Time sub-optimal nonlinear PI and PID controllers applied to longitudinal headway car control

Minh-Duc Hua*, Claude Samson**

* I3S UNS–CNRS, 2000 route des Lucioles, 06903 Sophia-Antipolis, France (e-mail: minh.hua@polytechnique.org)
** INRIA-Méditerranée, 2004 route des Lucioles, 06902 Sophia-Antipolis, France (e-mail: Claude.Samson@inria.fr)

Abstract: Simple nonlinear PI and PID controllers combining time-(sub)optimality with linear control robustness and anti-windup properties are presented for first-order and second-order integrator systems, without assuming that the control lower-bound and upper-bound are the opposite of each other. A complementary contribution is the introduction of an integral action with anti-windup properties into the control law, under the constraint of ensuring global asymptotic stability. For illustration purposes, the proposed PID solution is applied to the longitudinal headway control of a vehicle following another vehicle.

1. INTRODUCTION

Proportional-integral (PI) and proportional-integral-derivative (PID) controllers are at the heart of control engineering practice and, owing to their relative simplicity and satisfactory performance for a wide range of processes, have become the standard controllers used by industry. Following an estimation of Koivo and Tanttu (1991), perhaps only 5-10% of man-implemented control loops cannot be controlled by single-input single-output (SISO) PI or PID controllers. However, this widespread usage also goes with numerous problems due to either poor tuning practice or limited capabilities offered by standard PI-PID schemes. These problems have in turn periodically revived the interest from the academic research community in order to work out complementary explanations and solutions (cf., Aström and Hägglund (1995), O’Dwyer (2009)). In particular, a well-known source of degradation of performance is the occurrence of control saturation, when the boundedness of the “physical” control that can be applied to the system under consideration is no longer compatible with the application of the (theoretically unbounded) calculated control value. This has the consequence of invalidating the performance index established on the assumption of linearity of the controlled system, and can give rise to various undesired (and unnecessary) effects such as multiple bouncing between minimal and maximal values of the control, and important overshoots of the regulated error variables. The so-called integrator wind-up phenomenon, which worsens the overshoot problem and the reduction of which still motivates various research studies (cf., Aström and Hägglund (1995), Seshagiri and Khalil (2005)) is also commonly presented as a consequence of control saturation combined with the integral action incorporated in the control law in order to compensate for unknown (slowly varying) additive perturbations.

Compared to the already huge corpus of studies devoted to PI and PID controllers, the present paper has the limited ambition of proposing new nonlinear versions of these controllers that attempt to combine the constraints of control saturation with i) the objective of optimizing the control action in order to reduce the size of initially large tracking errors as fast as possible, and ii) the design of integral action terms with limited wind-up effects. The former issue is close to the line of research on “proximate” time-optimal for linear systems admitting closed-form time-optimal solutions (cf., Workman et al. (1987), Pao and Franklin (1993), Newman (1990)). The present study is restricted to the simplest first and second order linear systems. In particular, continuous nonlinear proportional (P) and proportional derivative (PD) state feedbacks depending continuously on an extra-parameter whose convergence to infinity yields the discontinuous time-optimal controls for these systems will be derived and will form the cores of the PI and PID controllers proposed subsequently. As for the latter issue, it is related to the work on anti-windup and “conditional integrators” (cf., Seshagiri and Khalil (2005)). The present work is also related to the theme of bounded control design based on the use of nested saturation functions, e.g., Teel (1992), Marconi and Isidori (2000), with the same concern of proving global asymptotic stability of the desired set-point, but with a different way of designing the control solutions.

In the second part of the paper, the proposed nonlinear PID controller is applied to the longitudinal headway control of a car following a leader. The reason for choosing this application is its good fit with the design constraints and objectives imposed on the control and its performance, namely the existence of different bounds on the car’s acceleration and deceleration capabilities, control effectiveness in terms of time of convergence to the desired inter-distance between the two vehicles, absence of bouncing transients—for the comfort of the passengers, fuel economy, and reduced wear-off of mechanical parts—, and very small overshoot in order to avoid collisions with the leader. This type of application has also motivated numerous studies in the last two decades (cf., Hatipoğlu
et al. (1996), Zhang et al. (1999), Martínez and de Wit (2007)). The results here presented are based on a simple model of the system’s dynamics which would obviously call for several refinements of practical relevance, and they by no means aim at covering the subject in depth. The purpose is just to point out a novel and simple PID solution which basically addresses the same issues as in Hatipoglu et al. (1996), with the economy of a switching strategy, and that experts on the subject might consider in the future.

