$H_\infty$ filtering for networked systems with stochastic protocols

Hongbo Song ∗ Li Yu ∗ Guoping Liu ∗∗ Wen-An Zhang ∗
Defeng He ∗

Department of Automation, College of Information Engineering,
Zhejiang University of Technology, Hangzhou, 310032, P. R. China
School of Electronics, University of Glamorgan,
Pontypridd CF37 1DL, UK

Abstract: This paper is concerned with the problem of $H_\infty$ filtering for networked systems with stochastic protocols. The communication process between the sensor nodes and the filter is governed by stochastic protocols. The filtering error system is modeled as a stochastic system and a sufficient condition is presented for the stochastic stability and guaranteed $H_\infty$ performance of the filtering error system. Based on the criterion, a design procedure is also presented for the desired $H_\infty$ filter. Moreover, the relation between the access probabilities of the nodes and the $H_\infty$ filtering performance is established, which helps assign the access probabilities to achieve better $H_\infty$ filtering performance. An illustrative example is given to show the effectiveness of the proposed method.

1. INTRODUCTION

State estimation (or filtering) for networked systems has recently been a research topic that receives increasing attention in the control and signal processing communities, see for example, Hespanha et al. [2007], Schenato et al. [2007], Gao and Chen [2007], Yue and Han [2006], Smith and Seiler [2003], Sahebsara et al. [2007], Song et al. [2009] and the references therein. This is partly due to the broad applications of networks in modern industry field, including remote sensing, space exploration, sensor networks, and so on (Hespanha et al. [2007], Yang [2006], and Zhang et al. [2001]). As well known, networked systems are often large-scaled distributed systems and thus the data collected by sensors have to be transmitted via separated sensor nodes. In the filtering problem of networked systems, only one sensor node (may contain more than one sensor) is allowed to communicate with the central node (filter) at each transmission instant because only one communication channel is provided. If more than one node attempts to get access to the exclusive channel simultaneously, collision will occur and this means no successful transmissions are provided at that time. This is often called the medium access constraint or communication constraint in the existing literature and we use the term medium access constraint in this paper without causing confusion. Apparently medium access constraint has negative impact on the system performance and should be properly handled (Brockett [1995] and Zhang [2005]).

One of the important features that make networked systems differ from traditional systems is that not only controllers or filters, but also scheduling protocols are involved. The main motivation of introducing scheduling protocols in networked systems is to handle the medium access constraint issue. As well known, a scheduling protocol is designed to reduce or prevent collisions in a network and it can be seen as a rule that determines which node gets access to the exclusive channel at what times (Tabbara and Nesic [2008]). Generally speaking, deterministic and stochastic protocols are used in networked systems. Examples of the network using deterministic protocols are Token Ring (TR) and Control Area Network (CAN). Up to now, the study of networked control systems with deterministic protocols from various aspects has received extensive research attention, see for example, Brockett [1995], Hristu and Morgansen [1999], Rehbinder and Sanfridson [2004], Ishii [2008], Walsh and Ye [2001], Nesic and Teel [2004], Tabbara et al. [2007], Dacic and Nesic [2007], Dacic and Nesic [2008], Zhang [2005] and the references therein. On the other hand, a typical example of networked systems with stochastic protocols is Industrial Ethernet (IE), which has found promising applications in industry field (Zhang [2005], Tabbara and Nesic [2008], and Zhang et al. [2004]). In IE, carrier sense multiple access with collision detection (CSMA-CD) mechanism and binary exponential backoff algorithm are used (Tanenbaum [2003]). Thus, one of the nodes may get access to the network (or equivalently communicate with the central node) and collisions (meaning packet dropout) may occur at any transmission instant. Moreover, the process is random and this makes such networked systems represent new features. One interesting feature is that the access probabilities of the nodes stand for the network resource allocated to them. Therefore, networked systems with stochastic protocols is an interesting topic that deserves investigation. However, few existing results were concerned with the filtering problem of networked systems with stochastic protocols. In Gupta et al. [2006], a stochastic sensor selection scheme was proposed to estimate a process state by minimizing the upper bound on the estimation error covariance. However, the external noise was assumed to be Gaussian with known statistical...
In this paper, we investigate the $H_\infty$ filtering problem of networked systems with medium access constraint was studied. However, the employed protocol was deterministic. Thus we focus on the problem of $H_\infty$ filtering for networked systems with stochastic protocols in this study. Another interesting question associated with this problem is that how to assign the access probabilities to each node to achieve better $H_\infty$ filtering performance. We will show that one possible way is to establish the relation between the access probabilities and the $H_\infty$ filtering performance.

