The Bloch Equation Revisited through Averaging under Adiabatic Passages

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Abstract:
In order to derive a simplified solution to a bilinear system originating from the Bloch equation under adiabatic excitation, we have employed a nonlinear averaging technique for the first time in magnetic resonance context. We achieve an averaged solution with an acceptable level of error by transferring the dynamics of the Bloch equation to a novel rotating frame of reference through a rotation and a proper time-scaling. In this frame, the states of the system which represent the components of magnetic resonance signal are slowly varying and thus can be averaged to achieve an approximate solution. Error analysis of the averaged solution as well as the simulation results clearly show that the error of the averaged solution is negligible. Therefore, the simplified solution presented in this paper in conjunction with adaptive control techniques, particularly extremum seeking, can be used to find optimal adiabatic modulation functions in a computationally efficient manner.

Keywords: Nonlinear averaging, Biomedical imaging systems, Magnetic resonance imaging, Bloch equation, Adiabatic passages.

1. INTRODUCTION
Radio Frequency (RF) pulses are used extensively as excitation patterns in Magnetic Resonance Imaging (MRI), Nuclear Magnetic Resonance (NMR), and optical resonance problems (Levitt, 2008; Allen and Eberly, 1975). Therefore, the design of RF pulses for controlling the bulk magnetisation has been and still is an active area of research (Garwood and DelaBarre, 2001). Typically, the RF pulse is applied at a constant resonance frequency, which is determined by the strength of the main static external field, and therefore it can manipulate the bulk magnetisation vector. In practice, these constant frequency pulses, referred to as non-adiabatic passages, are very sensitive to the field variations which results in a poor spectral resolution in NMR or low resolution images in MRI (Meriles et al., 2003; Li and Khaneja, 2006). To overcome this shortcoming of non-adiabatic passages a different class of pulses, called adiabatic passages, have been proposed (Garwood and DelaBarre, 2001; Meriles et al., 2003). Unlike non-adiabatic passages, during an adiabatic pulse excitation, the frequency of the RF field is varying with time in a proper manner such that the bulk magnetisation will follow the RF excitation field. The robustness of adiabatic passages to RF field inhomogeneities has made them popular in magnetic resonance context (Du et al., 2010).

In our previous work (Tahayori et al., 2008a; Tahayori, 2010) we introduced a novel rotating frame of reference known as the excitation dependent rotating frame of reference. We showed that the Bloch equation in this frame is much better behaved from a numerical point of view, leading to significant computational time savings in simulation studies under non-adiabatic excitations. Moreover, we used first order nonlinear averaging to derive approximate analytic solutions to the Bloch equation under typical non-adiabatic excitation patterns (Tahayori et al., 2008b; Tahayori, 2010). In this paper, a computationally efficient method of solving the Bloch equation by using nonlinear averaging technique is derived. Here, the Bloch equation is rotated, scaled, and subsequently averaged to find the magnetisation behaviour in a straight forward way with negligible error for adiabatic passages. An error analysis reveals that the derived solution is particularly suited to the adiabatic passages. The novel framework and the averaged solution presented here will be used in future work to improve the performance of adiabatic passages by adopting extremum seeking optimisation algorithm (Ariyur and Krstic, 2003; Tan et al., 2009).

In Section 2, a short introduction to the Bloch equation and adiabatic passages is presented. The behaviour of the magnetisation vector under adiabatic passages is visualised in Cartesian coordinates in Section 3. The preparation stage for use of the averaging technique is introduced in Section 4, including a novel rotating frame of reference and its mathematical framework are introduced and the dynamics of the Bloch equation is transferred to this new rotating frame. The averaged solution to the resultant equation and the related error analysis are studied in Section 5. The simulation results of the averaged method are presented in Section 6.