2. RECALLS: TIME-OPTIMAL CONTROLS (TOC) FOR FIRST-ORDER AND SECOND-ORDER INTEGRATORS AND CONTINUOUS FEEDBACK APPROXIMATIONS

2.1 First-order integrator system

Let \( M > 0 \) and \( m < 0 \) denote two real numbers. Let \( \text{sat}_{m}^{M} \) denote the saturation function defined on \( \mathbb{R} \) by

\[
\text{sat}_{m}^{M}(x) = \begin{cases} 
M & \text{if } x \geq M \\
n & \text{if } x \in [m, M[ \\
m & \text{if } x \leq m
\end{cases}
\]

(1)

To simplify the notation, we will write \( \text{sat}^{M}(x) \) instead of \( \text{sat}_{m}^{M}(x) \) when \( m + M = 0 \). The results described in this paper could in fact be adapted to a more general class of saturation functions, including those based on the use of \( \tanh \). Define also the discontinuous \( \text{sign}_{m}^{M} \) function as

\[
\text{sign}_{m}^{M}(x) = \begin{cases} 
M & \text{if } x > 0 \\
n & \text{if } x = 0 \\
m & \text{if } x < 0
\end{cases}
\]

(2)

One remarks that \( \text{sign}_{m}^{M}(x) = \lim_{k \to +\infty} \text{sat}_{m}^{M}(kx) \). Consider the first order integrator

\[
\dot{x} = u,
\]

(3)

with the control variable \( u \) such that \( m \leq u \leq M \). The TOC associated with this system that takes \( x \) to zero in minimal time can be written as

\[
u(x) = \text{sign}_m^M(-x).
\]

Due to the discontinuity at the desired set-point, this is not a good feedback law. Indeed, even though it theoretically stabilizes \( x = 0 \) asymptotically (when considering solutions of the controlled system defined in the sense of Filippov, for instance), it is excessively sensitive to measurement noise and chattering, and its discretization systematically renders the origin unstable. A continuous approximation of this optimal control, endowed with better robustness properties around the origin, is given by

\[
u(x) = \text{sat}_m^M(-k_p x), \quad k_p > 0,
\]

(4)

with the approximation improving uniformly (in terms of reaching a given small neighborhood of \( x = 0 \) from any initial condition) by increasing the value of \( k_p \). Locally, near the origin, this latter control is equal to the proportional feedback law \( u(x) = -k_p x \). It thus locally inherits the properties of this linear feedback control, whereas it approximates the TOC when the “error” \( x \) is initially large. In practice, the gain \( k_p \) can be tuned according to Linear Control Theory rules, typically in relation to control sampling, measurement noise, and additive perturbation issues addressed by control performance and robustness analyses.

2.2 Second-order integrator system

Consider now the second-order integrator

\[
\ddot{x} = u,
\]

(5)

with the same bound constraints as previously, i.e. \( m \leq u \leq M \). Set

\[
s_o(x, \dot{x}) = \begin{cases} 
x + 0.5x\dot{x}/a & \text{if } x + 0.5\dot{x}\dot{x}/a \neq 0 \\
0 & \text{if } x + 0.5\dot{x}/a = 0, x^2 + \dot{x}^2 \neq 0 \\
x + 0.5\dot{x}/a & \text{if } x = 0, x^2 + \dot{x}^2 = 0
\end{cases}
\]

The TOC associated with this system that takes \( x \) to zero in minimal time can be written as (see, e.g., Athans and Falb (1966), Kirk (1970))

\[
u(x, \dot{x}) = \text{sign}_m^M(-s_o(x, \dot{x})) \quad \text{with } a = \begin{cases} 
M + a & \text{if } x \geq 0 \\
-m & \text{if } x < 0
\end{cases}
\]

a simplification of which is

\[
u(x, \dot{x}) = \text{sign}_m^M(-x + \dot{x}/a) \quad \text{with } a = \begin{cases} 
M & \text{if } x \geq 0 \\
-m & \text{if } x < 0
\end{cases}
\]

