Fixed-Order $H_\infty$ Controller Design for Systems with Polytopic Uncertainty

Arash Sadeghzadeh ∗

∗ Department of Electrical Engineering, Faculty of Engineering, Razi University, Kermanshah, Iran (e-mail: a.sadz@razi.ac.ir)

Abstract: This note provides sufficient conditions for fixed-order $H_\infty$ controller design for SISO systems with polytopic uncertainty. New conditions are derived based on the concept of robust strict positive realness of an uncertain polynomial with respect to a parameter dependent polynomial. The main feature of the proposed method is that the non-convex set of fixed-order stabilizing controllers is approximated with an inner set which is not necessarily a convex set. The quality of this approximation depends on an initial stabilizing controller. The proposed conditions are in terms of solution to a set of linear matrix inequalities. The effectiveness of the proposed approach is demonstrated by comparing with the existing results.

Keywords: Robust control; Polytopic uncertainty; Convex optimization; Linear matrix inequality; Strictly positive real system; Fixed-order controller; $H_\infty$ performance

1. INTRODUCTION

Polytopic representation is a general way of describing the lack of knowledge on the physical system parameters. This kind of uncertainty includes the interval parametric uncertainty (Bhattacharyya et al. (1995)). Moreover, it can cover the ellipsoidal parametric uncertainty, deduced from the classical well-known identification methods (Sadeghzadeh and Momeni (2011)), as well as multi-model systems.

The robust controller synthesis problem for systems with polytopic uncertainty is in general NP-hard. The classical approach is $\mu$ synthesis which leads to a non-convex optimization problem and hence computationally intractable. To overcome the non-convexity, several fundamentally different approaches for solving these problems are presented. Although robust stability analysis for systems with polytopic uncertainty is considered by many authors (see e.g., Peaucelle et al. (2000), and de Oliveira et al. (1999)) but synthesis approaches are scarce. A popular approach to solving this problem is the Lyapunov approach. Sophisticated sufficient conditions may be derived in the context of Lyapunov theory (see, e.g., Apkarian et al. (2001)). The paper Geromel and Korogui (2006) provides sufficient stability conditions for continuous time polytopic systems using parameter dependent Lyapunov functions. In Cao and Lin (2004), a descriptor system approach is taken to deriving Linear Matrix Inequality (LMI) conditions for robust stability of polytopic systems and the affine parameter systems. In Rantzer and Megretski (1994), a synthesis method is presented that is based on infinite dimensional Youla-Kucera parametrization. This approach for synthesis problem cannot handle fixed-order controller design with an order less that that of the plant model and hence cause significant drawbacks in practical applications. The convex parametrization of fixed-order stabilizing controllers for polytopic systems is given in Henrion et al. (2003) and Khatibi et al. (2008). Robust fixed-order $H_\infty$ controller design for systems with polytopic uncertainty is investigated in Yang et al. (2007), Karimi et al. (2007), and Khatibi and Karimi (2010). In the existing methods for fixed-order $H_\infty$ controller design, a convex set of stabilizing controllers is parameterized such that the closed-loop characteristic polynomial divided by a so called central polynomial is a strictly positive real (SPR) transfer function. The quality of this parametrization is connected to the choice of the SPR-maker (central polynomial).

This note revisits the problem of fixed-order $H_\infty$ controller design for systems with polytopic uncertainty. Sufficient conditions for fixed-order $H_\infty$ controller design is provided in this paper, using the equivalence between robust stability and $H_\infty$ norm constraint, resulting in a perhaps simpler formulation with respect to Khatibi and Karimi (2010). The developed results can be viewed as a state space counterpart of the polynomial approach by Yang et al. However, the main contribution of this paper is to introduce the parameter dependent central polynomial for fixed-order robust controller design for polytopic systems. It is a well-known fact that the stability domain in the space of polynomial’s parameters is non-convex for polynomials with order higher than two (Ackermann (1993)). In the existing methods for robust fixed-order controller design, this non-convex domain is approximated with an inner convex set, using a fixed central polynomial. In our proposed method, the main point is that the approximation of the non-convex domain is not necessarily a convex set and the quality of approximation is related to an initial controller. However, the proposed conditions are established in terms of linear matrix inequalities. Undoubtedly, this may reduce the conservatism.