In this paper, we investigate the $H_\infty$ filtering problem of networked systems with stochastic protocols. The plant output is transmitted via separated sensor nodes and only one of them can communicate with the filter at each transmission instant. The communication process is governed by stochastic protocols. An independent and identically distributed (i.i.d) sequence is employed to model the communication process. Then the i.i.d sequence is mapped into a certain number of binary valued sequences, by which the filtering error system is modeled as a stochastic system. A sufficient condition is presented for the filtering error system to be stochastically stable with a prescribed $H_\infty$ performance. A filter design procedure is also presented based on the obtained condition. Moreover, the optimal $H_\infty$ filter design problem is casted into an optimization problem with linear matrix inequality (LMI) constraints, which can be efficiently solved by the existing LMI software. Based on the $H_\infty$ performance condition, the relation between the access probabilities and the $H_\infty$ filtering performance is established, which will help assign the access probabilities to achieve better $H_\infty$ filtering performance. An illustrative example is finally given to show the effectiveness of the proposed method.

2. MODELING OF THE FILTERING ERROR SYSTEM

Consider the networked filtering system shown in Fig. 1, where the plant is described by the following discrete-time state space model:

$$
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \\
    y(k) &= Cx(k) + Dw(k) \\
    z(k) &= Lx(k)
\end{align*}
$$

where $x \in \mathbb{R}^n$ is the plant state, $y \in \mathbb{R}^m$ is the plant output, $w \in \mathbb{R}^p$ is the disturbance input which belongs to $l_2[0, +\infty)$, and $z \in \mathbb{R}^l$ is the signal to be estimated. $A$, $B$, $C$, $D$, and $L$ are matrices with appropriate dimensions, and $A$ is assumed to be stable.

We are interested in the problem of estimating the signal $z$ via a filter which is connected to the plant by a network with stochastic protocols, and the considered filter is a full order one taking the following form:

$$
\begin{align*}
    x_f(k+1) &= A_f x_f(k) + B_f \hat{y}(k) \\
    z_f(k) &= C_f x_f(k)
\end{align*}
$$

where $x_f \in \mathbb{R}^n$ is the filter state, $\hat{y} \in \mathbb{R}^m$ is the filter input, and $z_f \in \mathbb{R}^l$ is the estimated signal. $A_f$, $B_f$, and $C_f$ are filter parameter matrices to be determined.

Without loss of generality, we assume that the plant output $y$ is transmitted to the filter via $m$ sensor nodes. Let $y = [y_1, y_2, \ldots, y_m]^T$ and $\hat{y} = [\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_m]^T$, and $y_i$ and $\hat{y}_i$, $i = 1, 2, \ldots, m$, are scalars. We denote the set consisting of the subscript of $y_j$, $j \in \{1, 2, \ldots, m\}$ which are transmitted via node $i$ by $N_i$, $i = 1, 2, \ldots, m$. To concentrate on the medium access constraint issue, it is assumed that the transmission delays are very small such that the effect can be neglected in this paper. Since the scheduling protocols employed here are stochastic ones, it can be seen that one of $\bar{m} + 1$ events randomly occur at any instants. These events are that node $i$, $i = 1, 2, \ldots, m$, communicates with the filter and none of the nodes successfully communicates with the filter due to collisions. An i.i.d sequence $\rho(k) \in M = \{0, 1, \ldots, \bar{m}\}$ is utilized to model the stochastic communication process, where $\rho(k) = i$, $i = 1, 2, \ldots, \bar{m}$, means that node $i$ communicates with the filter while $\rho(k) = 0$ implies that collision occurs. The occurrence probabilities of these events are denoted by $\text{Prob}(\rho(k) = i) = \theta_i$, $\forall i \in M$, $\sum_{i=0}^{\bar{m}} \theta_i = 1$.

