On Expansion of Gramians and Cross-Gramians of Continuous Control Systems over Spectra of System Dynamics Matrices

I.B. Yadykin
V.A. Trapeznikov Institute of Control Sciences, 65 Profsoyuznaya, Moscow 117997, Russia
(Tel: +7-495-3349031, e-mail: Jad@ipu.ru)

Abstract: The paper suggests a new approach to computing infinite and finite time Gramians and Cross-Gramians relying on the use of the Laplace transformation to matrix exponential time functions and expansion of the product of these functions. The expansions are bilinear and quadratic forms for sequences of Faddeev’s matrices generated by resolvents of original matrices. Matrix identities are obtained for bilinear and quadratic forms of these Faddeev’s sequences. Asymptotic expansions are found for expansion of controllability and observability Gramians of dynamic systems under control modes at time preceding the system arriving to the stability boundary. An illustrative example demonstrates a technique of computing finite time controllability Gramian.

Keywords: matrix integrals, linear continuous stationary systems, Lyapunov and Sylvester differential and algebraic equations, controllability and observability Gramians, Cross-Gramians, Faddeev’s sequences.

1. INTRODUCTION

In recent years we observe a growing interest to studying the properties of Gramians that can be computed in the time domain as an integral of over-weighted products of matrix exponents. These integrals play an important role in the stability theory and in various fields of the mathematical control theory. The integrals are well-known to be solutions to differential and algebraic Lyapunov and Sylvester equations (Ljapunoff, 1947, Sylvester, 1884, Fuhrmann, 1996, Andreyev, 1976). Solutions of these equations appear in the form of the LQR and LQG regulators (Kalman, 1965; Kwakernaak and Sivan, 1991), estimating covariance matrices (Kwakernaak and Sivan, 1991, Poznyak, 2008, Afanasiev et al., 1998), advanced stability theory (Willems and Fuhrmann, 1992, Poljak and Sheherbakov, 2002, Andrievsky and Fradkov, 1999), and mathematical models order reduction topics (Sorensen and Antoulas, 2005) and have been suggested representing a Gramian in the frequency domain on the basis of the dynamics matrix resolvent.

In the author’s paper (Yadykin, 2010) we have formulated the method to computing Gramians and Cross-Gramians expansion by spectra of the Lyapunov and Sylvester algebraic equations matrices, given the spectra is plain and the sum of all kinds the matrices eigenvalues is not a zero. In the next parts of the paper we shall try to give the method differentiation in the issues:

- the case of multiple matrices spectra,
- limit relations for bilinear and quadratic forms, formed by Faddeev’s matrices arrays,
- solving a solution of the Lyapunov and Sylvester matrix differential equations,
- applications to model order reduction problem.

2. PROBLEM STATEMENT

Let us take up two continuous dynamic systems

\[ \dot{x} = Ax + Bu, \quad x(0) = 0, \]
\[ y = Cx, \]

(1)
where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^w \), and
\[
\dot{x}_n = A_{nx} x_n + B_n u, \quad x_n(0) = 0,
\]
\[
y_m = C_n x_n,
\]
where \( x_n \in \mathbb{R}^r, u \in \mathbb{R}^m, y \in \mathbb{R}^w \). We shall consider real square finite dimension, matrices \( A_{nx}, A_{nxy} \), where \( r \) and \( n \) are any finite positive integers. Assume that the systems are completely controllable and observable. The system controllability Gramian of system (1) is known to be determined in the time and frequency domains (Sorensen and Antoulas, 2005) as
\[
P'(t) = \int_0^t e^{Bu} B^T e^{Bu} d\tau, \quad P(\infty) = \int_0^\infty e^{Bu} B^T e^{Bu} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega I - A)^{-1} B^T (\omega I - A^{-1})^{-1} d\omega,
\]
and the observability Gramians are known to be determined by similar expressions. In turn, finite and infinite time cross-Gramians for dynamic systems of form (1), (2) are determined by the expressions
\[
P(t) = \int_0^t e^{Bu} e^{Bu} d\tau, \quad P(\infty) = \int_0^\infty e^{Bu} e^{Bu} d\tau,
\]
provided that the corresponding integrals exist.