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2. A BRIEF REVIEW OF ADIABATIC PASSAGES

The Bloch equation, developed in 1946, describes the behaviour of the bulk magnetisation in the presence of external magnetic fields. The Bloch equation in the classical rotating frame of reference, denoted by $x'y'z'$, is written as

$$
\mathbf{M} = \begin{bmatrix} M_x' \\ M_y' \\ M_z' \end{bmatrix} = \begin{bmatrix} \frac{1}{T_2} & \frac{T_1}{T_2} & 0 \\ -\frac{T_1}{T_2} & \frac{1}{T_2} & 0 \\ 0 & 0 & \frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x' \\ M_y' \\ M_z' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_0 \end{bmatrix},
$$

(1)

where $\mathbf{M}$ is a three dimensional magnetisation vector, dependent on both position and time, $M_0 \mathbf{e}_z$ is the thermal equilibrium magnetisation created by an ideally-uniform static field oriented in the $z$-direction which is aligned with the static external field. Controls $u(t) = \gamma B_{y'}$ and $v(t) = \gamma B_{y'}$ are the inputs to the Bloch equation generated by an oscillating field applied in the transverse plain, and $\gamma$ is a constant known as the gyromagnetic ratio. $\Delta \omega(t) = \gamma \Delta B(t)$ represents the frequency deviation from the resonance frequency of the static field. $T_1$ and $T_2$ represent longitudinal and transverse relaxation time constants, respectively. For a short pulse excitation the relaxation terms may be dropped from the Bloch equation during the excitation period.

Consider a pulse excitation pattern with varying frequency over the pulse period. If the frequency modulation function is well chosen, which will be explained in detail later, these RF pulses may be used to excite a spin system. This class of excitation patterns are known as adiabatic passages or adiabatic pulses. The way that these pulses control the bulk magnetisation is inherently different from pulses with fixed frequency known as non-adiabatic passages, typical in MRI.

In the presence of a static field $B_0$, an excitation field is applied along the $x$-axis in the laboratory frame of reference, then it may be written as

$$
\mathbf{B}_1 = B_1(t) \cos \phi_{nx}(t) \mathbf{e}_x - B_1(t) \sin \phi_{nx}(t) \mathbf{e}_y.
$$

(2)

After transforming the Bloch equation into the classical rotating frame of reference the transverse components of the magnetic field shown in Figure 1 will be

$$
u(t) = u(t) e_{x'} \equiv \omega_1(t) e_{x'} = \gamma B_y e_{x'} = \gamma B_1(t) e_{x'},$$

(3a)

$$v(t) = v(t) e_{y'} \equiv \gamma B_y e_{y'} = 0.$$

(3b)

In the classical rotating frame of reference the net field in the $z'$-direction as represented by Figure 1 will be

$$
\mathbf{B}_{z'}(t) = \left( B_0 - \frac{\omega_{RF}(t)}{\gamma} \right) \mathbf{e}_{z'},
$$

(4)

where $\omega_{RF}(t) = \dot{\phi}_{nx}(t)$, and thus we may write

$$
\Delta \omega(t) = \gamma \mathbf{B}_{z'}(t) = (\omega_0 - \omega_{RF}(t)) e_{z'}.
$$

(5)

In the presence of field inhomogeneities, $\delta B_0$, and gradient fields, $B_g$ should be replaced by $B_0 + \delta B_0 + \mathbf{G} \cdot \mathbf{r}$, and therefore

$$
\Delta \omega(t) = \gamma \mathbf{B}_{z'}(t) = (\omega_0 + \delta \omega_0 + \gamma \mathbf{G} \cdot \mathbf{r} - \omega_{RF}(t)) e_{z'} = (\omega_0 + \Delta \omega_0 - \omega_{RF}(t)) e_{z'}.
$$

(6)

It is common to choose $\omega_{RF}(t) = \omega_0 + \omega_{rf}(t)$, where $\omega_{rf}(t)$ is called the frequency modulation function. Therefore, the bulk magnetisation will precess on a cone about the time dependent effective field shown in Figure 1 with the effective angular frequency, $\omega_{eff}(t)$, which may be written as

$$
\omega_{eff}(t) = \sqrt{\Delta \omega^2(t) + u^2(t)}.
$$

(7)

As shown in Figure 1, the orientation of the axis of rotation evolves with a time dependent angular velocity, $\dot{\alpha}(t)$, where

$$
\alpha(t) = \tan^{-1} \frac{u(t)}{\Delta \omega(t)}.
$$

(8)