Note that the stochastic parameter $\rho(k)$ takes values in $M$ and $M$ contains more than two values when $\bar{m} \geq 2$. Thus it is not easy to handle this situation. Motivated by the method used in Gao et al. [2008], we introduce the following $\bar{m} + 1$ binary valued functions:

$$
\sigma_i(k) = \begin{cases} 
1, & \text{if } \rho(k) = i \\
0, & \text{if } \rho(k) \neq i
\end{cases}, \forall i \in M,
$$

then we have that $\text{E}\{\sigma_i(k)\} = \theta_i$, $\forall i \in M$, where $\text{E}\{\cdot\}$ stands for the expectation operator. Then introduce $\Pi_i = \sum_{j \in \bar{N}_i} \text{diag}\{\delta(j - 1), \delta(j - 2), \ldots, \delta(j - m)\}, i = 1, 2, \ldots, \bar{m}$, where $\text{diag}\{\cdot\}$ represents a diagonal matrix and $\delta(h) = \begin{cases} 
0, & h \neq 0 \\
1, & h = 0
\end{cases}$.

**Remark 1:** The event of collision occurring means that none of the sensor nodes communicates with the filter, implying that the packets are lost. Thus the packet dropout probability is denoted by $\theta_0$ here. It should be pointed out that the considered problem is different from and not just an extension of the one in which the packet dropout issue is considered such as Seiler and Sengupta [2005]. However, they are the same when $\bar{m} = 1$, that is, the data collected by all the sensors can be transmitted via one sensor node.
It is clear that if node $i$ communicates with the filter at instant $k$, then $\hat{y}_j(k) = y_j(k)$, $\forall j \in N_i$. For the elements that are not updated in $\hat{y}$, the commonly used zero order holder (ZOH) scheme is employed, that is, $\hat{y}_j(k) = \hat{y}_j(k-1)$, $\forall j \in \{1, 2, \ldots, M\} \setminus N_i$. By the above analysis it can be readily obtained that

$$\hat{y}(k) = \Lambda_{\rho(k)}\hat{y}(k) + (I - \Lambda_{\rho(k)})\hat{y}(k-1)$$

(3)

where $\Lambda_{\rho(k)} = \sum_{i=1}^{m} \sigma_i(k)\Pi_i$.

Defining $\eta^T(k) = [x^T(k) \ x_j^T(k) \ y^T(k-1)]$ and $e(k) = \bar{z}(k) - z(k)$, we obtain by (1)-(3) the following filtering error system:

$$\eta(k+1) = \tilde{A}\eta(k) + \tilde{B}w(k)$$

(4)

$$\{ \begin{array}{l}
\eta(k+1) = \tilde{A}\eta(k) + \tilde{B}w(k) \\
e(k) = \tilde{C}\eta(k)
\end{array}$$

where

$$\tilde{A} = \begin{bmatrix} A + B\Lambda_{\rho(k)}C & 0 \\
B_{f}\Lambda_{\rho(k)} & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\
B_{f}\Lambda_{\rho(k)}D \end{bmatrix}$$

and $\tilde{C} = [-L \ C_f \ 0]$.

The considered $H_{\infty}$ filtering problem of the networked system with stochastic protocols is now formulated as the $H_{\infty}$ performance analysis and synthesis of the filtering error system (4). It can be seen that the filtering error system (4) is a stochastic system since stochastic parameters exist in the system representation. It is worthy pointing out that the filtering error system (4) can also be viewed as a switched system with the stochastic switching signal $\rho(k)$. Moreover, noticing the structure of $\tilde{A}$ and that $\Lambda_{\rho(k)}$ is a diagonal matrix, we can see that no matter what value $\rho(k)$ takes, $I - \Lambda_{\rho(k)}$ always has element equals one in the diagonal, thus all the subsystems of the filtering error system (4) are unstable.

