( \lambda^T_k \Delta t_k \frac{\partial F}{\partial u_k} \right) d\tau_k, \quad \forall k \in \Xi,
\] (16)
and the gradient of objective function w.r.t $\Delta t_k$ is
\[
\frac{dJ}{d \Delta t_k} = \int_0^1 \lambda^T_k \left( \sum_{i=k}^N \frac{\partial \Delta t_i F}{\partial \Delta t_k} \right) d\tau_k, \quad \forall k \in \Xi.
\] (17)

**Proof.** Calculate the variation of the objective function $J$ for problem (11).

\[
\delta J = \delta y^T_N \left( \frac{\partial z_N(1)}{\partial y_N} - \lambda_N(1) \right) + \sum_{k=1}^{N-1} \left( \delta y^T_{k+1}(0) (\lambda_{k+1}(0) - \lambda_k(1)) \right)
+ \sum_{k=1}^N \int_0^1 \left( \delta \hat{u}^T_k \left( \lambda^T_k \Delta t_k \frac{\partial \Delta t_i F}{\partial u_k} + \dot{\lambda}_k \right) + \delta \Delta t^T_k \left( \lambda^T_k \left( \sum_{i=k}^N \frac{\partial \Delta t_i F}{\partial \Delta t_k} \right) \right) \right) d\tau_k.
\] (18)

On the conditions of (13), (14), and (15), the first order variation (18) becomes
\[
\delta J = \sum_{k=1}^N \int_0^1 \left( \delta \hat{u}^T_k \left( \lambda^T_k \Delta t_k \frac{\partial \Delta t_i F}{\partial u_k} + \dot{\lambda}_k \right) + \delta \Delta t^T_k \left( \lambda^T_k \left( \sum_{i=k}^N \frac{\partial \Delta t_i F}{\partial \Delta t_k} \right) \right) \right) d\tau_k.
\] (19)

Since the decision variables $\hat{u}_k$ and $\Delta t_k$ are independent of each other and are time invariant in the time horizon, the gradients of the objective function satisfy (16) and (17).

If $F$ is a time invariant function, the gradient of objective function w.r.t segment size $\Delta t_k$ can be simplified as
\[
\frac{dJ}{d \Delta t_k} = \int_0^1 \lambda^T_k F d\tau_k, \quad \forall k \in \Xi.
\] (20)

Similarly, since the final time constraint function is constrained by the system dynamic model, the constraint function can be rewritten as the following augmented formation,
\[
g(\hat{y}_N(1)) = g(\hat{y}_N(1)) + \sum_{k=1}^N \int_0^1 \gamma_k^T (\Delta t_k F - \hat{y}_k) d\tau_k.
\] (21)

According to the maximum principle, the adjoint equation for the constraint function becomes
\[
\dot{\gamma}_k = -\gamma^T_k \Delta t_k \frac{\partial F}{\partial y_k}, \quad \forall k \in \Xi
\] (22)
for stage $k$, where
\[
\gamma_N(1) = \frac{\partial g(\hat{y}_N(1))}{\partial y_N},
\] (23)
\[
\gamma_k(1) = \gamma_{k+1}(0), \quad \forall k \in \Xi \setminus N.
\] (24)

**Theorem 2.** Under the conditions of (22), (23), and (24), the gradients of constraint function are
\[
\frac{dg(\hat{y}_N(1))}{d \hat{u}_k} = \int_0^1 \left( \gamma^T_k \Delta t_k \frac{\partial F}{\partial u_k} \right) d\tau_k, \forall k \in \Xi,
\] (25)
\[
\frac{dg(\hat{y}_N(1))}{d \Delta t_k} = \int_0^1 \gamma^T_k \left( \sum_{i=k}^N \frac{\partial \Delta t_i F}{\partial \Delta t_k} \right) d\tau_k, \forall k \in \Xi.
\] (26)

The proof of this theorem is similar to that of Theorem 2.

4. ALGORITHM IMPLEMENTATION

Based on the CVP method, the multi-stage optimization problem can be decomposed into two subproblems:

1. Initial value problem (IVP) of system and adjoint states.
2. The master subproblem namely NLP subproblem,
\[
\begin{array}{l}
\min_{\hat{u}_k, \Delta t_k, k \in \Xi} J = z_N(1)
\end{array}
\] (27)
\[
s.t. \quad g(\hat{y}_N(1)) \leq 0, \quad \sum_{k=1}^N \Delta t_k = t_f - t_0, \quad LB \leq \hat{u}_k \leq UB, \quad \Delta t_k > 0, \quad \forall k \in \Xi.
\]
run on a 2.60GHz Pentium Dual-Core E5300 processor, 2GB of RAM memory.

Algorithm 1. Step 1: \( i = 1 \). Initial state \( y_1(0) \), control \( u_{k,1} \), \( \Delta t_{k,1} \), \( k \in \Xi \), iterative tolerance \( \varepsilon > 0 \).

Step 2: Compute the values of objective and constraint functions and their gradients with respect to \( u_{k,i} \) and \( \Delta t_{k,i} \) at iteration \( i \).

Step 3: Solve the QP subproblem at iteration \( i \) and generate the new decision, \( u_{k,i+1} \) and \( \Delta t_{k,i+1} \). If
\[
\| J(\hat{u}_{k,i+1}, \Delta t_{k,i+1}) - J(\hat{u}_{k,i}, \Delta t_{k,i}) \| < \varepsilon,
\]
finish the algorithm, else \( i = i + 1 \), back to step 2 and begin a new iteration.