This feedback law is discontinuous at points \( (x, \dot{x}) \) where \( x + \frac{\dot{x}}{2a} = 0 \) \( (a = M, -m) \), and also on the line \( x = 0 \). It is in particular discontinuous at the desired equilibrium \( (x = 0, \dot{x} = 0) \). To ensure continuity at this point one may consider the following approximation:

\[
u(x, \dot{x}) = \text{sat}_m^M(-k_p(x + \dot{x}/a)) \quad \text{with } a = \begin{cases} 
M & \text{if } x \geq 0 \\
-m & \text{if } x < 0
\end{cases}
\]

where \( k_p \) plays the role of a “proportional gain”. Going further in this direction, an approximation which is continuous everywhere is given by

\[
u(x, \dot{x}) = \text{sat}_m^M(-k_p(x + \dot{x}/a + \varepsilon)) \quad \text{with } a = \begin{cases} 
M - m & \text{if } x \geq 0 \\
-m & \text{if } x < 0
\end{cases}
\]

(7)

and \( \varepsilon \) (a small) positive number. Note that \( a(x, \varepsilon) \) is constant and equal to \( M \) in the classical case when \( M + m = 0 \). Now, a shortcoming of the above approximations is that they do not yield a (local) rate of convergence uniformly as fast as exponential, due to the quadratic velocity correction term involved in the time-optimal feedback law. This can be taken care of by adding a complementary linear velocity term as follows

\[
u(x, \dot{x}) = \text{sat}_m^M(-k_p(x + \dot{x}/a + \varepsilon)) - \text{sat}_m^M(k_p \varepsilon),
\]

(8)

with \( l > 0 \) bounding the interval on which this linear term is the most active, and \( k_p > 0 \) playing the role of a “derivative gain”. Indeed, the linear approximation of the above feedback at the desired equilibrium \( (x = 0, \dot{x} = 0) \) is the classical PD controller \( u(x, \dot{x}) = -k_p x - k_v \dot{x} \), whose proportional and derivative gains, \( k_p \) and \( k_v \), can be determined by applying classical rules of Linear Control Theory. For instance, for the double integrator system \( \ddot{x} = u \), the choice \( k_v = 2\sqrt{k_p} \) yields two closed-loop poles equal to \( -\sqrt{k_p} \) and ensures a critically-damped response with no overshoot. As for the choice of the parameter \( l \), it corresponds to a compromise between (local) robustness –as provided by a linear PD feedback– and performance when starting far away from the desired equilibrium –as provided by the time-optimal nonlinear control. Newman (1990) studies the particular case where \( M + m = 0 \) and, by considering a sliding-mode formulation, derives a nonlinear PD controller which is essentially the same as (8).

2.3 Stability and convergence

Let us analyze the stability and convergence properties associated with the time sub-optimal controller (4) (resp.
applied to the first-order integrator system (3) (resp. second-order integrator system (5)). From the fact that the linear approximations of these controllers coincide with classical P and PD feedbacks one can already deduce that they are local exponential stabilizers. The following lemma points out that they are in fact global asymptotic stabilizers.

**Lemma 1.** The nonlinear proportional feedback control (4) globally asymptotically stabilizes \( x = 0 \) for the first-order integrator system (3). The nonlinear proportional-derivative feedback control (8), with \( a(x, \varepsilon) \) chosen either positive constant or according to relation (7) with \( 0 < \varepsilon < \min(-M, M)/k_p \), globally asymptotically stabilizes \( \bar{x} = 0 \) for the second-order integrator system (5).

**Proof:** For the P controller (4) and the PD controller (8) with \( a(\cdot) \) constant, the proofs of both lemma's statements is obtained by applying classical Lyapunov function techniques. For the first-order case, consider the function defined by \( \mathcal{V}_1(x) = 0.5x^2 \), whose time-derivative along any solution to the closed-loop system is given by \( \dot{\mathcal{V}}_1 = x\text{sat}_M^k(-k_p x) (\leq 0) \), with the time index omitted for the sake of notation simplification. The resulting boundedness of \( \mathcal{V}_1(x) \) along any solution to the controlled system yields the stability of the point \( x = 0 \), whereas the convergence of \( \mathcal{V}_1 \) to zero yields the convergence of \( x \) to zero, i.e. the desired convergence property.