**Remark 2:** It is well known that finding a switching law that render a switched system with all unstable subsystems stable with certain $H_{\infty}$ performance is challenging (Liberzon and Morse [2009]). Interestingly, we will show that switching laws satisfying the given probabilities distribution can render the filtering error systems (4) stable with a prescribed $H_{\infty}$ performance if the criterion presented in the next section is satisfied.

Our objective is then to design an $H_{\infty}$ filter such that the filtering error system (4) is stochastically stable with a prescribed $H_{\infty}$ performance. Before proceeding further, we introduce the following definition:

**Definition 1** (Yue and Han [2006]): The filtering error system (4) is said to be stochastically stable with a prescribed $H_{\infty}$ performance $\gamma$ if the following hold:

1. the filtering error system (4) with $w(k) = 0$ is stochastically stable, that is, for any initial condition $\eta(0)$, $E\{\sum_{i=0}^{\infty} \|\eta(i)\|^2\eta(0)\} < +\infty$ holds, where $\|\eta(k)\| = \sqrt{\eta^T(k)\eta(k)}$.
2. under zero initial condition, $E\{\|e\|_2^2\} \leq \gamma\|w\|_2^2$, where

$$\|e\|_2^2 = \sqrt{\sum_{i=0}^{\infty} \|\eta(i)\|^2}.$$

3. **MAIN RESULTS**

In this section, a sufficient condition is first presented for the filtering error system (4) to be stochastically stable with a prescribed $H_{\infty}$ performance.

**Theorem 1:** For given nonnegative scalars $\theta_i$, $\forall i \in M$, and a positive scalar $\gamma$, if there exists a matrix $P > 0$ such that the matrix inequality

$$E\{(\sigma_i(k) - \theta_i)(\sigma_j(k) - \theta_j)\} =
\begin{cases}
-\theta_i\theta_j, & i \neq j \\
\theta_i(1 - \theta_i), & i = j
\end{cases}

(9)$$

holds, then the filtering error system (4) is stochastically stable with an $H_{\infty}$ performance $\gamma$, where

$$\tilde{A}_1 = \begin{bmatrix} A & 0 & 0 \\
B_{f}\Phi C & A_f(J - \Phi) & 0 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} B \\
B_{f}\Phi D \end{bmatrix}, \Phi = \sum_{i=1}^{m} \theta_i\Pi_i, Q_1 = W_1\Pi_1W_2, \tilde{Q}_i = W_1\Pi_iD, i = 1, 2, \ldots, m, W_1 = [0 \ B_f^T J]^T$$

and $W_2 = [C \ 0 - I]$.

**Proof:**

Choose the Lyapunov function $V(k) = \eta^T(k)P\eta(k)$ for the filtering error system (4), then we have that

$$E\{\Delta V(k)\} = E\{(\tilde{A}_1\eta(k) + \tilde{B}_1w(k))^T P(\tilde{A}_1\eta(k) + \tilde{B}_1w(k))\} - \eta^T(k)P\eta(k)$$

(6)

First we assume that $w(k) = 0$ and consider the stochastic stability of the filtering error system (4). Note that $\tilde{A}$ and $\tilde{B}$ can be rewritten as

$$\tilde{A} = \tilde{A}_1 + \tilde{A}_2, \quad \tilde{B} = \tilde{B}_1 + \tilde{B}_2,$$

where

$$\tilde{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\
B_f(\Lambda_{\rho(k)} - \Phi)C & 0 & 0 \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} 0 \\
B_f(\Lambda_{\rho(k)} - \Phi)D \end{bmatrix}, \quad \tilde{A}_1 \text{ and } \tilde{B}_1 \text{ are shown in (5). It can be obtained that}

$$E\{\tilde{A}_1^T P\tilde{A}_1 - P\} = \tilde{A}_2^T P\tilde{A}_2 = 0$$

(8)