The rest of the paper is structured as follows. In the next section, fixed-order robust control design for a nominal system is investigated. Section 3, shows how to design $H_\infty$ controller for the polytopic systems. Two cases of fixed central polynomial and parameter dependent central poly-
2. FIXED-ORDER CONTROLLER DESIGN

In this section, the problem of fixed-order controller design for a nominal system is investigated. Consider the transfer function of a nominal discrete-time linear time-invariant SISO system

\[ G(z, \theta) = \frac{N(z, \theta)}{M(z, \theta)} = \frac{\theta_0 z^n + \theta_1 z^{n-1} + \cdots + \theta_p}{z^n + \theta_1 z^{n-1} + \cdots + \theta_{n-1}}, \]

where \( N(z, \theta) \) and \( M(z, \theta) \) are polynomials and parameter \( \theta = [\theta_0 \ \theta_1 \ \cdots \ \theta_{n-1}]^T \in \mathbb{R}^n \) is a vector that parameterizes \( G \). We consider a standard negative feedback configuration shown in Fig. 1. The goal is to design a fixed-order controller

\[ K(z) = \frac{X(z)}{Y(z)} = \frac{x_0 z^m + x_1 z^{m-1} + \cdots + x_m}{z^m + y_1 z^{m-1} + \cdots + y_m}, \]

such that stabilizes the system and the closed-loop system achieves the performance specification.

The performance for a closed-loop system is often defined via the modulus of the frequency of the four different closed-loop sensitivity functions \( Z \) (Zhou et al. (1995)). Usually, it is needed to shape these sensitivity functions to achieve certain performance specifications and to cope with unstructured uncertainty. In this paper, for simplicity of presentation, the control objective is to guarantee a certain upper bound \( \gamma \) on the \( H_{\infty} \) norm for only one weighted transfer function. However, the results can straightforwardly be applied when more than one sensitivity function should be shaped. In this case, the LMIs developed in this paper should be duplicated for each sensitivity function and only the controller parameters are common in the LMIs. Therefore, performance specification will be formulated as:

\[ ||W(z)H(z)||_\infty = \frac{||S(z)||}{||L(z)||_\infty} < \gamma. \]

where \( H(z) \) is a sensitivity function and \( W(z) \) is a desired weighting function. \( S(z) \) and \( L(z) \) are two polynomials.

The following lemmas are required to proceed:

**Lemma 1.** (KYP Lemma (Landau et al. (1997))) The transfer function \( H(z) = C(zI-A)^{-1}B + D \) is SPR if and only if there exists matrix \( P = P^T > 0 \) such that:

\[ \begin{bmatrix} A^T PA - P & A^T PB - C^T & C^T \\ B^T PA - C & B^T PB - D - D^T \end{bmatrix} < 0. \]

**Lemma 2.** Consider an SPR transfer function

\[ H(z) = \frac{S(z)}{L(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \cdots + s_0}{z^n + l_0 z^{n-1} + \cdots + l_0} \]

with the controllable state space realization \((A, B, C, D)\) that satisfies the inequality constraint (4), then Lyapunov equation for the stable polynomial \( S(z) \) is satisfied with the same \( P \) as in (4).

**Proof.** See Appendix A.

In virtue of the Small Gain Theorem, robust stability of uncertain polynomial

\[ L(z) + \delta S(z), \quad ||\delta||_\infty < \gamma^{-1} \]

is equivalent to the \( H_{\infty} \) performance constraint (3). On the other hand, in the case of one block of the uncertainty, the notations of robust stability for complex parameter variations and quadratic stability for real parameter variations are equivalent (Rotea et al. (1993), Khargonekar et al. (1990)). Therefore, similar to proof of Corollary 1 in Yang et al. (2007), we can consider \( \delta \) being a real parameter and then apply the quadratic stability condition on the uncertain polynomial \( L(z) + \delta S(z) \).