It has been shown that if the precession frequency of the bulk magnetisation about the effective field is much smaller than the rate of change in orientation of the axis of rotation, then the bulk magnetisation will follow the effective field (Bernstein et al., 2004; Garwood and DelaBarre, 2001; Taams and Garwood, 1998). This is recognised as the adiabatic passage principle or adiabatic condition, which mathematically is described by

$$
|\omega_0(t) | \equiv |\dot{\alpha}(t)| \ll \omega_{eff}(t) \quad \forall t \in [0, \tau_p],
$$

(9)

where $\tau_p$ represents the duration of the pulse. An adiabatic factor (Bernstein et al., 2004) can be defined to be

$$
\eta(t) \equiv \frac{\omega_{eff}(t)}{|\dot{\alpha}(t)|},
$$

(10)

which should be much larger than unity during the pulse period to satisfy the adiabatic condition. The major advantage of adiabatic pulses over non-adiabatic pulses is their robustness to amplitude variation of the excitation field.

Adiabatic passages, similar to non-adiabatic passages, may be used to tip the bulk magnetisation into the transverse plain, refocus, or invert the bulk magnetisation vector (Shen and Rothman, 1997; Bernstein et al., 2004). In this context the former is called Adiabatic Half Passage (AHP) and the latter is known as Adiabatic Full Passage (AFP). Without loss of generality, the focus of our work in this paper is on adiabatic full passages.

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1. An inversion pulse changes the initial condition $\mathbf{M}_0 = [0 \ 0 \ M_0]^T$ to $\mathbf{M}(\tau_p) = [-0 \ 0 \ M_0]^T$. In terms of quantum mechanics and density matrix, this is called population inversion.
3. VISUALISING ADIABATIC PASSAGES IN CARTESIAN COORDINATES

Two common adiabatic full passages are chirp and hyperbolic secant excitation. In this paper we only look at the hyperbolic secant passage. Other forms of adiabatic passages are addressed in (Garwood and DelaBarre, 2001). The amplitude and the frequency modulation functions for a typical hyperbolic secant excitation may be written as

\[ u(t) = u_0 \text{sech} \left( \beta \left( 1 - \frac{2t}{\tau_p} \right) \right), \]

\[ \omega_{rf}(t) = A \tan \left( \beta \left( 1 - \frac{2t}{\tau_p} \right) \right), \]

where \( \tau_p \) represents the duration of the applied pulse. The simulation results of this excitation for \( u_0 = 2\pi \times 5700\text{rad/s}, A = 2\pi 10^3\text{rad/s}, \beta = 6, \) and \( \tau_p = 2.5\text{ms} \) are as in (Garwood and DelaBarre, 2001) are represented by Figure 2. In Figure 2a the result is shown when \( \Delta \omega_0 = 0 \) and Figure 2b corresponds to \( \Delta \omega_0 = 6\pi \text{krad/s}. \)

4. TRANSFERRING THE BLOCH EQUATION INTO A NOVEL ROTATING FRAME OF REFERENCE

For a short pulse, if the excitation is initially applied along the \( x' \)-direction, meaning \( v(t) = 0 \), the Bloch equation in the classical rotating frame of reference, characterised by prime notation, may be written as

\[
\begin{bmatrix}
\dot{M}_{x'} \\
\dot{M}_{y'} \\
\dot{M}_{z'}
\end{bmatrix}
= \begin{bmatrix}
0 & \Delta \omega(t) & 0 \\
-\Delta \omega(t) & 0 & u(t) \\
0 & -u(t) & 0
\end{bmatrix}
\begin{bmatrix}
M_{x'} \\
M_{y'} \\
M_{z'}
\end{bmatrix},
\]

with the initial condition \( M(0) = [0 0 M_0]^T \).

The objective is to rewrite the Bloch equation in a frame whose \( z' \)-axis is in aligned with the effective field, \( B_{eff} \), and its transverse plane is rotating at the angular frequency \( \omega_{eff} \) defined by (7). Figure 3 shows the steps of this process. Initially the Bloch equation must be presented in the double prime frame represented by Figure 3a and then resultant equation must be presented in the triple prime frame whose transverse plane is rotating at the instantaneous angular frequency \( \omega_{eff}(t) \).