It follows by $w(k) = 0$ and (6)-(8) that

$$E\{\Delta V(k)\} = E\{\eta^T(k)(\tilde{A}_1^T P\tilde{A}_1 - P)\eta(k)\}$$

$$= \eta^T(k)\tilde{A}_2^T P\tilde{A}_2 + \eta^T(k)\tilde{A}_2^T P\tilde{A}_2 - P)\eta(k)$$

Note that $\tilde{A}_2$ can be written as $\tilde{A}_2 = W_1(\sum_{i=1}^{m} (\sigma_i(k) - \theta_i)\Pi_i)W_2$, where $W_1$ and $W_2$ are shown in (5). Then we see that $E\{(\sigma_i(k) - \theta_i)\eta(k) - \eta(k)\}$ is involved. By analyzing all the possible events and the corresponding probabilities, we calculate expectations and have that

$$E\{(\sigma_i(k) - \theta_i)\eta(k) - \eta(k)\} = \begin{cases}
-\theta_i\theta_j, & i \neq j \\
\theta_i(1 - \theta_i), & i = j
\end{cases}

(9)$$
Thus it follows by (9) and some matrix manipulations that
\[
E\{\tilde{A}_T^TP\tilde{A}_2\} = -\Gamma + \sum_{i=1}^{m} \theta_i Q_i^TPQ_i
\] (10)
where \(\Gamma_1 = (W_1^T\Phi W_2)P(W_1^T\Phi W_2)\) and \(Q_i, i = 1, 2, \ldots, m\) are shown in (5). Since \(\Gamma_1\) is semi-definite positive, we have by (7)-(8) and (10) that
\[
E\{\Delta V(k)\} \leq \eta^T(k)\Gamma\eta(k)
\] (11)
where \(\Gamma = \tilde{A}_T^TP\tilde{A}_1 + \sum_{i=1}^{m} \theta_i Q_i^TPQ_i - P\). By Schur complement, we have that (5) implies that \(\Gamma < 0\) holds. Thus for nonzero \(\eta(k)\), if (5) holds, then \(E\{V(k+1)|\eta(k)\} < V(k)\). It is clear that there always exists a scalar \(0 < \varepsilon < 1\) such that \(E\{V(k+1)|\eta(k)\} \leq \varepsilon V(k)\). We have by induction that \(E\{V(k)|\eta(0)\} \leq \varepsilon^k V(0)\), which implies that
\[
E\{\sum_{i=0}^{\infty} V(i)|\eta(0)\} \leq \frac{1}{1-\varepsilon}V(0).
\]
Denoting the minimum eigenvalue of \(P\) by \(\alpha = \lambda_{\text{min}}(P)\) and letting \(k \to +\infty\), we have that \(E\{\sum_{i=0}^{\infty} \|\eta(i)\|^2\eta(0)\} \leq \frac{1}{\alpha(1-\varepsilon)}(\eta(0)|P\eta(0) < +\infty\). It is by Definition 1 that the filtering error system (4) is stochastically stable.