Then let us take up a matrix integral of the form
\[
\frac{dP(t)}{dt} = -AP(t) - P(t)A^T + R, \quad P(0) = 0,
\]
\[
AP(\infty) + P(\infty)A^T + R = 0,
\]
\[
\frac{dP(t)}{dt} = -AP(t) - P(t)A_n^T - R, \quad P(0) = 0,
\]
\[
AP(\infty) + P(\infty)A_n^T + R = 0,
\]
and assuming that \( A = A_n, A_n = A_n^T \), substituting expressions of the matrix integral \( P(t) = \int_0^t e^{Bu} e^{Bu} d\tau \) in equation (1) and using its properties one can see that the matrix function \( P(t) \) is a solution to the associated Lyapunov and Sylvester equations (Ljapunoff, 1947; Andreev, 1976; Hanzon and Peeters, 1996). An expansion of matrix resolvents is known to have the form (Faddeev and Faddeeva, 1963):
\[
(A - s)^{-1} = \sum_{j=0}^{\infty} \frac{1}{s^j} A^j, \quad (sI - A_n)^{-1} = \sum_{j=0}^{\infty} \frac{1}{s^j} A_{nj} N_j^{(-1)}(s),
\]
\[
A_j = \sum_{j=1}^{\infty} a_1 A^{j-1}, \quad A_{nj} = \sum_{j=1}^{\infty} a_1 A^{j-1}.
\]

Here \( N(s), N_n(s) \) denote the characteristic polynomials of the matrices
\[
N(s) = a_0 s^n + \ldots + a_1 s + a_0, \quad a_0 = 1,
\]
\[
N_n(s) = a_m s^n + \ldots + a_1 s + a_0, \quad a_0 = 1.
\]

The matrices \( A_{nx}, A_{nxy} \) are referred as the Faddeev’s matrices and can be found by using the Faddeev’s-Leverrier algorithm (Faddeev and Faddeeva, 1963, Hanzon and Peeters, 1996).

We shall study properties of the Gramians and cross-Gramians through studying matrix time functions \( e^{Bu} Re^{Bu} d\tau, P(t) = \int_0^t e^{Bu} Re^{Bu} d\tau \), and this requires an analysis of bilinear and quadratic forms of Faddeev’s-like matrices
\[
\sum_{j=1}^{\infty} A_j R A_{nj}, \sum_{j=1}^{\infty} A_j R A_{nj}, \quad R = \text{rectangular or quadratic matrix of size } [n \times r] \text{ or } [n \times n].
\]

A study of properties of these forms can be conveniently moved to the complex domain and using the Laplace transform of these functions (Yadykin, 2010).

3. A GENERAL APPROACH TO SOLVING DIFFERENTIAL AND ALGEBRAIC LYAPUNOV AND SYLVESTER EQUATIONS

**Theorem 1.** Let us take up a matrix integral of the form
\[
P(t) = \int_0^t e^{Bu} Re^{Bu} d\tau \quad \text{with the real quadratic matrices of finite dimensions } A_{nx} = [a_{ij}], \quad A_{nxy} = [a_{ij}], \quad R_{xy} = [r_{ij}].
\]

Assume that the conditions \( s_k + s_l \neq 0, \forall k = 1, 2, \ldots, n, \forall \lambda = 1, 2, \ldots, r \) are met. Then the following statements are true:

i) if the spectrum of the matrix \( A \) includes multiples of eigenvalues \( s_k \) of multiplicity \( m_k \), the following identity holds
\[
P(s) = s^{m_k} \left( \sum_{j=1}^{\infty} (-1)^j \sum_{l=0}^{j-1} K_{k,l} \frac{d^{j-l}}{ds^{j-l}} \left( \sum_{j=0}^{m_k} \sum_{j=0}^{n_k} A \right)(s) \right) \times A R_{k,l}.
\]

where the matrix \( K_{k,l} \) is found from the formula (Gardner and Barnes, 1942)
\[
K_{k,l} = \frac{1}{(\rho - 1)!} \frac{d^{\rho-1}}{ds^{\rho-1}} \left( (s - s_k)^{\rho} \sum_{j=0}^{m_k} \sum_{j=0}^{n_k} A \right)(s) \left|_{s=s_k} \right.
\]

ii) for all finite times \( t \in [0, \infty) \) and all values of the matrix elements \( A, A_n \), except those ones in which the eigenvalues of both matrices become multiples, and cases in which the condition \( s_k + s_l \neq 0, \forall k = 1, 2, \ldots, n, \forall \lambda = 1, 2, \ldots, r \) does not
hold, in the time and frequency domains the following identities are hold.