As for the second-order system, let us assume that \( a(\cdot) \) is positive constant. We note that the control may then be written as \( u = \text{sat}_m^k((-k_p x + g(\dot{x}))) \), with \( g(s) = k_p |s|/2a + sat^k(bu) \).

Therefore, \( g'(s) > \min\{k_v, k_p |/(k_v a) \} > 0, \forall s \), with \( g'(s) \) denoting the (right) derivative of \( g \) at \( s \). Moreover, \( \text{sat}_m^k(s) > 0, \forall s \neq 0 \). Consider the positive function

\[
\mathcal{V}_2(x, \dot{x}) = \frac{k_v \dot{x}^2}{2} + \int_{k_p x + g(\dot{x})}^{k_v |x|/a} \text{sat}_m^k(s) ds,
\]

whose time-derivative along any solution to the closed-loop system is given by

\[
\dot{\mathcal{V}}_2 = -g'(|\dot{x}|)(\text{sat}_m^k(-k_p x + g(\dot{x})))^2 \leq 0.
\]

The resulting boundedness of \( \mathcal{V}_2(x, \dot{x}) \) along any solution to the controlled system yields the stability of the point \( (x, \dot{x}) = (0, 0) \). The convergence of \( \mathcal{V}_2 \) to zero implies that \( \text{sat}_m^k(-k_p x + g(\dot{x})) \) tends to zero, and thus that \( k_p x + g(\dot{x}) \) tends to zero. Therefore, since \( |\dot{x}| \) is bounded, \( \dot{\mathcal{V}}_2 \) and \( \text{sat}_m^k(k_v |x|/a - k_p x + g(\dot{x})) \) tend to zero. Using the fact that \( g(\dot{x})/\dot{x} \) is bounded, this in turn implies that \( x \) and \( \dot{x} \) tend to zero. The proof of the lemma in the case where \( a(\cdot) \) is defined by (7) is more involved and is reported in Hua and Samson (2010).

We have seen so far that the controllers (4) and (8) are exponential stabilizers of the origins of the first-order and second-order integrator systems respectively, that they are continuous approximations of corresponding discontinuous TOCs, and that the approximations are all the better (in terms of functional approximation) than \( k_p \) is large and \( \varepsilon \) is small. In fact, these feedback controls are well-conditioned alternative to the original TOCs when it comes to simulate the solutions to the controlled systems by using classical Runge-Kutta numerical integration packages. In practice, the use of large control gains poses a number of well-known robustness problems in relation to various implementation issues (modeling errors, control discretization, measurement noise, etc.), so that the tuning of these gains is needed to reach an acceptable performance/robustness compromise. Nevertheless, an important practical shortcoming of these controllers is that they do not preserve the convergence to the desired equilibrium as soon as a non-zero constant—or slowly varying, in practice—additive perturbation acts on the system. Adding an integral action to the control law is the common way to correct this problem. In the next section, we propose a technique to complement the previously derived P and PD controllers with such an action, by taking into account the bounds imposed on the control magnitude, with the concerns of limiting wind-up effects and of preserving the global asymptotic stability properties of the original controllers.

3. INTEGRAL ACTION COMPLEMENTATION

3.1 First-order integrator system

We consider the first-order integrator with a complementary constant (unknown) perturbation input \( c \)

\[
\dot{x} = u + c.
\]

We further assume that the perturbation magnitude is not too large. More precisely, we assume that \( |c| < \min\{M, -m\} \) to ensure that the problem of global asymptotic stabilization of \( x = 0 \) has a solution despite the bounds imposed on the control input. Rather than using a pure integrator of \( x \) in the control law, Seshagiri and Khalil (2005) propose to use a “bounded” integral term \( z \) calculated as follows

\[
\dot{z} = k_z (z + \text{sat}^\delta(z + x)), \quad |z(0)| < \delta_z,
\]

with \( k_z \) and \( \delta_z \) denoting positive numbers. This relation indicates that we have a pure integrator \( \dot{z} = k_z x \) as long as \( |z + x| \leq \delta_z \), and also that \( |z(t)| \leq \delta_z \) and \( |\dot{z}(t)| \leq 2k_z \delta_z \), \( \forall t \geq 0 \). Therefore, by defining \( \dot{z}_{\text{max}} \) as the maximal value that \( |\dot{z}(t)| \) is allowed to take, one has