Now we consider the \(H_\infty\) performance of the filtering error system (4). Consider the performance index \(E\{e_T(k)e(k) - \gamma^2 w_T(k)w(k) + \Delta V(k)\}\). Noting that \(\tilde{B}_2\) can be rewritten as \(\tilde{B}_2 = W_i(\sum_{i=1}^{m}(\sigma_i(k) - \theta_i)\Pi_i)D\), we have by (9) and some matrix manipulations that
\[
E\{e_T(k)e(k) - \gamma^2 w_T(k)w(k) + \Delta V(k)\} = \xi^T(k)(-\Omega_1 + \Omega_2)\xi(k)
\] (12)
where \(\xi(k) = [\eta^T(k) w^T(k)]^T\), \(\Omega_1 = \Omega_3^T W_T WP_1 \Omega_3\), \(\Omega_4 = [\Phi W_2 \Phi]\), \(\Omega_2 = [\tilde{A}_T^T \tilde{B}_1^T] P [\tilde{A}_1 \tilde{B}_1] + [\tilde{C}^T \tilde{C} - P - 0 - \gamma^2] + \sum_{i=1}^{m} \theta_i [Q_i^T Q_i]^T\). \(P\) is chosen such that \(\Omega_4 < 0\). By Schur complement, \(\Omega_2 < 0\) is equivalent to the following inequality:
\[
\begin{bmatrix}
-P & 0 & \tilde{C}^T & \sqrt{\theta_1} Q_1 & \cdots & \sqrt{\theta_m} Q_m \\
* & -\gamma^2 & 0 & \sqrt{\theta_1} Q_1 & \cdots & \sqrt{\theta_m} Q_m \\
* & * & -P & 0 & \cdots & 0 \\
* & * & * & -I & 0 & \cdots & 0 \\
* & * & * & * & -P & \cdots & 0 \\
* & * & * & * & * & \cdots & -\gamma^2 \\
\end{bmatrix} < 0
\] (13)
Performing congruence transformations to both sides of (13) by \(\text{diag}(I, I, P, I, \ldots, P)\), we obtain (5). Then it can be obtained that (5) implies that
\[
E\{e_T(k)e(k) - \gamma^2 w_T(k)w(k) + \Delta V(k)\} < 0
\] (14)
Setting \(k = 0\) and \(k = 1\) in (14) lead, respectively, to
\[
E\{e_T(0)e(0) - \gamma^2 w_T(0)w(0) + E\{V(1)|\eta(0)\} - V(0)\} < 0
\] (15)
\[
E\{e_T(1)e(1) - \gamma^2 w_T(1)w(1) + E\{V(2)|\eta(0)\} - V(0)\} < 0
\] (16)
Taking expectation \(E\{\eta(0)\}\) on both sides of (16) and by (15) we have \(E\{e_T(i)e(i) - \gamma^2 w_T(i)w(i) + E\{V(i)|\eta(0)\} - V(0)\} < 0\) for all \(i\).

By induction we have that \(E\{e_T(i)e(i) - \gamma^2 w_T(i)w(i) + E\{V(i+1)|\eta(0)\} - V(0)\} < 0, \forall \alpha \geq 1\). Letting \(\alpha \to +\infty\), it follows by the stochastic stability of the filtering error system (4) and the zero initial condition that \(E\{e_T(i)e(i) - \gamma^2 w_T(i)w(i)\} < 0\). By definition 1 this is sufficient to show that the filtering error system (4) is stochastically stable with an \(H_\infty\) performance \(\gamma\). The proof is completed.

Remark 3: Note that the condition (5) is not only related to the filter parameter matrices \(A_f, B_f, C_f\), but also the access probabilities \(\theta_i, i = 1, 2, \ldots, m\), and the packet dropout probability \(\theta_0\). Thus the relation among the access probabilities, packet dropout probability and the \(H_\infty\) filtering performance can be established. Moreover, the relation between the access probabilities and the \(H_\infty\) filtering performance can help assign the access probabilities to the nodes to achieve better \(H_\infty\) filtering performance. Since the filtering parameter matrices can not be obtained directly from (5), a design procedure is also presented for the desired \(H_\infty\) filter based on the obtained condition.