ii-a) for cases, where the spectrum of the matrix $A_m$ is multiple

\[ L \{P(t)\} = s^{-1} \left( \sum_{k=1}^{n} \text{Res}(Is - A)^{-1} \left| \int_{s_j} R[I(s - s_k) - A]^{-1} \right| \right), \]

\[ P(t) = L^{-1} \left( s^{-1} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j(s - s_k)^q}{N(s_j) N_m(s - s_k)} A_m R_{mq} \right) , \]

and Res above stands for a residue of the matrix function, while $N'(s) = \frac{dN(s)}{ds}$.

ii-b) for cases where the spectrum of the matrix $A_m$ is plain, the following identities are true:

\[ P(s) = L \{P(t)\} = s^{-1} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \frac{s_j(s - s_k)^j}{N(s_j) N_m(s - s_k)} A_m R_{mq} + \]

\[ \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j s_j^q}{(s_j + s_j) N(s_j) N_m(s - s_k)} \frac{dN_m(s - s_k)}{ds} \bigg|_{s_j} \]

\[ \times \frac{1}{s - s_j - s_k} A_m R_{mq} , \]

\[ P(t) = L^{-1} \left[ s^{-1} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \frac{s_j(s - s_k)^j}{N(s_j) N_m(s - s_k)} A_m R_{mq} + \right] \]

Theorem 1 is proved in Appendix 1. In effect, Theorem 1 specifies a common form for representing solutions to the Lyapunov and Sylvester equations in the frequency and time domains. Relation (4) provides the most general representation of the solutions in the frequency domain without restrictions being imposed on the matrix spectra. Relations (5) to (8) specify a semi-expansion of the solutions in the frequency domain on the spectrum of the matrix that is plain while that of the other matrix may be multiple. The identities (9) and (10) specify a complete spectral solution of the Lyapunov and Sylvester equations in the time and frequency domains on the spectrum of combinational eigenvalues of both matrices provided that the spectra of both matrices are plain.

4. LIMIT RELATIONS AND MATRIX IDENTITIES FOR BILINEAR AND QUADRATIC FORMS OF FADDEEV’S MATRICES, GRAMIANS AND CROSS-GRAMIANS

Another goal of the paper is to study properties of the Gramians and Cross-Gramians that arise in the stability theory and to study the problem of reducing the system model order. Theorem 1 leads to the following corollaries:

**Corollary 1.** Let real matrices $A, A_m$ be Hurwitz with plain spectra. Then for bilinear forms of Faddeev’s matrix generated by matrices $A, A_m$ of the form

\[ \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j(s - s_k)^j}{N(s_j) N_m(s - s_k)} A_m R_{mq} = \]

\[ \equiv \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j(s - s_k)^j}{N(s_j) N_m(s - s_k)} A_m R_{mq} , \]

The identity is proved by applying the theorem on the finite value of the Laplace transformation and properties of being Hurwitz to the identities (9), (10). New matrix identities may be obtained by using the theorem on the initial value of the Laplace transformation and giving up the requirement that the matrices $A, A_m$ are Hurwitz

\[ \lim_{x \to 0} P(t) = \lim_{x \to 0} sL[P(t)] = 0. \]

Let us apply limit relationship (12) to identities (9).

**Corollary 2.** Let real matrices $A, A_m$ be Hurwitz with plain spectra. Then for bilinear forms of the Faddeev’s matrix the following identities hold

\[ \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j(s - s_k)^j}{N(s_j) N_m(s - s_k)} A_m R_{mq} = \]

\[ \equiv -\sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j(s - s_k)^j}{(s_j + s_j) N(s_j) dN_m(s - s_k)} \bigg|_{s_j} A_m R_{mq} , \]

Matrix identities for controllability and observability Gramians may be obtained from the identities (11) and (13) by substituting into them,,..,.,, through $A_j = A_j, A_{mq} = A_{mq}, R = BB^T, n = r$, or

\[ A_j = A_j, A_{mq} = A_{mq}, R = BB^T, n = r . \]

For example,

\[ P^r = \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j(s - s_k)^j}{(s_j + s_j) N(s_j) dN_m(s - s_k)} \bigg|_{s_j} A_m R_{mq} , \]

\[ P^r = \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{s_j(s - s_k)^j}{N(s_j) N_m(s - s_k)} A_m R_{mq} . \]
Theorem 2. Let a linear continuous stationary dynamic system be described by equations of form (1), where the matrices \( A, B, \) and \( C \) are real and the matrix \( A \) is Hurwitz, the roots \( s_k \) of the characteristic equation of the matrix \( A \) be plain and the condition \( s_k + s_{k+1} \neq 0, \forall k = 1, 2, \ldots, n, \forall \lambda = 1, 2, \ldots, n \) be valid.