By defining

\[
\sigma_3(t) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

then linear equation between the double prime frame and the classical rotation frame denoted by prime may be written as

\[
\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \cos \alpha(t) & 0 & -\sin \alpha(t) \\ 0 & 1 & 0 \\ \sin \alpha(t) & 0 & \cos \alpha(t) \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix},
\]

where \( \alpha(t) = \tan^{-1}(u(t)/\Delta \omega(t)) \). In order to rewrite the Bloch equation in this frame we define \( M'' = \sigma_3(t)M' \).

Thus we may write

\[
\dot{M}'' = (\sigma_3(t)^{-1}(t) + \sigma_3(t)\Omega(t)\sigma_3^{-1}(t))M'',
\]

or equivalently

\[
\begin{bmatrix}
\dot{M}_{x''} \\
\dot{M}_{y''} \\
\dot{M}_{z''}
\end{bmatrix}
= \begin{bmatrix}
0 & \omega_{eff}(t) - \dot{\alpha}(t) & 0 \\
-\omega_{eff}(t) & 0 & 0 \\
\dot{\alpha}(t) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_{x''} \\
M_{y''} \\
M_{z''}
\end{bmatrix},
\]

in which \( \omega_{eff}(t) = \sqrt{\Delta \omega^2(t) + u(t)^2} \).

As expected intuitively the effective angular frequency arises in Equation (15). To remove this fast component from the equation we transfer Equation (15) to the rotating frame distinguished by triple prime notation in Figure 3b. The relation between the double prime and the triple prime frame can be expressed by

\[
\begin{bmatrix} x''' \\ y''' \\ z''' \end{bmatrix} = \begin{bmatrix}
\cos \int_0^t \omega_{eff}(t')d\tau & -\sin \int_0^t \omega_{eff}(t')d\tau & 0 \\
\sin \int_0^t \omega_{eff}(t')d\tau & \cos \int_0^t \omega_{eff}(t')d\tau & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}.
\]

By defining

\[
M''' = \sigma_3(t)M'',
\]
we may write
\[ M''' = \left( \dot{\sigma}^2(t) \sigma^{-1}(t) + \sigma^2(t) \Omega''(t) \sigma^{-1}(t) \right) M''' , \] (18)
that leads to
\[ M'''(t) = M'''(0^+) = -M_0 \sin \alpha_1 \begin{bmatrix} 0 & 0 \\ M_0 \cos \alpha_1 \end{bmatrix} \] \begin{bmatrix} 0 \\ M_0 \cos \alpha_1 \end{bmatrix}, \] (19)
in which
\[ f(t) = \dot{\alpha}(t) \cos \int_0^t \omega_{eff}(\tau) d\tau, \] (20a)
\[ g(t) = \dot{\alpha}(t) \sin \int_0^t \omega_{eff}(\tau) d\tau. \] (20b)

The initial condition of the above equation is calculated to be
\[ M'''(0^+) = \sigma_2(0^+) \sigma_1(0^+) M(0^+) = \begin{bmatrix} -M_0 \sin \alpha_1 \\ 0 \end{bmatrix}, \] where we have defined \( \alpha_1 = \alpha(t = 0^+) \).

4.1 Constant Excitation Pattern Revisited in the Novel Frame

For an excitation pattern with constant envelope and constant frequency, Equation (19) simplifies to
\[ M''' = 0, \] (21)
and thus,
\[ M'''(t) = M'''(0^+) = \begin{bmatrix} -M_0 \sin \alpha_1 \\ 0 \end{bmatrix} \] \begin{bmatrix} 0 \\ M_0 \cos \alpha_1 \end{bmatrix}, \] (22)

As a result, the solution of the Bloch equation in the classical rotating frame of reference can be calculated through a linear transformation as follows
\[ M(t) = \sigma_1^{-1}(t) \sigma_2^{-1}(t) M'''(t) = M_0 \begin{bmatrix} \sin \alpha_1 \cos \omega_{eff} t \\ \sin \alpha_1 \cos \omega_{eff} t \\ \cos^2 \alpha_1 + \sin^2 \alpha_1 \cos \omega_{eff} t \end{bmatrix}. \] (23)