\[
\delta_z = \dot{z}_{\text{max}}/(2k_z),
\]

and it is possible to modify the magnitudes of \( z \) and \( \dot{z} \) at will via the choice of the parameters \( \dot{z}_{\text{max}} \) and \( k_z \). We will see that the first of these parameters characterizes the importance given to the integral action at the control level, whereas \( k_z \) enters in a simple way the calculation of the gains associated with the linear PI controller of which the proposed nonlinear controller is a local approximation at \( (x, z) = (0, 0) \). Let us proceed with the control design itself. A way to complement the nonlinear proportional feedback (4) with and integral action consists in conceptually replacing the initial state \( x \) by the modified state

\[
\bar{x} \equiv x + z,
\]

determining a control which asymptotically stabilizes the augmented state \( (\bar{x}, z) = (0, 0) \) when \( c \equiv 0 \). In this case, and with the above definition of \( z \), the augmented control system writes as

\[
\begin{cases}
\dot{\bar{x}} = u + v(\bar{x}, z) \\
\dot{\bar{z}} = w(\bar{x}, z)
\end{cases}
\]

with \( v(\bar{x}, z) = k_z(\bar{x} + \text{sat}^\delta(\bar{x})) \). The time-suboptimal feedback (4) can be considered to asymptotically stabilize \( \bar{x} = 0 \). Set

\[
\begin{cases}
\dot{\bar{z}}_{\text{max}} < \min\{M, -m\} \\
\bar{M} \equiv M - \dot{\bar{z}}_{\text{max}} (> 0) \\
\bar{m} \equiv m + \dot{\bar{z}}_{\text{max}} (< 0)
\end{cases}
\]

(12)
This yields the feedback controller
\[ u(\bar{x}, z) = \text{sat}_{\bar{M}}(\bar{k}_p \bar{x}) - v(\bar{x}, z) = \text{sat}_{\bar{M}}(\bar{k}_p \bar{x}) - \bar{z}(z + \text{sat}_{\bar{M}}(\bar{x})) \] (13)

which, by construction, takes its values in the interval \([m, M]\), and whose linear approximation at \((\bar{x}, z) = (0, 0)\) is the linear PI controller \(u = -\bar{k}_p \bar{x} - k_f f x \) with \(k_p = k_p + k_z = k_p k_z\). The corresponding closed-loop poles are real negative and equal to \(-k_p\) and \(-k_z\) respectively.

**Lemma 2.** The nonlinear PI feedback control \((10) - (13)\) globally asymptotically stabilizes \((x, z) = (0, c/k_p)\) for the perturbed augmented system \((9) - (10)\), provided that \(0 < z_{\text{max}} < \min\{-m, M\} - |c|\) and \(0 < \bar{k}_z < \frac{1}{2\max(-m, M)}\).

**Proof:** Define \(\tilde{x} \equiv \bar{x} - c/k_p\) and \(\tilde{z} \equiv z - c/k_p\). The desired stability property is equivalent to the global asymptotic stabilization of \((\bar{x}, \bar{z}) = (0, 0)\). From the system’s equation and control expression, using the fact that
\[ \text{sat}_{\bar{M}}(x - \varepsilon) + \varepsilon = \text{sat}_{\bar{M} + \varepsilon}(x), \]

one easily verifies that \(\dot{x} = \text{sat}_{\bar{M} + \varepsilon}(-k_p \bar{x})\), along any closed-loop solution. Using the condition on \(z_{\text{max}}\) one deduces that the time-derivative of \(\tilde{z}\) is negative whenever \(\tilde{x} \neq 0\). This in turn implies that \(\dot{z}\) tends to zero and \(z = 0\) is globally asymptotically stable. Skipping technical arguments of minor importance, it remains to show that \(\tilde{z}\) is asymptotically stable on the zero dynamics defined by \(\dot{x} = 0\). From (11), using the fact that the condition upon \(k_z\) implies \(\delta_z > |c|/k_p\) and thus \(\text{sat}_{\bar{M}}(c/k_p) = c/k_p\), the evolution of \(\tilde{z}\) on zero dynamics is given by \(\dot{\tilde{z}} = -k_z \tilde{z}\). The desired property follows directly.