Then the following limit relationships are valid:

i) if the root \( s_k \) of the characteristic equation is real and \( \Re s_k = \alpha \to -0 \), then

\[
\lim_{Re s_k \to 0} P^\ast = \frac{1}{2\alpha} A_k^T B B^T A_k^T \quad \text{(16)}
\]

ii) if the roots \( s_k, s_{k+1}, s_k = j\omega_k - \alpha, s_{k+1} = -j\omega_k - \alpha \), of the characteristic equation and \( \Re s_k, s_{k+1} = -\alpha \to -0 \), then

\[
\lim_{Re s_k, s_{k+1} \to 0} P^\ast = \frac{1}{2\alpha} \Re \left[ \sum_{k=0}^{\infty} \frac{1}{(\omega)^2} A_k^T B B^T A_k^T \right] \quad \text{(17)}
\]

iii) if the roots \( s_k = \alpha_k, \forall k = 1, 2, \ldots, n \), where \( \alpha_k \) is a negative real root. Then

\[
\lim_{Re s_k \to 0} P^\ast = \frac{1}{2\alpha} \sum_{k=1}^{\infty} \frac{1}{(\alpha_k)^2} A_k^T B B^T A_k^T - \frac{1}{2\alpha} \sum_{k=1}^{\infty} \frac{1}{(\alpha_k)^2} N(\alpha_k) \quad \text{(18)}
\]

The characteristic polynomial of the matrix \( A \) and \( A_0 \) denotes a Faddeev matrix.

Corollary 3. Let the conditions of Theorem 2 be met and also \( s_k = \alpha_k, \forall k = 1, 2, \ldots, n \), where \( \alpha_k \) is a negative real root. Then

\[
\lim_{Re s_k \to 0} P^\ast = \frac{1}{2\alpha} \sum_{k=1}^{\infty} \frac{1}{(\alpha_k)^2} A_k^T B B^T A_k^T \quad \text{(19)}
\]

Theorem and corollaries are proved in Appendix 2.

5. AN ILLUSTRATIVE EXAMPLE

Let us take up the case of control a two-zone furnace (Andreyev, 1976). The plant model is:

\[
\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -0.5 & 0 \\ 0 & -1.0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x(0) = 0, \quad y = Cx.
\]

For this case we have

\[
\text{det}(Is - A) = N(s) = a_2 s^2 + a_1 s + a_0 = (s - s_1)(s - s_2), \quad s_1 = -0.5, s_2 = 1,
\]

\[
a_2 = 1, a_1 = 1.5, a_0 = 0.5, N(s) = 2a_2 s + a_1,
\]

\[
(\text{Is} - A)^{-1} = (A s + A_0) N_0^{-1}(s),
\]

\[
\begin{bmatrix} s + 0.5 & 0 \\ 0 & s + 1 \end{bmatrix}^{-1} = \begin{bmatrix} s + 1 & 0 \\ 0 & s + 0.5 \end{bmatrix} \times (s^2 + 1.5 s + 0.5)^{-1},
\]

\[
A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad BB^T = \begin{bmatrix} 1.25 & 1.5 \\ 1.5 & 4.25 \end{bmatrix},
\]

\[
N_0(s_i) N_0^T(-s_i) = 0.75, \quad N_0(s_i) N_0^T(-s_i) = -1.5.
\]

For this example, the expression of the controllability Gramian takes the form

\[
P^\ast = \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{(s_j)^q}{N(s_j) N(-s_j)} A_j BB^T A_j^T, P^\ast = P^T + P^T. \quad \text{(21)}
\]

Here in the right hand part of the equality one can see a quadratic form made up by the weighted Faddeev’s matrices \( A_j, A_j^T \). Let us compute the values of the matrices \( A_j BB^T A_j^T \).