Theorem 2: For given nonnegative scalars \(\theta_i, \forall i \in M\), and a positive scalar \(\gamma\), if there exist matrices \(P_{11}, P_{13}, P_{221}, P_{22}, P_{33}, A_{f_1}, A_{f_2}, B_{f_1}, B_{f_2}\) with appropriate dimensions such that the following matrix inequality
\[
\begin{bmatrix}
-Z_{11} & 0 & \Xi_{13} & 0 & \Xi_{15} & \cdots & \Xi_{15m} \\
* & -\gamma^2 I & 0 & \Xi_{25} & \cdots & \Xi_{25m} \\
* & * & -\Xi_{11} & 0 & \cdots & 0 \\
* & * & * & -1 & 0 & \cdots & 0 \\
* & * & * & * & -\Xi_{11} & \cdots & 0 \\
* & * & * & * & * & \cdots & 0 \\
\end{bmatrix} < 0
\] (17)
holds, then the filtering error system (4) is stochastically stable with an \(H_\infty\) performance \(\gamma\), where
\[
\begin{align*}
Z_{11} &= \begin{bmatrix}
P_{11} & P_{22} & P_{221} \\
P_{11} & * & P_{13} \end{bmatrix}, \quad Z_{13} = \begin{bmatrix}
P_{31} A_1 + B_f \Phi C_1 + P_{21} C_1 A_{f_1} (B_{f_1} + P_{21})I - \Phi \end{bmatrix}P_{22}^T A_1 + B_f \Phi C_2 + P_{21} C_2 A_{f_2} (B_{f_2} + P_{22})I - \Phi \\
P_{31} A_1 + B_f \Phi C_1 + P_{21} C_1 A_{f_1} (B_{f_1} + P_{21})I - \Phi \end{bmatrix}P_{13}^T A_1 + B_f \Phi C_2 + P_{21} C_2 A_{f_2} (B_{f_2} + P_{22})I - \Phi \\
Z_{23} &= \begin{bmatrix}
P_{22} B_f + B_{f_1} \Phi D + P_{221} \Phi D \\
P_{22} B_f + B_{f_1} \Phi D + P_{221} \Phi D \\
\end{bmatrix}, \quad Z_{15i} = \begin{bmatrix}
(B_{f_1} + P_{21}) \Pi_i C_0 & 0 \\
(B_{f_1} + P_{21}) \Pi_i C_0 \\
\end{bmatrix}
\end{align*}
\]
\[ \Xi_{254} = \sqrt{V_i} \left[ \begin{array}{c} (B_f + P_{13})\Pi_i D \\ (B_f + P_{21})\Pi_i D \\ (B_f + P_{31})\Pi_i D \end{array} \right], \ i = 1, 2, \ldots, m \]

\[ \bar{A}_f = P_{22}^T A_f, \ A_{f1} = \left[ \begin{array}{c} \bar{A}_f \\ \bar{A}_{f2} \end{array} \right], \ B_f = \left[ \begin{array}{c} \bar{B}_f \\ \bar{B}_{f2} \end{array} \right]. \]

Moreover, the filter parameter matrices are given by \( A_f = P_{22}^{-1} A_f, B_f = P_{22}^{-1} B_f \) and \( C_f \).

**Proof:** Note the structure of \( \bar{A}_f, \bar{B}_f, Q_i, \) and \( \bar{Q}_i, i = 1, 2, \ldots, m \), we restrict the structure \( P \) as \( \Xi_{11} \). Then letting \( \bar{A}_f = P_{22}^T A_f, \ A_{f2} = P_{22}^T A_f, \ B_f = P_{22}^T B_f, \)

\[ \tilde{B}_f = P_{22}^T B_f, \ A_f = \left[ \begin{array}{c} \tilde{A}_f \\ \tilde{A}_{f2} \end{array} \right], \ B_f = \left[ \begin{array}{c} \tilde{B}_f \\ \tilde{B}_{f2} \end{array} \right], \]

and some matrix manipulations we obtain (17) from (5). Since inequality (17) is linear in variables involved and thus can easily be solved by the existing LMI solvers. Based on Theorem 1, we can see that if (17) has solutions, then the filtering error system (4) is stochastically stable with an \( H_\infty \) performance \( \gamma \). Moreover, the filter parameter matrices are given by \( A_f = P_{22}^{-1} A_f, B_f = P_{22}^{-1} B_f \) and \( C_f \). The proof is completed.

**Remark 4:** Notice that the condition (17) is linear not only in the matrix variables involved, but also in the scalar \( \gamma^2 \). Thus the optimal \( H_\infty \) performance of the filtering error system (4) with given \( \theta_i, \forall i \in M \), can be obtained by the following optimization problem with LMI constraint: minimize \( \gamma_i \), with \( \gamma_i^2 = \gamma \), subject to (17). If the optimization problem has solution \( \hat{\gamma} \), then the optimal \( H_\infty \) performance is \( \gamma^* = \sqrt{\hat{\gamma}} \).

