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# Management of a Large Dam via Optimal Price Control ⋆

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**Abstract:** This paper considers the optimal management of a dam with its level approximated by $N$ discrete states and modeled via a continuous-time controlled Markov chain on a finite time horizon. The inflow process is seasonal, and so non-stationary, as are the customer demands. The control takes the form of price, the bounds of which are set by regulators, and we take into account the individual sectoral demands of customers. The general approach to the solution of this problem is to consider the stochastic optimization problem in the average sense and solve it via dynamic programming. We show that an optimal price can be obtained for each state based on minimizing the difference between desired and optimal consumption, and minimizing the probability that the dam level falls below a prescribed level both on the control period and at the terminal time. The result is illustrated by a numerical example.

Keywords: Stochastic control, optimal control, optimization problems, dynamic programming, Markov decision problems.

## 1. INTRODUCTION

Effective water management will be one of the major challenges of this coming century as climate patterns change and populations become more urbanized. For many countries this will require the optimal management of already existing systems of large dams.

Problems of this type are often approached as continuous-time Markov decision processes (MDP) and have produced a large amount of literature (see for example Altman (1999),Bertsekas and Shreve (1996) and Kitaev and Rykov (1995)). Specifically with respect to dams, a range of papers have been written where the input process is a Weiner process or a compound Poisson process and the dam has either finite or infinite capacity (see for example Faddy (1974),Yeh and Hua (1987),Abdel-Hameed (2000),Bae et al. (2003) and Abramov (2007)). Faddy Faddy (1974) makes the point that a Weiner process is not a very realistic model of dam inflows but simplifies some calculations. The more recent papers use a compound Poisson process to describe inflows. All of these authors use very simple threshold control models. These are not optimal on a finite horizon and cannot be applied to more complicated problems with different criteria and true management mechanisms. In all of these papers the main optimality criteria is the long-run expected average criterion. In terms of the long term health of the dam such a criterion makes sense, however, for short term dam management this has real drawbacks. Short term water availability is extremely important for dam management. Also, a long time horizon requires the inflow to be greater than or equal to the outflow, or the system will break down, but in the short term outflow may well exceed inflow. It is usually assumed that the inflow data is stationary but this is not the case with short term seasonal inflows to dams. Finally, a long term average criterion does not take into account the costs of transient states and the resources needed to reach them on a finite interval (Miller, 2009).

Our problem concerns the optimal management of a dam through the medium of a price control over a finite time period. We consider a large dam with its level approximated by a finite number of discrete states and model the dam level dynamics with a continuous-time controlled Markov chain. The general approach is to reduce this stochastic optimization problem to a deterministic control problem with integral and terminal optimality criteria (Miller et al., 2009). We assume that the seasonal demand functions of the customers, the intensity of natural losses and the inflow intensity are deterministic and show via the dynamic programming approach and the numerical solutions of the corresponding system of differential equations, that we can find an optimal price function $p(t)$ for each state. We will set a maximum and minimum bound on the price, $p_{max}$ and $p_{min}$ respectively, such that the control is bounded.

In section 2 we describe how a large dam can be modeled using a controlled Markov chain and give specific details for this application. This will be followed in section 3 by a description of the non-stationary outflow dynamics, which are driven by the interaction of the sectoral demanded consumption and the price placed on water. Section 4

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gives the general form of the performance criteria and the general solution to this type of optimal control problem. Section 5 gives the specific performance criteria used in this application. Section 6 applies the results of the previous sections to the optimal control of a large dam with active sectoral demands. In section 7 we give a numerical example which demonstrates the type of price control obtained and compares the actual demand with the weighted average of the controlled demand on the control interval We conclude with future research directions in section 8.