\[
A_1 BB^T A_1^T = \begin{bmatrix} 0.6125 & 1.5 \\ 0.375 & 2.125 \end{bmatrix}, \quad A_2 BB^T A_2^T = \begin{bmatrix} 1.25 & 0.75 \\ 1.5 & 2.125 \end{bmatrix},
\]

\[
A_1 BB^T A_1^T = \begin{bmatrix} 1.25 & 1.5 \\ 0.75 & 2.125 \end{bmatrix}, \quad A_2 BB^T A_2^T = \begin{bmatrix} 1.25 & 1.5 \\ 1.5 & 4.25 \end{bmatrix}. \quad \text{(22)}
\]

Substituting the expressions (19) and (20) into the formula (21), we obtain infinite time controllability Gramian

\[
P^\ast = \begin{bmatrix} 1.25 & 1 \\ 1 & 2.125 \end{bmatrix}, P^T = \begin{bmatrix} 1.25 & 1 \\ 0 & 2.125 \end{bmatrix}, P^G = \begin{bmatrix} 0 & 0 \\ 1 & 2.125 \end{bmatrix}. \quad \text{(23)}
\]

It follows from expressions (21), (22) that the controllability Gramian is a sum of two semi-positively definite matrices, each of them being correspond one eigenvalue of the matrix \( A \). Each semi-positively definite matrix is a quadratic form in the positively definite matrices \( A_j, A_j^T \). It is easy to see that expression (21) is a solution to the Lyapunov equation of the form \( AP^\ast + P^\ast A^T = -BB^T \).

The matrix \( P^\ast \) is symmetrical and positive-definite. Let us take up now the problem of determining a finite time controllability Gramian. Relation (9) have the form

\[
P^\ast(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{q=0}^{\infty} \frac{s_j^q}{N(s_j) N(-s_j)} A_j BB^T A_j^T + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{q=0}^{\infty} \frac{s_j^q}{N(s_j) N(-s_j)} \frac{dN(s-s_j)}{ds} \eta_{s_j} \eta^T \quad \text{(24)}
\]

Substituting in (24) the values of the roots of the characteristic equation and expressions (19) and (20) we have
\[ \mathcal{P}(t) = \\
\begin{align*}
&= \left( 1.25 - 4e^{-t} + \frac{16}{3} e^{-1.5t} - 2e^{-2t} \right) A_n B B^T A^T t + \\
&\quad + \left( 1 + 2e^{-t} - 4e^{-1.5t} + 2e^{-2t} \right) A_n B B^T A^T t + \\
&\quad \left( 1 + 2e^{-t} - 4e^{-1.5t} + 2e^{-2t} \right) A_n B B^T A^T t + \\
&\quad + \left( 2.125 - 1e^{-t} + \frac{8}{3} e^{-1.5t} - 2e^{-2t} \right) A_n B B^T A^T t.
\end{align*}
\]

As it follows from (25), \( \lim_{t \to \infty} \mathcal{P}(t) = 0 \), identities (11) are true, when one substitutes in them \( A_j = A_n, A_{\infty} = A^\top n \), \( R = B B^\top \), \( n = r \). The Laplace transformation for the finite time Gramian has the form

\[ \mathcal{P}(s) = \\
\begin{bmatrix}
1.25s^2 + 4.5s^3 + 3.75 & 1.5s^2 + 4.5s^3 + 3 \\
1.5s^2 + 4.5s^3 + 3 & 4.25s^3 + 10.625s^4 + 6.375 \\
1.5s^2 + 4.5s^3 + 3 & 4.25s^3 + 10.625s^4 + 6.375 \\
1.5s^2 + 4.5s^3 + 3 & 4.25s^3 + 10.625s^4 + 6.375
\end{bmatrix}.
\]

It can be pointed out that plant matrix transfer function has only two plain poles, but the Laplace transformation for the finite time Gramian has five poles: one pole is zero, and four poles are combined ones, arranged from all kind combinations of plant characteristic equation roots while two poles of them are equal.

6. CONCLUSION

This paper has suggested a general approach to solution of differential or algebraic Lyapunov and Sylvester equations by computing integrals over products of matrix exponents by using the Laplace transforms for matrix functions of time and expansion of one or two functions being multiplied into common fractions. New forms are obtained for controllability and observability Gramians and also for finite and infinite time Cross-Gramians. The paper proves that limits of the matrix forms generate, as \( T \to -0 \) and as \( T \to \infty \), a family of matrix identities of bilinear and quadratic forms for the above sequences of the Faddeev’s matrices. This is valid for any values of the original matrices entries at which the sum of their all kind eigenvalues is not equal to zero (this condition is found to be true for Hurwitz matrices). Asymptotic expansions for finite time controllability and observability Gramians are found to analyze the stability of dynamic system in using control modes at times, when a system is reaching the stability boundary (Sorensen and Antoulas, 2005).