**Theorem 3.** Given a stable polynomial \( E(z) \), a fixed-order controller \( K(z) \), given by (2), stabilizes the system and \( H_{\infty} \) performance constraint (3) is satisfied for the closed-loop system, if there exist a symmetric matrix \( P = P^T \) and a scalar \( \tau \) such that

\[ \begin{bmatrix} A^T PA - P & A^T PB - C^T & C^T \\ B^T PA - C & B^T PB - D - D^T \end{bmatrix} \begin{bmatrix} J^T - \delta C^T \\ J - \delta C_s \end{bmatrix} < 0, \]

where, \((A, B, C_s, D_s)\) and \((A, B, C_l, D_l)\) are controllable canonical form realizations of transfer functions \( S(z)/E(z) \) and \( L(z)/E(z) \), respectively.

**Proof.** Suppose that \((A, B, C_s, D_s)\) and \((A, B, C_l, D_l)\) are controllable canonical form realizations of transfer functions \( S(z)/E(z) \) and \( L(z)/E(z) \), respectively. Therefore, \((A, B, C_l + \delta C_s, D_l + \delta D_s)\) is controllable canonical form realization of uncertain transfer function \((L(z) + \delta S(z))/E(z)\). In virtue of Lemma 2, \( L(z) + \delta S(z) \) will be quadratic stable if state space realization of transfer function \((L(z) + \delta S(z))/E(z)\) satisfies the inequality constraint (4) with a common Lyapunov matrix \( P \) for all admissible real \( \delta (\delta^2 < \gamma^{-2}) \). Therefore, substituting \((A, B, C_l + \delta C_s, D_l + \delta D_s)\) in the inequality constraint (4) results

\[ \begin{bmatrix} A^T PA - P & A^T PB - C^T & C^T \\ B^T PA - C & B^T PB - D - D^T \end{bmatrix} \begin{bmatrix} J^T - \delta C^T \\ J - \delta C_s \end{bmatrix} < 0, \]

where, \( J = B^T PA - C_l \). The Schur complement formula (Boyd et al. (1994)) gives that (7) is equivalent to the following two conditions

\[ Z < 0, \]

\[ B^T PB - D_l - D_l^T - \delta D_s - \delta D_s^T < 0, \]

\[ -(J - \delta C_s)Z^{-1}(J^T - \delta C_s^T) < 0, \]

where, \( Z = A^T PA - P \). The later inequality can be written as

\[ \begin{bmatrix} \delta & \frac{1}{1} \\ \frac{1}{1} & \frac{1}{1} \end{bmatrix} \begin{bmatrix} -C_sZ^{-1}C_s^T & C_sZ^{-1}J^T - D_s \\ JZ^{-1}C_s^T - D_s^T \end{bmatrix} \begin{bmatrix} \delta \\ 1 \end{bmatrix} < 0, \]

where, \( Y = B^T PB - D_l - D_l^T - JZ^{-1}J^T \). We thus have shown that (7) is equivalent to (8) and (10). On the other hand, the inequality constraint \( \delta^2 < \gamma^{-2} \) can be considered as
This means that we would like (10) to hold for all $\delta$ for which (11) holds. Therefore, (10) and (11) will be combined using the S-procedure (Boyd et al. (1994)). Thus, We have

$$\tau > 0,$$

$$\left[ -\tau \gamma^2 - C_s z^{-1} C_s^T, C_s z^{-1} j \tau - D_s \right] \Delta < 0,$$

where, $\Delta = \mathbf{Y} + \tau$. Another use of Schur complement formula results that (13) is equivalent to $\Delta < 0$, 

$$-\tau \gamma^2 - C_s z^{-1} C_s^T - (C_s z^{-1} j \tau - D_s) \Delta^{-1} (J Z_s^{-1} C_s^T - D_s^T) < 0.$$  

(15)

Conditions (8) and (14) are equivalent to

$$\Pi = \left[ \begin{array}{ccc} Z & J & J^T \\ B^T P B & D_l & D_l^T + \tau \end{array} \right] < 0.$$  

(16)

The inverse of matrix $\Pi$ (Zhou et al. (1995)) is

$$\Pi^{-1} = \left[ \begin{array}{ccc} Z^{-1} + Z^{-1} j \tau \Delta^{-1} j \tau Z_s^{-1} & \Delta^{-1} & -Z_s^{-1} j \tau \Delta^{-1} \\ -\Delta^{-1} & -\Delta^{-1} Z_s^{-1} \end{array} \right]$$