## 2. THE CONTROLLED DAM MODEL

To model this large dam, assume that we have a dam with one inflow process, an outflow process made up of natural losses and an outflow process made up of water consumed by dam users. We assume that the consumption outflow is controlled by a time and dam level dependent price. We approximate the level of the dam by dividing it into $N+1$ levels, where $N < \infty$ is in $\mathbb{N}$. We then let $L(t) \in \{0, ..., N\}$ be an integer valued random variable which describes the level of the dam at time $t$. The set of $N+1$ possible states of the dam, $S$, can be represented by the set of unit vectors $S = \{e_0, ..., e_N\}$, where $e_i$ is null for $j \neq i$ and 1 at the $i^{th}$ position, in $\mathbb{R}^{N+1}$(Aggoun et al., 1995).

We define all of the processes related with this dam on the probability space $\{\Omega, \mathscr{F}, \mathbb{P}\}$ and specifically $X_t$, where $\{X_t \in S, t \in [0,T]\}$ for $T < \infty$, as a controlled jump Markov process with piecewise constant right-continuous trajectories. With respect to the price control $p(t)$, we make the following assumption.

*Assumption 1.* Assume that the set of admissible controls, $\bar{P} = \{p(\cdot)\}$, is the set of $\mathscr{F}_t^X$-predictable controls taking values in $P = \{p \in [p_{min}, p_{max}]\}$. This ensures that if the number of jumps in the process up to time $t \in [0,T]$ is $N_t$, $\tau_k$ is the time of the $k^{th}$ jump and

$$X_0^t = \{(X_0, 0), (X_1, \tau_1), ..., X_{N_t}, \tau_{N_t})\}$$

is the set of states and jump times, then for $\tau_{N_t} \le t < \tau_{N_{t+1}}$ the price control $p(t) = p(t, X_0^t)$ is measurable with respect to $t$ and $X_0^t$ (Aggoun et al., 1995).

Now suppose that we can approximate the inflows and outflows of the dam by $\mathscr{F}_t^X$-predictable counting processes with unit jumps (we assume this for sake of simplicity). The approximate counting process $Y_{in}(t)$ has an intensity $\lambda(t)I\{L(s) < N\} \ge 0$ and so it is a counting process with the representation

$$Y_{in}(t) = \int_0^t \lambda(s)I\{L(s) < N\}ds + M_{in}(t),$$

where $M_{in}(t)$ is a square integrable martingale with quadratic variation

$$\langle M_{in}\rangle_t = \int_0^t \lambda(s)I\{L(s) < N\}ds.$$

The approximate outflow process, $Y_{out}(t)$, is a function of the controllable consumption rate, $C(t) = C(t, p(t), X)$, which depends on the current price of water and the seasonal demands of customers, and natural losses due to evaporation, which has the deterministic intensity for each

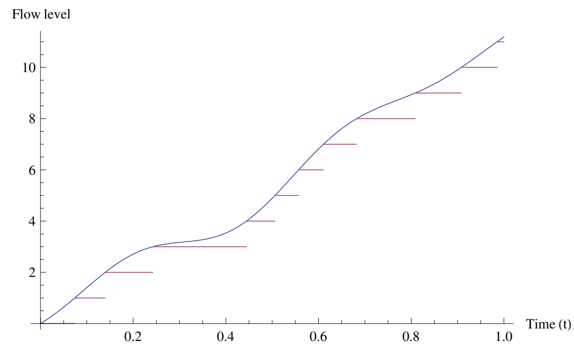


Fig. 1. Approximation of the flow process.

time and state, $\mu(t) = \mu(t, X_t)$. So this also is a counting process and has the representation

$$Y_{out}(t) = \int_0^t (C(s) + \mu(s))I\{L(s) > 0\}ds + M_{out}(t),$$

where $M_{out}(t)$ is a square integrable martingale with quadratic variation

$$\langle M_{out}\rangle_t = \int_0^t (C(s) + \mu(s))I\{L(s) > 0\}ds.$$

One can visualize these approximations as in figure 1, which shows a monotonically increasing flow over a one year period along with piecewise constant approximation. In essence by splitting the dam into $N$ levels we are saying that the mean time between level changes of the continuous process corresponds with the mean time between jumps in our counting process approximations. The martingale terms provide the random perturbation about this mean and, importantly, the mean of the martingale terms is zero, such that,

$$\mathbb{E}[Y_{in}(t)] = \mathbb{E}\left[\int_0^t \lambda(s)I\{L(s) < N\}ds\right],$$

and

$$\mathbb{E}[Y_{out}(t)] = \mathbb{E}\left[\int_0^t (C(s) + \mu(s))I\{L(s) > 0\}ds\right].$$