Choosing \(z_{\text{max}}\) small does not impede the compensation of perturbations almost as large as the control bounds, and also limits the degradation of the control in terms of time-(sub)optimality. On the other hand, this imposes to use a small gain \(k_z\) with the risk of much penalizing the integral action embedded in (15). From these equations one can already deduce that the evolution of \(\tilde{z}\) is globally asymptotically stable on the zero dynamics defined by \(\dot{x} = 0\), when \(\bar{k}_z < k_z\) chosen as control bounds, yields the nonlinear PID feedback control
\[ u = \text{sat}_{\bar{M}}(\bar{k}_p \bar{x} + \frac{\bar{x} k_z}{2\bar{M}(\bar{x}, \bar{z})}) + \text{sat}(k_z \bar{z}) - v(\bar{x}, \bar{z}), \] (17)

with \((\bar{x}, \bar{z})\) either positive constant or calculated according to (7) with \(M\) and \(m\) replaced by \(M\) and \(m\) respectively. Since \(|v(x, z)| < z_{\text{max}}\) along any solution to the controlled system, this control takes its values in the interval \([m, M]\). One easily verifies that the linear approximation of the (augmented) closed-loop system at \((\bar{x}, \tilde{x}, \bar{z}) = (0, 0, 0, 0)\) is
\[ \begin{cases} \dot{x} = -k_p \bar{x} - k_z \bar{z}^2 \\ \dot{\bar{z}} = -k_z \bar{z}^2 + k_p \bar{z} + k_z \bar{z} \end{cases} \]

From these equations one can already deduce that the above controller is a (local) exponential stabilizer of \((\bar{x}, \tilde{x}, \bar{z}) = (0, 0, 0, 0)\), and thus also of \((x, \tilde{x}, \bar{z}) = (0, 0, 0)\), when \(c \equiv 0\). The following lemma establishes a stronger asymptotic stability property when \(a\) is positive constant and the perturbation \(c\) is not exceedingly large (to the point of rendering the stabilization problem untractable).

**Lemma 3.** With \(a(x, \varepsilon)\) chosen either positive constant or according to relation (7) with \(M\) and \(m\) replaced by \(\bar{M}\) and \(\bar{m}\) respectively, the nonlinear PID feedback control \((16) - (17)\) globally asymptotically stabilizes \((x, \tilde{x}, \bar{z}) = (0, 0, 0, 0)\) for the perturbed augmented system \((14) - (15)\), provided that \(0 < \tilde{z}_{\text{max}} < \min\{-\bar{M}, -\bar{m}\} - |c|\), \(\delta_z > |c|/k_p\), and \(0 < \varepsilon < \frac{\min(|-\bar{M}, -\bar{m}|, |c|)}{k_p}\).

**Proof:** Let us first assume that \(a(\cdot, \cdot) = 0\) is a positive constant. Define \(\tilde{x} = \bar{x} - c/k_p\). One easily verifies that along any solution to the controlled system
\[ \dot{\tilde{x}} = \text{sat}_{\bar{M} + \varepsilon}(\bar{k}_p \tilde{x} + \frac{\tilde{x} k_z}{2\bar{M}(\tilde{x}, \tilde{z})}) - \text{sat}(k_z \tilde{z}). \]

From this equation and the condition imposed on \(z_{\text{max}}\) one deduces, via a minor adaptation of the proof of Lemma 2, that \((\bar{x}, \tilde{x}) = (0, 0)\) is globally asymptotically stable. It then suffices to work on the zero dynamics defined by \(\dot{x} = 0\), i.e. \(x = c/k_p\), to prove the global asymptotic stability of \((\bar{z}, \bar{z}) = (c/k_p, 0)\). The technical arguments which justify the previous statement rigorously are classical and omitted for the sake of concision. Define \(\tilde{z} = z - c/k_p\). In view of (15), when \(\bar{x} = c/k_p\) and \(\delta_z > |c|/k_p\), the evolution of \(\tilde{z}\) is given by

\[ \tilde{z} = \frac{\bar{M}}{k_p} x \text{ when } |\bar{x}| \text{ is small. This latter equation points out the integral action embedded in (15). Note also that, if } x + z = 0 \text{ and } |z| < \tilde{z}_{\text{max}}, \text{ then the evolution of } z \text{ is given by the autonomous second-order equation } \ddot{z} + k_z z + k_p z = 0. \]
\[ \ddot{z} = -kv_z \dot{z} + \text{sat}\frac{\dot{z}}{\delta_{z}} \dot{z} - h(z) \dot{z}, \]