4.2 Revisiting Piecewise Constant Excitation Patterns in the Novel Frame

For a piecewise constant excitation pattern, \( u(t) \) will be piecewise constant as shown in Figure 4. For the adiabatic passages, \( \Delta \omega(t) \) will be discretised to \( \Delta \omega_i \) as well. To solve the Bloch equation, at each step Equation (19) must be reinitialised. It can be shown that the bulk magnetisation may be calculated as follows
\[ M = \sigma_1(-\alpha_1) \sigma_2(-\omega_{eff} \Delta T) \sigma_1(-\alpha_1) \sigma_2(-\omega_{eff} \Delta T) \sigma_1(-\alpha_1) \sigma_2(-\omega_{eff} \Delta T) \sigma_1(\alpha_1) \] (24)
where
\[ \alpha_i = \tan^{-1} \frac{u_i}{\Delta \omega_i}, \quad \omega_{eff} = \sqrt{\Delta \omega_i^2 + u_i^2}. \] (25)

For piecewise constant excitation, Equation (19) may be rewritten in the following form
\[ \begin{bmatrix} M_{\alpha'''} \\ M_{\beta'''} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -f_i(t) \\ 0 & 0 & -g_i(t) \end{bmatrix} \begin{bmatrix} M_{\alpha'''} \\ M_{\beta'''} \end{bmatrix}, \] (26)
where
\[ f_i(t) = \alpha_1 \delta(t) + (\alpha_2 - \alpha_1) \delta(t - \Delta T) \cos \omega_{eff} \Delta T + \frac{1}{N+1} \sum_{i=0}^{N} \delta((i-1) \Delta T) \cos \omega_{eff} \Delta T, \] (29a)
\[ g_i(t) = (\alpha_2 - \alpha_1) \delta(t - \Delta T) \sin \omega_{eff} \Delta T + \frac{1}{N+1} \sum_{i=0}^{N} \delta((i-1) \Delta T) \sin \omega_{eff} \Delta T, \] (29b)
in which we have defined \( \omega_{eff} \equiv 0, \quad \alpha_0 \equiv 0, \quad \alpha_{N+1} \equiv 0. \) (27)

In the above equations \( \delta(t) \) is the Dirac delta function. It can be easily shown that the answer to Equation (26) may be expressed by
\[ M''' = \exp(A_2) \exp(A_1) M_0 = \prod_{i=1}^{N+1} \exp(A_i) M_0, \] (28)
where
\[ A_i = \begin{bmatrix} 0 & 0 & -f_i \\ 0 & 0 & -g_i \end{bmatrix}. \] (29)

It should be mentioned that the \( A_i \) matrices do not commute, therefore the product of exponentials in Equation (28) cannot be simplified to the exponential of summation of \( A_i \) matrices. However, for adiabatic passages it is possible to simplify Equation (28) by using averaging technique which is the subject of Section 5.

5. ADIABATIC PASSAGES REVISITED IN THE NOVEL FRAME THROUGH AVERAGING

The variation rate of the angle \( \alpha(t) \) shown in Figure 1, which is denoted by \( \dot{\alpha}(t) \), may be interpreted as an angular frequency \( \omega_{\alpha} \) as indicated by (9). This angular frequency can be considered as a result of a virtual field in the \( y' \)-direction with the following magnitude
\[ B_{\alpha}(t) = \frac{\omega_{\alpha}(t)}{\gamma} \equiv \frac{\dot{\alpha}(t)}{\gamma}. \] (31)
If the following condition is satisfied
\[ |B_z(t) - B_{eff}(t)| \kappa \leq 0, \tau_p, \]  
then the bulk magnetisation is dominantly controlled by the effective field, \( B_{eff}(t) \). This implies that in the triple frame shown in Figure 3b the magnetic resonance signal represents a slowly varying signal and hence first order averaging may be adopted to find an approximate solution to (26).

It should be noted that the inequality indicated by (32) is equivalent to \( |\alpha(t)| \ll \omega_{eff}(t) \) which is the adiabatic passage principle. Therefore, it is expected that averaging provides an approximation and therefore simplification to Equation (26) with an acceptable level of error.