Then, for our approximation of the dam dynamics, the evolution of this process is governed by the equation,

$$L(t) = Y_{in}(t) - Y_{out}(t) + M(t).$$

*Proposition 2.* Given the inflow and outflow processes as defined, the controlled process is represented by a controlled Markov chain with $N+1$ states and has the $(N+1) \times (N+1)$ generator matrix $A(t, p) = A(t, C(t))$, where

$A(t, C) =$

$$\begin{bmatrix} -\lambda & C+\mu_1 & \dots & 0 & 0 \\ \lambda & -(C+\mu_1+\lambda) & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -(C+\mu_{N-1}+\lambda) & C+\mu_N \\ 0 & 0 & \dots & \lambda & -(C+\mu_N) \end{bmatrix}$$

**Proof.** A detailed proof of this follows the same procedure as that written in Miller (2009) for the generator of a controlled Markov chain for a queuing system. The idea is to look at the increments of $L(t)$ in terms of the increments of the counting processes which approximate it and then use the fact that $I\{L(t) = i\} = I\{X(t) = e_i\}$

to rewrite the increment process in terms of matrices and unit vectors.

*Remark 3.* Each $\mathscr{F}_t^X$-predictable control $p(.)$ induces a probability measure, $\mathbb{P}^p$, on the space $SD$, the space of $S$-valued *cadlag* functions. Then, for each control $p(\cdot)$, the process $\{X_t\}$ satisfies a system of stochastic differential equations given in the following form (Aggoun et al., 1995):

$$X_t^p = X_0 + \int_0^t A(s, p(s))X_{s-}^p ds + \mathscr{M}_t^p \quad (1)$$

where $X_0$ is known or its distribution given and $\mathscr{M}_t^p := \{\mathscr{M}_t^1, ..., \mathscr{M}_t^n, ...\}$ is a square integrable $(\mathscr{F}_t^X, \mathbb{P}^p)$ martingale with quadratic variations

$$\langle \mathscr{M}^p \rangle_t = \int_0^t diag(A(s, p(s))X_{s-}^p)ds -$$

$$\int_0^t [A(s, p(s))(diag X_{s-}^p) + (diag X_{s-}^p)A^T(s, p(s))]ds.$$

Here, diag $X$ is a matrix with diagonal entries $X^i$,$i = 1, ..., N$, the components of $X$ (for details see Aggoun et al. (1995), Appendix B, Lemmas 1.1-1.3).

## 3. OUTFLOW AS THE PRICE CONTROLLED SEASONAL DEMAND

The innovation that makes this model different from other optimal control models is the active interplay between the seasonal demands of each sector and the state and time dependent price of water. The resulting controlled consumption is denoted $C(t, p(t, X_t)) = C(t, p(t))$ for brevity. The derivation of $C(t, p(t))$ involves the current price, $p(t)$, at each state and the active seasonal demands of the sectors. For this model we consider $n$ sectors, which we will designate $\bar{x}_i$, $i = 1, 2, ..., n$, each with their own seasonal demand function, $\bar{x}_i(t)$. For the term sector, it would be useful to think of it as industry, agriculture and private consumption, for example. For the purposes of control, we also have a target level of demand for each sector, $x_i$, $i = 1, 2, ..., n$. Suppose we want to set the target for each sector such that the utility function

$$f_i(x_i) = \alpha(x_i - (1 - r)\bar{x}_i(t))^2 + p(t)x_i(t) \longrightarrow \min_{x_i}, \quad (2)$$

where $\alpha$ is a parameter which governs how much water can be consumed above net natural flow, and $r$ is a minimum consumption reduction target. Differentiating and solving for $x_i$ gives

$$x_i(t, p(t)) =$$

$$\left((1 - r)\bar{x}_i(t) - \frac{p(t)}{2\alpha}\right) I \left((1 - r)\bar{x}_i(t) - \frac{p(t)}{2\alpha} \geq 0\right), \quad (3)$$

which is the optimal usage for each sector. Then the total controlled consumption is simply defined as

$$C(t, p(t)) = \sum_{i=1}^n x_i(t, p(t)). \quad (4)$$

We also note here that for each state there is an optimal consumption function because $p(t)$ depends on the current state. So $C(t, p(t))$ is a vector of optimal consumption functions.