4. ILLUSTRATIVE EXAMPLE

Consider the spring-mass system in Song et al. [2010], where the matrices for the system model are given by

\[
A = \begin{bmatrix}
0.9617 & 0.0191 & 0.1878 & 0.0012 \\
0.0370 & 0.9629 & 0.0025 & 0.1789 \\
-0.3732 & 0.1853 & 0.0678 & 0.0179 \\
0.3528 & -0.3553 & 0.0357 & 0.7840
\end{bmatrix}, \quad L = \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0.0193 & 0 \\
0.0187 & 0 \\
0.1890 & 0 \\
0.1813 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0.1 \\
0.1 & 0.1
\end{bmatrix}
\]

It is assumed that the plant is equipped with two sensor nodes labeled by node 1 and node 2, respectively, and that \( y_1 \) and \( y_2 \) are transmitted via node 1 and node 2, respectively. We will first show that the proposed method is effective in dealing with the considered \( H_\infty \) filtering problem of networked systems with stochastic protocols. Choosing \( \theta_0 = 0.2, \theta_1 = \theta_2 = 0.4 \), and by solving the optimization problem formulated in the above section, we obtain the optimal \( H_\infty \) filtering performance is \( \gamma^* = 1.3274 \). The associated filter parameter matrices are given by:

\[
A_f = \begin{bmatrix}
0.0490 & 0.0401 & 0.1147 & 0.2977 \\
0.0757 & 0.0582 & -0.6931 & 0.7452 \\
-0.1453 & -0.0533 & 0.5831 & 0.2350 \\
-0.1739 & -0.0713 & 0.1295 & 0.8790
\end{bmatrix}
\]

Table 1. Optimal \( H_\infty \) filtering performance \( \gamma^* \) with different \( \theta_0 \)

<table>
<thead>
<tr>
<th>( \theta_0 )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma^* )</td>
<td>1.2161</td>
<td>1.3274</td>
<td>1.4722</td>
<td>1.6663</td>
<td>1.9212</td>
</tr>
</tbody>
</table>

In Fig. 2 we present the random access sequence \( \rho(k) \)

\[
B_f = \begin{bmatrix}
-1.3974 & -0.2433 \\
-0.4756 & -1.3073 \\
-0.1747 & -0.0643 \\
-0.2591 & -0.1085
\end{bmatrix}, \quad C_f = \begin{bmatrix}
-0.5759 \\
-0.3807 \\
0.6990 \\
-0.7526
\end{bmatrix}
\]

Next we consider the effects of the parameters \( \theta_i, i = 0.1, 2, 4 \), namely, packet dropout probability and access probabilities of node 1 and node 2, on the optimal \( H_\infty \) filtering performance \( \gamma^* \). First, let \( \theta_1 = \theta_2 \) and the values of \( \gamma^* \) with \( \theta_0 \) varying from 0.1 to 0.9 is shown in Table 1, from which we can see that \( \gamma^* \) increases with larger \( \theta_0 \). This means that a larger packet dropout probability leads to a worse \( H_\infty \) filtering performance. Then let \( \theta_0 = 0 \) and see how \( \gamma^* \) changes according to \( \theta_1 \) and \( \theta_2 \). Fig. 4 depicts the trajectory of \( \gamma^* \) with \( \theta_1 \) varying from 0.01 to 0.09. We can see that \( \theta_1 = 0.58 \) and \( \theta_2 = 0.42 \) result in the smallest \( \gamma^* = 1.1228 \). This indicates that assigning node 1 and node 2 with access probabilities 0.58 and 0.42, respectively, leads to the best \( H_\infty \) performance.

5. CONCLUSIONS

In this paper, the \( H_\infty \) filtering problem of networked systems with stochastic protocols was studied. The communication process between the sensor nodes and the filter was
Fig. 3. State trajectory for the filtering error

Fig. 4. $H_{\infty}$ filtering performance with different $\theta_1$

modeled as a stochastic sequence. Then the filtering error system was modeled as a stochastic systems. A sufficient condition was presented for the stochastic stability and guaranteed $H_{\infty}$ performance of the filtering error system. A design procedure for the desired $H_{\infty}$ was also presented. An example is finally given to show the effectiveness of the proposed method.

REFERENCES