One can consider a set of asymptotically stable linear systems, with the property that for each system represented in the canonical form the corresponding observability and controllability Grammians are equal, diagonal and positively definite. From identities (11), (13) and identities (14), (15) it follows that for given Gramians balanced realization of their spectral expansions are simplified, so if matrices \( A, A_n \) are diagonal, Faddeev’s matrices \( A_j, A_{\infty} \) are diagonal ones also. Motivation for studying balanced realizations is their close relation to the model reduction (Hanzon and Peeters, 1996, Sorensen and Antoulas, 2005), which is, in turn, closely related to the robust control theory (Boyd et al, 1994).

REFERENCES


APPENDIX 1

Proof of Theorem 1. Let \( g_1(t) = e^{\sigma_1 t} \), \( g_2(t) = e^{\sigma_2 t} \). In the light of the properties of the Laplace transform, we have

\[
L\left[ e^{\sigma_1 t} e^{\sigma_2 t} \right] = s^N L[g_1(t)g_2(t)],
\]

where \( L \) denotes computation of the Laplace transform of the product of matrix functions inside the parenthesis. In line with the theorem on convolution in a complex domain we have

\[
L[g_1(t)g_2(t)] = \frac{1}{2\pi j} \int_{c_1-j\infty}^{c_2+j\infty} G_1(s)G_2(s)ds,
\]

for the functions \( g_1(t) \) and \( g_2(t) \) may be Laplace transformed, then

\[
e^{\sigma_1 t} < M_1 e^{\sigma_2 t} \quad \text{or} \quad e^{\sigma_2 t} < M_2 e^{\sigma_1 t},
\]

with being \( c_1 \) the degree of the first exponent growth, \( c_2 \) the degree of the second exponent growth. The above inequalities should be read in the sense of inequalities between the matrices entries. The next inequalities are valid

\[
c_1 > \max \Re s_k, \quad k = 1, 2, \ldots, n, \quad c_2 > \max \Re s_k, \quad \lambda = 1, 2, \ldots, r.
\]

In accordance to the above theorem about the convolution in the complex domain, we have \( c_2 < c' < c - c_1 \), where \( c' \) is an absicissa of absolute convergence of the matrices exponent.

\[
L[g_1(t)g_2(t)] = \frac{1}{2\pi j} \int_{c_1-j\infty}^{c_2+j\infty} G_1(s)G_2(s)ds.
\]

Let us represent the resolvent of a matrix with a plain spectrum as an expansion into plain fractions and use in computing the integral (26) the theorem on bias in a complex field; this will result in formulae (5) to (8). With the above assumptions the function \( G_1(s) \) is a regular fractional rational matrix function and so

\[
L[g_1(t)g_2(t)] = \sum_{j=1}^{\frac{p^*}{2}} \sum_{j=1}^{\frac{p^*}{2}} s_j A R_{j+\frac{1}{2}}(s - s_j) A^T_{j+\frac{1}{2}}.
\]

Substituting (28) into (27) and in the light of (26), we have

\[
L\left[ e^{\sigma_1 t} e^{\sigma_2 t} \right] = \sum_{j=1}^{\frac{p^*}{2}} \sum_{j=1}^{\frac{p^*}{2}} s_j A R_{j+\frac{1}{2}}(s - s_j) A^T_{j+\frac{1}{2}} +
\]

\[
\sum_{j=1}^{\frac{p^*}{2}} \sum_{j=1}^{\frac{p^*}{2}} s_j s_j^* A R_{j+\frac{1}{2}}(s - s_j) A^T_{j+\frac{1}{2}}
\]

\forall t: 0 \leq t < \infty.