## 4. DYNAMIC PROGRAMMING AND OPTIMAL CONTROL

Consider a general performance criterion, $J[p(\cdot)]$, which defines some restriction on the way the Markov chain behaves in order to be optimal. Let $f_0(s, p(s), X_s)$ be the running cost when the Markov chain is in state $X_s$ at time $s \in [0, T]$, then the criterion has the form

$$J[p(\cdot)] = \mathbb{E}\left[\phi_0(X_T) + \int_0^T f_0(s, p(s), X_s)ds\right] \to \min_{p(\cdot)}, \quad (5)$$

where if $\langle \cdot, \cdot \rangle$ is the inner product and $\phi_0 \in \mathbb{R}^n$, then $\phi_0(X_T) = \langle \phi_0, X_T \rangle$, $f_0(s, p(s), X_s) = \langle f_0(s, p(s)), X_s \rangle$. From this we may define

$$f_0^*(s, p(s)) = (f_0(s, p(s), e_1), ..., f_0(s, p(s), e_n))$$

as the vector of running cost functions of the Markov chain.

*Assumption 4.* For each $i$, $i = 1, ..., N$, the elements of $f_0^*(s, p(s))$ are continuous on $[0, T] \times [p_{min}, p_{max}]$ and bounded below.

This assumption allows us to consider the cost function

$$V(t, x) = \inf_{p(\cdot)} J[p(\cdot)|X_t = x], \quad (6)$$

where

$$J[p(\cdot)|X_t = x] =$$

$$\mathbb{E}\left[\phi_0(X_T) + \int_t^T f_0(s, p(s), X_s)ds | X_t = x\right], \quad (7)$$

and be certain that this infimum exists. Then we can represent $V(t, x)$ as $V(t, x) = \langle \phi(t), x \rangle$, where $\phi(t) = (\phi^1(t), ..., \phi^n(t))^T \in \mathbb{R}^n$ is some measurable function. So we now consider the *dynamic programming equation* with respect to $\hat{\phi}(t)$(Aggoun et al., 1995), (Bertsekas, 1976), (Howard, 1960):

$$\begin{aligned} 0 &= \langle \hat{\phi}'(t), x \rangle - \min_{p \in \bar{P}}[\langle \hat{\phi}(t), A(t, p)x \rangle + \langle f_0(t, p), x \rangle] \\ &= \langle \hat{\phi}'(t), x \rangle - \min_{p \in \bar{P}} H(t, \hat{\phi}(t), p, x) \\ &= \langle \hat{\phi}'(t), x \rangle - \mathscr{H}(t, \hat{\phi}(t), x), \end{aligned} \quad (8)$$

with boundary condition $\hat{\phi}(T) = \hat{\phi}_0$. Since $H(t, \hat{\phi(t)}, p, x)$ is continuous in $(t, p)$ and affine in $\hat{\phi}$, for any $(t, x) \in [0, T] \times S$, $\mathscr{H}(t, \hat{\phi}(t), x)$ is Lipschitz in $\hat{\phi}$ (Pliska, 1975).

*Proposition 5.* With Assumption 4 holding, equation (8) has a unique solution on $[0, T]$.

*Remark 6.* By setting $x = e_i$, $i = 1, ..., N$, we get a system of ordinary differential equations

$$\frac{d\hat{\phi}^i(t)}{dt} = -\mathscr{H}(t, \hat{\phi}(t), e_i), \ i = 1, ..., N. \quad (9)$$

The optimal control is then characterized as in the following theorem (Aggoun et al., 1995), (Pliska, 1975).