5.1 Averaging

We use first order averaging (Sanders et al., 2007) to approximate Equation (26) resulting in the following linear time-invariant system
\[
\begin{align*}
\Omega''_{avg} &= \begin{bmatrix} M''_{avg} & -g_{avg} \\ f_{avg} & -M''_{avg} \end{bmatrix} \cdot \\
&= \begin{bmatrix} 0 & 0 & -M''_{avg} \\ 0 & 0 & -g_{avg} \\ f_{avg} & g_{avg} & 0 \end{bmatrix}.
\end{align*}
\]

At an arbitrary observation point in time, \( t = T_{obs} \), where
\[ T_{obs} = n\Delta T, \quad n = 1, 2, ..., N, \]
the averaged functions for \( f_s(t) \) and \( g_s(t) \) are calculated to be
\[
\begin{align*}
f_{avg} &= \frac{1}{T_{obs}} \int_0^{T_{obs}} f_s(\tau) d\tau \\
&= \frac{1}{T_{obs}} \sum_{i=1}^{n+1} (\alpha_i - \alpha_{i-1}) \cos(\omega_{eff}(\Delta T) t_i, \\
g_{avg} &= \frac{1}{T_{obs}} \int_0^{T_{obs}} g_s(\tau) d\tau \\
&= \frac{1}{T_{obs}} \sum_{i=1}^{n+1} (\alpha_i - \alpha_{i-1}) \sin(\omega_{eff}(\Delta T) t_i,
\end{align*}
\]
where \( f_i \) and \( g_i \) were defined in Equation (30). Therefore, \( \Omega''_{avg} \) simplifies to
\[
\begin{align*}
\Omega''_{avg} &= \frac{1}{n\Delta T} \begin{bmatrix} 0 & 0 & -\sum_{i=1}^{n+1} f_i \\ 0 & 0 & -\sum_{i=1}^{n+1} g_i \\ \sum_{i=1}^{n+1} f_i & \sum_{i=1}^{n+1} g_i & 0 \end{bmatrix} = \frac{1}{n\Delta T} \sum_{i=1}^{n+1} A_i,
\end{align*}
\]
for \( A_i \) matrices defined by Equation (29).

The solution to the linear time-invariant system in Equation (33) is found from linear systems theory to be
\[
\begin{align*}
M''_{avg}(t = T_{obs}) &= \exp(\Omega''_{avg}\eta T) M_0 \\
&= \exp(\sum_{i=1}^{n+1} A_i) M_0. \quad (36)
\end{align*}
\]

Specifically, at the end of the pulse where \( T_{obs} = N\Delta T \), the averaged magnetisation vector can be calculated to be
\[
M''_{avg}(N\Delta T) = \exp \left( \sum_{i=1}^{N+1} A_i \right) M_0. \quad (37)
\]

The above equation states that the solution to the averaged system can be found by one exponential of summation of matrices. If the error of the averaged solution is small enough, then averaging techniques justifies that the error made by writing Equation (28) in form of exponential of summation of matrices is negligible regardless of the fact that the \( A_i \) matrices do not commute. Simulation results show that the error of averaging is small enough for adiabatic passages as expected.

5.2 Error Analysis of Averaging

Equation (19) which represents the Bloch equation in the triple prime frame before discretising the excitation pattern may be rewritten as
\[
\begin{align*}
\begin{bmatrix} \dot{M}''_x \\ \dot{M}''_y \\ \dot{M}''_z \end{bmatrix} &= \begin{bmatrix} \frac{\dot{\alpha}(t)}{\omega_{eff}(t)} \\ \frac{\dot{\alpha}(t)}{\omega_{eff}(t)} \\ \frac{\dot{\alpha}(t)}{\omega_{eff}(t)} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\tilde{f}(t) \\ 0 & 0 & -\tilde{g}(t) \\ \tilde{f}(t) & \tilde{g}(t) & 0 \end{bmatrix} \begin{bmatrix} M''_x \\ M''_y \\ M''_z \end{bmatrix}. \quad (38)
\end{align*}
\]

in which
\[
\begin{align*}
\tilde{f}(t) &= \omega_{eff}(t) \cos \int_0^t \omega_{eff}(\tau) d\tau, \\
\tilde{g}(t) &= \omega_{eff}(t) \sin \int_0^t \omega_{eff}(\tau) d\tau.
\end{align*}
\]