Then the identities (9) and (10) follow. Let us consider the case of multiple roots of Laplace transforms for the function \( g_1(t) \). In accordance to the theorem about the Laplace transformation for two real-valued function product (Gardner and Barnes, 1942), we have the Theorem:

Let functions \( f_1(t) \) and \( f_2(t) \) have been Laplace transformed, have \( F_1(s) \) and \( F_2(s) \) as the Laplace transformations, and let \( F_i(s) \) be rational algebraic fraction, having \( n \) various poles \( s_1, s_2, \ldots, s_n \) of the orders \( m_1, m_2, \ldots, m_n \) and also \( m_1 + m_2 + \cdots + m_n = q \). Then

\[
L\left[ f_1(t)f_2(t) \right] = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \left( -1 \right)^{n-k} K_{bj} \left[ \frac{d^{n-k}}{ds^{n-k}} F_i(s) \right]_{s=s_j} \sum_{j=1}^{m_k} \frac{K_{bj}}{(m_j - j)!} \left[ \frac{d^{j-1}}{ds^{j-1}} (s - s_j)^{m_j} F_i(s) \right]_{s=s_j},
\]

where

\[
K_{bj} = \frac{1}{(j-1)!} \left[ \frac{d^{j-1}}{ds^{j-1}} (s - s_j)^{m_j} F_i(s) \right]_{s=s_j}.
\]

Using the resolvent expansions of the matrices \( A, A^T \), reconstituting characteristic polynomial of matrix \( A \) as

\[
N(s) = (s - s_1)^{m_1} \cdots (s - s_n)^{m_n},
\]

rename \( k \) as \( \delta, j \) as \( \rho \), \( f_1 \) as \( g_1, f_2 \) as \( g_2 \) and substituting the above expressions in formulae (29), (30), we obtain formula (4).

APPENDIX 2

Let \( k = 1, \ s_i = -\alpha, \alpha \to -0 \). By virtue of equation (14), we have

\[
P = P_1 + P_2 + P_3, \ P_1 = \frac{1}{2\pi j} \sum_{j=1}^{\frac{p^*}{2}} \sum_{j=1}^{\frac{p^*}{2}} s_j (s - s_j) s_j A B B^T A^T, \ (31)
\]
As all the roots starting with the second one remain unchanged,
\[ \lim_{\alpha \to 0} P_{11} = P_{11}^*, \]
(34)
Substitution of the root value in equation (31) leads to
\[ \lim_{\alpha \to 0} P_{11}^* = \lim_{\alpha \to 0} \frac{\alpha A_b B^T A \alpha + (-\alpha) A_b B^T A \alpha + A_b B^T A \alpha}{N'(-\alpha)N(\alpha)} \]
(35)
Since \( s_1 = -\alpha \), is not a root of the characteristic equation and zero is not the equation’s multiple root because the system is stable, it follows that
\[ \lim_{\alpha \to 0} N'(s_1) \sim N'(-\alpha) \neq 0, \]
\[ \lim_{\alpha \to 0} N(s_1) \sim 2\alpha (-s_2) \cdots (-s_n). \]
(36)
Thence
\[ \lim_{\alpha \to 0} P_{11}^* \sim 1 \frac{A_b B^T A \alpha}{(-2\alpha) N'(-\alpha)(s_1) \cdots (s_n)}. \]
As the matrix \( P_{21}^* \) includes senior powers of \( \alpha \) starting with \( \alpha^2 \), it is true that
\[ \lim_{\alpha \to 0} P_{11}^* = 0_{n \times n}. \]
In the light of (33) and (36), we have (16). Now let us take up case ii). For specificity assume that \( k=1 \), \( s_1 = j \omega - \alpha \), \( s_2 = -j \omega - \alpha \), all the other roots not changing their positions as a pair of complex root does, the addends of the matrix in the right-hand part of equation (31) take the form
\[ P_{11} = \sum_{j=0}^{n-1} \sum_{q=0}^{n-1} \frac{s_j'(-s_q)^0}{N'(s_j)N(-s_q)} A_b B^T A \alpha \]
\[ + \sum_{j=0}^{n-1} \sum_{q=0}^{n-1} \frac{s_j'(-s_q)^0}{N'(s_j)N(-s_q)} A_b B^T A \alpha \]
\[ + \sum_{j=0}^{n-1} \sum_{q=0}^{n-1} \frac{s_j'(-s_q)^0}{N'(s_j)N(-s_q)} A_b B^T A \alpha \]
As before, equality (31) holds. Since the roots \( s_1 \) and \( s_2 \) are complex-conjugate, it is true that
\[ P_{11} = 2\text{Re} \sum_{j=0}^{n-1} \sum_{q=0}^{n-1} \frac{s_j'(-s_q)^0}{N'(s_j)N(-s_q)} A_b B^T A \alpha, \]
(37)