(17)

condition (15) can be reformulated as

$$-\tau \gamma^2 - [C_s D_s] \Pi^{-1} \left[ \begin{array}{c} C_s^T \\ D_s^T \end{array} \right] < 0.$$  

(18)

Finally, conditions (17) and (18) are equivalent to

$$\left[ \begin{array}{ccc} A^T P A & A^T P B - C_s^T \\ B^T P A - C_i & B^T P B - D_l - D_l^T + \tau \end{array} \right] < 0.$$

(19)

Since $E(z)$ is a stable polynomial, we can ignore the condition $P > 0$. Also condition (19) includes $\tau > 0$. This concludes the proof.

Controller parameters appear linearly in matrices $C_s$ and $C_i$. Since, condition (6) is an LMI with respect to $\tau$, $P$ and $x_0, x_1, \ldots, x_m, y_1, \ldots, y_m$, this condition can be used for fixed-order $H_\infty$ controller design.

Remark 4. The condition (6) is an alternative formulation for the proposed condition in Yang et al. (2007), in the state space realization. Moreover, in comparison with the result of Khatibi and Karimi (2010), the above theorem provides a simpler formulation for fixed-order $H_\infty$ controller design.

3. CONTROLLER DESIGN FOR SYSTEMS WITH POLYTOPIC UNCERTAINTY

3.1 Fixed Central Polynomial Approach

In this section, we consider fixed-order $H_\infty$ controller design problem for systems with polytopic uncertainty. A polytopic system is defined with its $q$ vertices such that the $i$-th vertex is $\theta_i = [\theta_{0i} \theta_{1i} \cdots \theta_{(n-1)i}]^T \in \mathbb{R}^n$, that represents the transfer function

$$G(z, \theta_i) = \frac{N(z, \theta_i)}{M(z, \theta_i)} = \frac{\theta_{0i} z^p + \theta_{1i} z^{p-1} + \cdots + \theta_{pi}}{z^q + \theta_{(p+1)i} z^{q-1} + \cdots + \theta_{(n-1)i}}.$$  

Therefore the whole polytope is given by

$$\mathcal{P} = \{ G(z, \theta) = \frac{N(z, \theta)}{M(z, \theta)} \mid \theta = \sum_{i=1}^{q} \lambda_i \theta_i, \lambda_i \geq 1, \sum_{i=1}^{q} \lambda_i \}.$$  

The parameter $\theta$ is a positive linear combination of the parameters of the vertices. Therefore, if the condition of Theorem 3 for fixed-order controller design for nominal systems is satisfied for all the vertices, we can conclude that the $H_\infty$ performance specification would be fulfilled for all the systems in the model set.

Proposition 5. Given a stable polynomial $E(z)$ (Central polynomial), a fixed-order controller $K(z)$, given by (2), stabilizes the uncertain closed-loop system and the $H_\infty$ performance

$$\| W(z) H(z, \theta) \|_{\infty} = \| S(z, \theta) L(z, \theta) \|_{\infty} < \gamma$$

is satisfied for the polytopic system, given by (20), if there exist symmetric matrices $P_i = P_i^T$ and scalars $\tau_i$ such that for $i = 1, \ldots, q$

$$\left[ \begin{array}{ccc} A_i^T P_i A_i & A_i^T P_i B - C_i^T \\ B_i^T P_i A_i & B_i^T P_i B - D_i - D_i^T + \tau_i \end{array} \right] < 0,$$

(21)

where, $(A_i, B_i, C_i, D_i)$ and $(A, B, C_i, D_i)$ are controllable canonical form realizations of transfer functions $S(z, \theta)/E(z)$ and $L(z, \theta)/E(z)$ for $i = 1, \ldots, q$, respectively.

This proposition gives only sufficient conditions for fixed-order robust controller design. The main source of conservatism is the choice of the central polynomial, $E(z)$. For the choice of the central polynomial some rules of thumb are suggested in Henrion et al. (2003).