5. NUMERICAL EXAMPLE

Consider the following optimal control problem of polymer flooding one-dimensional laboratory experiment model (Ramirez, 1987).

\[
\begin{align*}
\text{max} & \quad J = \int_0^{t_f} (1 - f_w|z=1 - Ru) dt \\
\text{s.t.} & \quad \frac{\partial S_w}{\partial t} = -\frac{\partial f_w}{\partial x}, \quad \frac{\partial C}{\partial t} = -\frac{f_w}{D} \frac{\partial C}{\partial x} + \frac{1}{Pe} \frac{\partial^2 C}{\partial x^2}, \\
& \quad \int_0^{t_f} \mu \frac{w}{v_w} dt \leq Q_{\text{max}}, \quad C_{\text{min}} \leq u \leq C_{\text{max}}.
\end{align*}
\]

Initial conditions: \( C|_{t=0} = 0, \quad S_w|_{z=0} = 1 - S_{\text{or}} \).

Boundary conditions:
\[
\begin{align*}
& \frac{\partial C}{\partial x} |_{x=0} = Pe(C - u), \quad S_w|_{x=0} = 1 - S_{\text{or}}, \\
& f_w|_{z=0} = 1, \quad \frac{\partial C}{\partial z} |_{z=1} = 0.
\end{align*}
\]

where \( C \) is the mass concentration of polymer in core, \( S_w \) is the water saturation, \( u \) denotes the mass concentration percent of injected polymer solution and acts as control variable, \( f_w \) is water ratio and is a function of \( S_w \) and \( C \), \( t \) denotes the volume rate between injected solution volume and pore volume of core, \( x \) is a normalized length \( x \in [0, 1] \), \( v_w \) is flooding rate, \( Q_{\text{max}} \) is maximum polymer cost, \( C_{\text{min}}, C_{\text{max}} \) are the lower and upper bounds of the polymer concentration respectively, \( Pe \) is a constant. The detail formulation of \( f_w \), \( D \), and other parameter values can be found in Ramirez (1987).

According to polymer flooding recovery experience, the bounded constraints of polymer concentration are set to \( C_{\text{min}} = 0, C_{\text{max}} = 2.5g/L \) in this example. The uniform flooding rate is set as \( v_w = 0.0045 \). Final time \( t_f = 2 \). The iteration termination condition of the algorithm is chosen as \( \varepsilon = 10^{-8} \). The maximum NPV acts as optimization index. Single-stage slug, three-stage slug, and polymer loading constrained three-stage slug polymer flooding optimal control problems are discussed in the following sections, respectively.

5.1 Single-Stage Slug

Initial polymer concentration is set to 1.8 g/L, injection time 0.3PV, and initial NPV value \( J_0 = 0.1580 \). Based on the multi-stage optimization approach, only two variables, injection concentration and slug size of the single slug, need to be optimized. The optimized NPV value is \( J = 0.1618 \), and the number of iteration is 17, the iteration time is 17.19s. The optimal injection concentration is 1.37g/L, and slug size is 0.349PV. The injection strategy is shown in Fig. 2.

Fig. 2. Polymer injection strategy of single-stage slug optimization

5.2 Three-Stage Slug

Initial polymer concentration is set to 1.8 g/L, injection time 0.4PV, and initial NPV value \( J_0 = 0.1427 \). Using the multi-stage approach, we need to optimize six variables for the three-stage slug strategy. After 21 iterations, algorithm terminates and the related NPV value is \( J = 0.16279 \), the iteration time is 60.44s. The optimal injection concentrations are \([1.74, 1.08, 2.42]g/L\), and slug sizes are \([0.046, 0.225, 0.067]PV\). The injection strategy of polymer concentration is shown in Fig. 4.

In this example, we also consider the ordinary CVP approach for polymer flooding optimization problem. The control profile is divided into equal 20 pieces by using piecewise constant method. The iteration termination condition is chosen as \( \varepsilon = 10^{-8} \). After 27 iterations, algorithm terminates and the related NPV value is \( J = 0.16247 \), the iteration time is 347.26s.

5.3 Three-Stage Slug with Polymer Loading Constraint

Initial polymer concentration is set to 1.8 g/L, injection time 0.3PV, and initial NPV value \( J_0 = 0.1580 \). On the basis of three-stage slug optimization conditions, we add polymer loading constraint into the optimization problem. The polymer cost is \( Q_{\text{max}} = v_w \times 1.8 \times 0.3PV \). After
37 iterations, algorithm terminates and the related NPV value is $J = 0.16178$, the iteration time is 109.49s. The optimal injection concentrations are $[2.5, 1.03, 1.96]$g/L, and slug sizes are $[0.028, 0.289, 0.088]$PV.

6. CONCLUSIONS

Based on the proposed multi-stage optimization approach, polymer flooding optimal injection problem is discussed in this paper. According to the slug structure of polymer injection, the polymer flooding model is reformulated as a multi-stage system. In each stage, we only need optimize injection concentration and slug size. Thus we realize the polymer injection optimization by using the least decision variables. The optimization results of experiment example indicate that the multi-stage optimization approach has better optimized solution and faster solution speed than that of the ordinary CVP approach. Thus the proposed multi-stage optimization approach is effective in solving the polymer flooding optimal control problems.

REFERENCES