with \( h(z) \equiv -\frac{1}{2} \text{sat}\frac{\dot{z}}{\delta_{z}} \dot{z} (> 0, \forall z) \). Consider the positive function \( V \) defined by \( V(z, \dot{z}) = 0.5 \dot{z}^2 + \int_0^z h(s) \text{d}s \). Using the above equation of evolution of \( \dot{z} \), the calculation of the time-derivative of this function yields

\[ \dot{V} = -kv_z \dot{z}^2 \text{sat}\frac{\dot{z}}{\delta_{z}} + \frac{1}{2} \left( \frac{\dot{z}}{\delta_{z}} \dot{z} \right)^2. \]

As for the convergence issue, \( V \) tends to zero, and so does \( \dot{z} \). From there, one shows that \( \ddot{z} \) is uniformly continuous and thus, by application of Barbalat’s Lemma, that \( \dot{z} \) tends to zero. In view of the equation of evolution of \( z \), the convergence of \( \ddot{z} \) and \( \dot{z} \) to zero in turn implies that \( \ddot{z} \) tends to zero. For the proof we refer to Appendix A for details.

A few words concerning the choice of the parameters \( \delta_z \) and \( \delta_{z_{\text{max}}} \) are in order. The term \( \delta_z \) should be chosen larger than \( c/k_p \), but not much larger so as to avoid the possible occurrence of useless large values of \( z \) favoring large overshoots. As for \( \delta_{z_{\text{max}}} \), a compromise has to be found between a small value which minimizes the important of the integral action, and thus also its negative effects (overshoot and performance degradation in terms of time optimality, in particular), and a larger value which allows for faster desaturation of the integral term \( z \).

4. APPLICATION TO LONGITUDINAL HEADWAY CAR CONTROL

Alike other studies on this subject, the control design is here addressed by considering a simple model of the car’s dynamics with motorization and braking components schematized to the extreme, the idea being to work out a rough sketch of solutions before going to the stage of adaptation to an actual physical system. The problem statement and modeling equations here considered are basically those of Hatipoglu et al. (1996), with the noticeable exception of aerodynamic drag forces which are not modeled in this reference. We believe that it is important to take these forces into account because the intensity of their sum rapidly increases approximately like the square of the vehicle’ velocity until its reaches its maximal value, corresponding to the vehicle’s maximal velocity, when it exactly matches the maximal traction force produced by the vehicle’s engine. Let us briefly recall the simplified longitudinal headway control problem that we are addressing:

- the control variable \( u \) is the vehicle’s acceleration/deceleration capacity. Assuming that the maximal motor-traction force \( F_m \) and braking force \( F_b \) are constant and known, together with the vehicle’s mass, one has \( u \in [m, M] \) with \( m = -F_b \) and \( M = \frac{F_m}{\delta_{z_{\text{max}}}^2} \);
- the longitudinal dynamics of the controlled vehicle is given by Newton’s law: \( \ddot{z} = u - k_d |u| \dot{v}, \) with \( u \) denoting the vehicle’s abscessa along the road, measured from an arbitrary fixed point, \( v = \dot{z} \) the vehicle’s velocity, and \( k_d \) the drag coefficient related to the vehicle’s maximal velocity \( v_{\text{max}} \) by the relation \( v_{\text{max}}^2 = M/k_d \);
- the leading vehicle’s abscessa and velocity are denoted as \( d \) and \( v_\ell \), respectively, and the desired inter-distance between the two vehicles, here assumed constant and independent of \( v_\ell \) for the sake of simplification, is denoted as \( \Delta_v \).