3.2 Parameter Dependent Central Polynomial Approach

One of the key points in the robust fixed-order controller design by convex optimization is the existence of a central polynomial. This polynomial should be chosen such that if divided by the characteristic closed-loop polynomial, the resulting transfer function is SPR. Choice of the central polynomial is the main source of conservatism for the fixed-order controller design. In the fixed-order robust control design for systems with polytopic uncertainty, usually a common central polynomial is considered for all the vertices that causes more conservatism (see e.g. Yang et al. (2007), Karimi et al. (2007)). In this section, instead of a fixed central polynomial, we consider an affine central polynomial, which will reduce the conservatism of the approach. The subsequent example reveals the importance of the choice of the central polynomial.

Example 1. Consider the following closed-loop characteristic polynomial:

$$R(z, \theta, k) = z^3 + 70z^2 + kz + \theta$$

for an uncertain system. The uncertain parameter $\theta$ is a member of $U$. Where,$$
U = \{ \theta \mid 0.28 < \theta < 0.3 \ or \ -0.3 < \theta < -0.28 \}.$$  

$R(z, \theta, k)$ is Schur stable for all values of $1.4 \leq k \leq 1.4704$. Now, suppose that the goal is to find the minimum value of $k$ such that $R(z, \theta, k)$ is Schur stable, using the concept
of SPRness. It can be easily verified that for example for two members $R(z, 0.29, 1.4)$ and $R(z, -0.29, 1.4)$, the phase difference exceeds $\pi$ for some values of $0 \leq \omega \leq \pi$. Consequently, one cannot find a fixed central polynomial (SPR-maker) $E(z)$ such that $R(z, \theta, 1.4)/E(z)$ is SPR for all values of $\theta \in U$. This means that $k = 1.4$ would not be found using a fixed central polynomial. Now, assume that the initial value $k_0 = 1.45$ which results a stable characteristic polynomial is available ($k_0$ is not necessarily minimum admissible value of $k$). Parameter dependent polynomial $R(z, \theta, 1.45) = z^3 + 70z^2 + 1.45z + \theta$ makes the polynomial $R(z, \theta, 1.4)$ SPR for all values of $\theta \in U$ (i.e. $R(z, \theta, 1.4)/R(z, \theta, 1.45)$ is SPR for all values of $\theta \in U$). Obviously, considering the parameter dependent central polynomial $R(z, \theta, 1.45)$, utilizing line search, leads to the minimum value of $k = 1.4$, whereas a fixed central polynomial is not applicable. This example shows the effectiveness of considering the parameter dependent central polynomial. For more examples on this subject see Sadeghzadeh et al. (2011).

Now, suppose that a fixed-order stabilizing controller $K_0$ is available, which stabilizes the polytopic system (without any specific performance). For each of the vertices, this stabilizing controller results the following closed-loop characteristic polynomial

$$E_i(z) = L(z, \theta)|_{K=K_0, \theta=\theta_i}$$

(22)

In the sequel, we consider the above stable polynomials for fixed-order controller design. Utilizing this set, one can construct a parameter dependent central polynomial for each of the systems in the model set.

To proceed, we need the following lemma:

**Lemma 6.** (Peaucelle and Arzelier (2001)) Let $I$, $\Phi$, and $\Sigma$ be matrices of appropriate dimensions. Then, the following two statements are equivalent.

(1) $\begin{bmatrix} I & \Sigma \end{bmatrix}^T \Phi \begin{bmatrix} I \\ \Sigma \end{bmatrix} > 0$.

(2) There exists a matrix $Q$ such that

$$\Sigma + \Phi^T Q^T + Q \Phi - I > 0.$$

**Proof.** This lemma is a particular case of elimination lemma.

The inequality constraint (6) can be reformulated as

$$0 < \begin{bmatrix} I & \Sigma \end{bmatrix}^T \begin{bmatrix} P & C_i^T \\ C_i & D_i + D_i^T - \tau - D_i^T \end{bmatrix} \begin{bmatrix} I \\ \Sigma \end{bmatrix}$$

(23)

In virtue of Lemma 6, there exists a matrix $Q$ such that the above inequality can be written as

$$+ Q \begin{bmatrix} A B 0 \end{bmatrix} > 0.$$

The following proposition introduces the sufficient conditions for the fixed-order $H_\infty$ controller design method for the systems with polytopic uncertainty based on the parameter dependent central polynomial concept.