The adiabatic Condition states that
\[
\epsilon(t) \triangleq \frac{|\dot{\alpha}(t)|}{\omega_{eff}(t)} \ll 1 \quad \forall t \in [0, \tau_p), \quad (40)
\]
then in the following system
\[
\begin{align*}
\dot{M}'' &= \frac{\dot{\alpha}}{\omega_{eff}} \tilde{M}''(t) M'' = \epsilon(t) \tilde{\Omega}''(t) M''.
\end{align*}
\]

If we define
\[
\epsilon_M \triangleq \max \{|\epsilon(t)|\} \quad \forall t \in [0, \tau_p), \quad (42)
\]
then in the following system
\[
\begin{align*}
\dot{M}''_{avg}(t) &= \epsilon_M \tilde{\Omega}''_{avg}(t) M''_{avg} \quad (43)
\end{align*}
\]
the main matrix, \( \tilde{\Omega}''_{avg} \), is independent of \( \epsilon_M \), and averaging can be applied to this system. The averaging approximation is \( O(\epsilon_M) \) on a time scale of \( O(\frac{1}{\tau_{\alpha}}) \) (Sanders et al., 2007). Since we have chosen the maximum value for \( \epsilon(t) \), thus for any \( \epsilon < \epsilon_M \) the error of averaging is smaller than this case, which is the worst case, based on the comparison lemma.\(^3\) This error analysis clearly states that as the adiabatic factor, \( \eta(t) \), for an adiabatic pulse increases the error of averaging decreases.

The averaged solution derived in Section 5.1 was performed after discretising the excitation pattern. The error in this case consists of the averaging error and the discretisation error which is proportional to the step size, \( \Delta T \). Therefore, the total error is of the form \( O(\epsilon_M) + O(\Delta T^2) \).

\(^3\) \( \tilde{\Omega}'' \) is a skew symmetric matrix, and therefore the sign of \( \dot{\alpha}(t) \) does not make a difference in the argument presented here.
In this paper we studied the behaviour of adiabatic passages through a first order averaging technique used in classical rotating frame of reference. Denote by the single prime notation, is negligible as well.

This error analysis will be illustrated through simulations in the following section. It is worth mentioning that since matrices $\sigma_1(t)$ and $\sigma_2(t)$ indicated by Equations (13) and (16) are both rotation matrices with unity norm, the error caused by averaging in $M^{\sigma_1}(t)$ and $M^{\sigma_2}(t)$ are equal in the norm sense. Therefore, the error of the averaged solution in classical rotating frame of reference, denote by the single prime notation, is negligible as well.

6. SIMULATION RESULTS
In order to observe the behaviour of the magnetisation vector in the novel rotating frame and to investigate the efficiency of the averaged solution the Bloch equation and the averaged Bloch equation in the introduced rotating frame are simulated for a hyperbolic secant adiabatic pulse. Figure 5 show the simulation results for the hyperbolic secant pulse introduced in Section 3. As can be observed from these simulations the error for the averaged solution of the hyperbolic secant pulse is less than one percent. The simulation results agree with the error analysis presented in Section 5.2.

7. CONCLUSIONS
In this paper we studied the behaviour of adiabatic passages through a first order averaging technique used in nonlinear dynamical systems theory. A surprising result is that in this novel representation of the Bloch equation, the solution is given by a single matrix exponential, and therefore it is extremely computationally efficient to observe the behaviour of bulk magnetisation under adiabatic pulse excitations. The error analysis and the simulation results demonstrate the negligible error of averaging technique for adiabatic passages. It was shown that as the adiabatic passage becomes more efficient, the averaged solution results in a more accurate answer. The method can be directly applied to aid the design of adiabatic passages in MRI as well as NMR through optimising the adiabatic modulation functions which will result in adiabatic passages with larger adiabatic factors. In our future work, we will use extremum seeking technique along with the averaged solution presented here to design optimal adiabatic modulation functions leading to novel adiabatic passages.

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