Define \( x \equiv d - d_v + \Delta_v \), the control objective is to asymptotically stabilize \( x = 0 \) as efficiently as possible via the calculation of \( u \). Combining the above model of the vehicle’s dynamics with the definition of \( x \) yields the control system \( \ddot{x} = u - k_d |u| \dot{v}, \) \( |u| \dot{v} \equiv -k_d |d + v_\ell - v_\ell| - \dot{x} \). Although the perturbation \( c \) is not constant in this case, it should tend to the constant value \(-k_d |v_\ell - v_\ell| \) when the leader’s velocity \( v_\ell \) is constant. If \( v_\ell \) varies slowly, it should also vary slowly. This observation, combined with the fact that \( v_\ell \) is not known in advance, intuitively justifies the idea of applying nonlinear PID controller (16)-(17) of the previous section. The simulation results reported below have been performed by assuming that the inter-distance \( d - d_v \) and the difference of velocities \( v - v_\ell \) between the two vehicles are (precisely) measured on-line, so that \( x \) and \( \dot{x} \) are also measured and can be used directly in the control calculation. In practice, the measurement of the inter-distance can be obtained by using various optical devices (cameras, laser range-finders, etc.), but its time-derivative often has to be estimated. Such an estimator is given in Hua and Samson (2010).

For the simulations, we have used the function \( a \) given by (7) with \( M = \frac{v_{\text{max}}}{\delta_{z_{\text{max}}}} = \frac{M}{k_d} = 40 \text{m/s}^2 \) replaced by \( \bar{M} \) and \( \bar{n} \) respectively, and the various parameters involved in the system’s dynamics and control calculation have been chosen as follows:

- Control bounds: \( M = 3 \text{m/s}^2, m = 9 \text{m/s}^2 \);
-_drag coefficient and maximal velocity: \( k_d = 1.875 \times 10^{-3} \text{m}^{-1}, v_{\text{max}} = \sqrt{M/k_d} = 40 \text{m/s} \);
- Control gains and other parameters: \( k_p = k_{p_2} = 2, k_v = k_{v_2} = 2 \sqrt{K_p}, \delta = 1, l = 20, \bar{z}_{\text{max}} = 0.1, \delta_z = (M - \bar{z}_{\text{max}})/k_d = 1.45 \).

Moreover, knowing that the integral correction term \( z \) is useful only when the error \( x \) is not too large, we have modified the calculation of \( z \), initially given by (15), as

\[ \bar{z} = -kv_z \dot{z} + \text{sat}\frac{\dot{z}}{\delta_{z}}(-z + \text{sat}\dot{z} \bar{z} + x \text{bell}_{\nu_{\ell}}(x))). \]

with \( \text{bell}_{\nu_{\ell}}(x) = \frac{\tanh((x + \nu)/\nu) + \tanh((-x + \nu)/\nu)}{2\tanh(\nu/\nu)} \)

a bell-shaped symmetric function with a peak value equal to one at \( x = 0 \) and which tends to zero when \( |x| \) tends to +\( \infty \). The parameter \( \nu \) characterizes the width of the bell, i.e. the size of the interval in which the contribution of \( x \) to the calculation and evolution of \( z \) is the most important, whereas \( s \) characterizes the steepness of the bell’s sides. For the reported simulations we have chosen \( \nu = 10 \) and \( s = 1 \). One easily verifies that this modification only changes the function \( h \) in the proof of Lemma 3 without affecting its positivity property on which the result relies.

The results shown in the figures correspond to three situations involving three distinct leader’s velocities (\( v_\ell \) \( = 20, 35 \text{m/s} \)). The initial inter-distance error is equal to 100 meters in all cases, and the follower’s velocity is initially equal to the leader’s velocity. The simulation figures show the time-evolution of \( i) \) the inter-distance (Fig. 1 and 2), \( ii) \) the difference of velocities between the follower and the leader (Fig. 3), \( iii) \) the control intensity (Fig. 4), and \( iv) \) the correction integral term (Fig. 5). They illustrate important performance differences between the proposed nonlinear PID controller and a (saturated) classical linear PID controller. In particular, the latter would yield important overshoots and several ineffective control-sign changes. One can also observe the consistency and robustness of the system’s response for
different velocities that involve a large spectrum of drag forces, and the quasi absence of overshoot in all situations, despite the integral correction term which allows for the convergence of tracking error to zero. We also note that for large values of \(|x|\) the sign of \(x\) is always negative. This is coherent with the context of the application according to which the control is used to “catch up” with the leader, prior to stabilizing the inter-distance at zero. Therefore, the control can be implemented with \(a(\cdot, \cdot)\) constant and equal to \(-\bar{a}\) without a significant change in performance. However, in this latter case the control would not be optimal for (large) positive initial values of \(x\) – a virtual possibility, since it means that the leader is initially (far) behind the follower.

---

**REFERENCES**