**Proposition 7.** Given a stabilizing controller $K_0(z)$ (without any specific performance), a fixed-order controller $K(z)$, given by (2), stabilizes the uncertain closed-loop system and the $H_\infty$ performance

$$\|W(z)H(z, \theta)\|_\infty = \left\| \frac{S(z, \theta)}{L(z, \theta)} \right\|_\infty < \gamma$$

is satisfied for the polytopic system, given by (20), if there exist symmetric matrices $P_i = P_i^T$ and scalars $\tau_1$ and a matrix $Q$ such that for $i = 1, \ldots, q$

$$\begin{bmatrix} P_i & C_i^T \\ C_i & D_i + D_i^T - \tau_1 - D_i^T \end{bmatrix} > 0 + Q \begin{bmatrix} A I 0 \end{bmatrix} > 0$$

(25)

where, $(A_i, B, C_i, D_i)$ and $(A_i, B, C_i, D_i)$ are controllable canonical form realizations of transfer functions $S(z, \theta)/E_i(z)$ and $L(z, \theta)/E_i(z)$ for $i = 1, \ldots, q$, respectively.

**Remark 8.** The stabilizing controller $K_0(z)$ can be computed e.g. by the approach in Yang et al. (2007) or can be the conservative controller computed by a fixed central polynomial (the presented approach in Section 3.1).

**Remark 9.** Since $E_i(z)$ for $i = 1, \ldots, q$ are polynomials with constant coefficients, therefore the above matrix inequalities, given by (25), are LMIs with respect to controller parameters, matrix $Q$, $P_i$ and $\tau_1$ for $i = 1, \ldots, q$.

**Remark 10.** For a typical system $G(z, \theta_0)$ in the polytopic set, given by (20), where $\theta_0 = \sum_{i=1}^{q} \lambda_i \theta_i$, the considered central polynomial (SPR-maker) for the fixed-order controller design is $\sum_{i=1}^{q} \lambda_i E_i(z)$, which is a parameter dependent central polynomial.

4. NUMERICAL ILLUSTRATION

This section provides an example that demonstrates how effective the parameter dependent central polynomial approach is and that it is an improvement over existing methods. Consider the problem of robust controller design for the same third order system as in Karimi et al. (2007), which is affected by the polytopic uncertainty. The problem is to design a robust controller for the following system,

$$G(z) = \frac{z + \theta_0}{z^3 + \theta_1 z^2 + \theta_2 z + \theta_3},$$

(26)

with $\theta_0 = -0.2, \theta_1 = -1.2, \theta_2 = 0.5$ and $\theta_3 = -0.1$, where all the parameters are uncertain up to $\pm 20\%$ of their nominal values. Therefore, the parametric uncertainty is in the form of a polytope with $2^q = 16$ vertices.
The goal is to design a second order robust controller, given by (2), based on the following optimization problem:

$$\min_{z_0, z_1, z_2, y_0, y_1, y_2, \gamma} \gamma$$

Subject to:

$$\left\| \frac{1}{1 + K(z)G(z, \theta)} \right\|_\infty < \gamma,$$

for all the systems in the polytopic set.

Where the results of section 3 is used. The smallest feasible $\gamma$, which is obtained by a line search, is the optimal solution to this optimization program, which can be solved using YALMIP (Löfberg (2004)) with SDPT3 (Toh et al. (1999)).

At the first step, the following stabilizing controller (without any specific performance) is computed by the presented approach in Yang et al. (2007), using a central polynomial $(z - 0.1)^5$.

$$K_0(z) = \frac{0.4476z^2 - 0.3564z + 0.1615}{z^2 + 0.5019z - 0.2075}$$

Based on the results of section 3.2, the optimization program (27)-(28) results the following controller:

$$K(z) = \frac{0.5233z^2 - 0.3375z + 0.1467}{z^2 + 0.7119z - 0.2155},$$

and the optimal value of $\gamma_{opt.} = 1.75$ is obtained. Utilizing the designed controller $K(z)$, the sensitivity function magnitude Bode diagrams related to all the 16 vertices of the polytope are shown in Fig. 2.

Now, for the comparison purposes, we solve this problem with the proposed approaches in Khatibi and Karimi (2010) and Yang et al. (2007). In addition, the approach of Section 3.1 is employed for robust controller design for the polytopic system. We consider two different central polynomials for each one. The first one is the computed characteristic polynomial using the stabilizing controller $K_0(z)$ with the nominal system $G(z)$.

$$E_{nom.} = z^5 - 0.6981z^4 + 0.1379z^3 + 0.133z^2 - 0.0637z + 0.05304$$

The later is the central polynomial $E_0 = (z - 0.1)^5$ used for computing the stabilizing controller. The results are summarized in Table 1. In the case of a fixed central polynomial, it seems that these three approaches provide a same formulation for fixed-order robust controller design however it has not been clearly proved yet.

### Table 1. $\gamma_{opt.}$ for different approaches for robust controller design

<table>
<thead>
<tr>
<th>Approach</th>
<th>$E_{nom.}$</th>
<th>$E_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Khatibi and Karimi (2010)</td>
<td>2.25</td>
<td>1.95</td>
</tr>
<tr>
<td>Yang et al. (2007)</td>
<td>2.25</td>
<td>1.95</td>
</tr>
<tr>
<td>Fixed Cen. Poly. App. (Section 3.1)</td>
<td>2.25</td>
<td>1.95</td>
</tr>
<tr>
<td>Far. Dep. Cen. Poly. App. (section 3.2)</td>
<td>1.75</td>
<td></td>
</tr>
</tbody>
</table>

The presented approaches in Khatibi and Karimi (2010), Yang et al. (2007) and also the proposed method in Section 3.1 cause the same results for $\gamma_{opt.}$. Whereas, using the parameter dependent central polynomial approach for the robust controller design causes a better result. It is worthwhile to mention that we do not pretend to give a complete answer to the question of the choice of the central polynomial. However, if a stabilizing controller is available, we can build a better central polynomial for improving the performance. We believe that if the performance specifications are relaxed and we seek only a stabilizing controller the choice of the central polynomial is less restrictive. For more details on the subject of parameter dependent central polynomials see Sadeghizadeh et al. (2011).

### 5. CONCLUSION

A new LMI-based robust $H_\infty$ performance condition with respect to the polytopic uncertainty has been proposed. We have improved the results perviously available for the fixed-order $H_\infty$ controller design for SISO systems introducing a parameter dependent SPR-maker. The quality of the established conditions depends on an initial stabilizing controller. Since, controller parameters are decision variables, any controller structure, such as PID, can be considered. The capability of the proposed method has been illustrated via a numerical example.

### ACKNOWLEDGEMENTS

I am grateful to Professor Alireza Karimi for the helpful discussions on the subject of robust control design for the systems with parametric uncertainty.

### REFERENCES


**Appendix A. PROOF OF LEMMA 2**

**Proof.** Controllable state space realization of the transfer function $H(z)$ is given by

$$
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& & & & \\
-l_0 & -l_1 & -l_2 & \cdots & -l_{r-1}
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
& & & & \\
1
\end{bmatrix},
C = \begin{bmatrix}
s_0 - s_r l_0 & s_1 - s_r l_1 & \cdots & s_{r-1} - s_r l_{r-1}
\end{bmatrix},
D = s_r.
\tag{A.1}
$$

The inequality constraint (4) can be written as

$$
-PC^T - C^T - D - D^T P [A B] < 0
\tag{A.2}
$$

We pre- and post-multiply (A.2) by matrix

$$
\begin{bmatrix}
I_r & \vdots \\
\vdots & \ddots \\
0 & \cdots & 0 & 1
\end{bmatrix},
$$

and its transpose, respectively. We obtain Lyapunov equation

$$
\bar{A}^T P \bar{A} - P < 0,
$$

where,

$$
\bar{A} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& & & & \\
-s_0 s_r^{-1} & -s_1 s_r^{-1} & -s_2 s_r^{-1} & \cdots & -s_{r-1} s_r^{-1}
\end{bmatrix}
$$

is the block companion matrix associated with polynomial $S(z)$. Therefore, we can see that the Lyapunov equation is satisfied with the same $P$ as in (4).