*Theorem 7.* Let $\hat{\phi}(t)$ be the solution of the system of equations (9), the for each $(t, x) \in [0, T] \times S$ there exists $p_0(t, x) \in \bar{P}$ such that $H(t, \hat{\phi}(t), p(t), x)$ achieves a minimum at $p_0(t, x)$. Then

(1) There exists an $\mathscr{F}_t^X$-predictable optimal control, $\hat{p}(t, X_0^t)$ such that $V(t,x) = J[\hat{p}(\cdot)|X_t = x] = \langle\hat{\phi}(t), x\rangle$.
(2) The optimal control can be chosen as Markovian, that is

$$\hat{p}(t, X_0^t) = p_0(t, X_{t-}) = \arg\min_{p\in\bar{P}} H(t, \hat{\phi}(t), p(t), X_{t-}).$$

## 5. PERFORMANCE CRITERION

The goal of optimization in the context of this problem is to minimize some cost function of the Markov chain states and the price controls. Such a function should minimize the difference between the customer demand and the optimal demand, the average probability that the dam falls below a prescribed level during the control interval, and the probability that the dam is below a prescribed level at the terminal time.

The three specific criteria for this dam problem have the same form as in equation (5). The first is the mean square deviation of the total customer demand for water and the actually supplied water:

$$J_1[p(\cdot)] = \mathbb{E}^p\left\{\int_0^T \left(C(s,p(s)) - \sum_{i=1}^n \bar{x}_i(s)\right)^2 ds\right\}. \quad (10)$$

The second gives the average probability that the dam level falls below level $M \le N$ over the interval $[0,T]$:

$$J_2[p(\cdot)] = \mathbb{E}^p\left\{\int_0^T \sum_{i=1}^M X_i(s)ds\right\}. \quad (11)$$

The third criterion gives the probability that the dam is below level $M \le N$ at time $T$:

$$J_3[p(\cdot)] = \mathbb{E}^p\left\{\sum_{i=1}^M X_i(T)\right\}. \quad (12)$$

The linear combination of $J_1$ and $J_2$ gives us an integral cost function

$$J_1[p(\cdot)] + J_2[p(\cdot)] = \mathbb{E}^p\left\{\int_0^T f_0(t,p(t),X_t)dt\right\}, \quad (13)$$

where

$$\begin{aligned} f_0(t,p,X) &= \sum_{i=1}^N \langle (C(t,p) - \sum_{i=1}^n \bar{x}_i(t))^2, X_t\rangle \\ &+ \sum_{i=1}^M \langle \mathbf{l}, X_t\rangle, \end{aligned} \quad (14)$$

where $\mathbf{l} = (1,1,...,1,0,...,0)$ with $M$ first units.

This integral cost function appears in the dynamic programming equation and the terminal criterion is accounted for as the initial conditions in the solution of the dynamic programming equations.

If we are optimizing under constrained control resources, then the Lagrangian approach is employed to find the optimal weighting for each of the criterion. This has yet to be done with this application, however, the paper by Miller et al. (2010) gives details of how this may be achieved and we expect to implement this in further research.

## 6. THE DYNAMIC PROGRAMMING EQUATIONS AND SOLUTIONS FOR A LARGE DAM

Using the results of the previous sections we can now show how the optimal control problem for a large dam is solved. Let us take equations (9) and evaluate these with the matrix $A(t,p(t))$ and the function $f_0(t,p(t),X_t)$. Doing so, we get the following dynamic programming equations, where the dependencies have been omitted for the sake of clarity:

$$\begin{aligned} 0 &= \hat{\phi}_1' + \min_{C_1}\{\lambda(\hat{\phi}_2 - \hat{\phi}_1) + (C_1 - \sum_{i=1}^n \bar{x}_i)^2\} + 1 \\ 0 &= \hat{\phi}_2' + \min_{C_2}\{(C_2 + \mu_2)(\hat{\phi}_1 - \hat{\phi}_2) + \\ &\quad \lambda(\hat{\phi}_3 - \hat{\phi}_2) + (C_2 - \sum_{i=1}^n \bar{x}_i)^2\} + 1 \\ &\dots \dots \\ 0 &= \hat{\phi}_M' + \min_{C_M}\{(C_M + \mu_M)(\hat{\phi}_{M-1} - \hat{\phi}_M) + \\ &\quad \lambda(\hat{\phi}_{M+1} - \hat{\phi}_M) + (C_M - \sum_{i=1}^n \bar{x}_i)^2\} + 1 \\ &\dots \dots \\ 0 &= \hat{\phi}_N' + \min_{C_N}\{(C_N + \mu_N)(\hat{\phi}_{N-1}(t) - \hat{\phi}_N(t)) + \\ &\quad (C_N - \sum_{i=1}^n \bar{x}_i)^2\}. \end{aligned} \quad (15)$$

Here we take $C(t,p(t))$ as the control for ease of calculation. Since $C(t,p(t))$ depends linearly on price, $p(t)$ can be recovered for each state after solution.

As mentioned in section 5, the terminal conditions account for the $J_3$ criterion by attaching a reasonable but significant cost to all states less than the prescribed state $M$. That is, for $X_i(T) \le M$, $i = 1,...,M$, $\hat{\phi}_i(T) = K$, where $K$ the cost penalty for ending in this state. For all $X_i(T) > M$, $i = M+1,...,N$, $\hat{\phi}_i(T) = 0$. The size of $K$ must be significant enough to ensure that the solution is sensitive to the criterion whilst still being reasonable.

Now, $C(t,p) = \sum_{i=1}^n x_i(t,p(t))$, where $x_i(t,p(t))$ is given by (3). We need the minimum of this function in each of the above equations (17) for each $t$. The absolute maximum of $C(t,p(t))$ occurs when the price is the stipulated minimum set by the regulator, that is

$$\begin{aligned} C_{max}(t,p) &= \sum_{i=1}^n ((1-r)\bar{x}_i(t) - \frac{p_{min}}{2\alpha})\times \\ &\quad I((1-r)\bar{x}_i(t) - \frac{p_{min}}{2\alpha} \ge 0), \end{aligned} \quad (16)$$

where $p_{min}$ is the minimum price. The absolute minimum occurs with the maximum stipulated price and so is

$$\begin{aligned} C_{min}(t,p) &= \sum_{i=1}^n ((1-r)\bar{x}_i(t) - \frac{p_{max}}{2\alpha})\times \\ &\quad I((1-r)\bar{x}_i(t) - \frac{p_{max}}{2\alpha} \ge 0), \end{aligned} \quad (17)$$

where $p_{max}$ is the maximum price. If the minimizing function is below or above the absolute minimum or maximum respectively, then we take the absolute minimum or maximum as the minimizing function. So, considering each equation, we minimize it and find the conditions which give us the correct minimizing function.

From the first, differentiating with respect to $C_1(t,p)$ gives us

$$C_1(t,p) = \sum_{i=1}^{n}(1-r)\bar{x}_i(t) \tag{18}$$

which implies that $C(t,p)$ is equal to the minimum reduced demand target. For $C_i(t,p)$, $i = 2, ..., N$, the minimizing equations are given by

$$C_i(t,p) = \sum_{j=1}^{n}(1-r)\bar{x}_j(t) + \frac{\hat{\phi}_i(t) - \hat{\phi}_{i-1}(t)}{2}. \tag{19}$$

The functions $C_i(t,p)$, $i = 1, ..., N$ can be summarized as follows:

(1) $C_1(t,p) = \sum_{j=1}^{n} \bar{(}1-r)x_j(t)$ is greater than $C_{max}(t,p)$ for all $t$ and nonzero price $p(t)$, so we take $C_{max}(t,p)$ as the minimizing function.
(2) For the remainder of the equations, the minimizing function is given by the following definition:

$$C_i(t,p) = \begin{cases} C_{max}(t,p), \quad \text{if} \quad \sum_{j=1}^{n}(1-r)\bar{x}_j(t) + \\ \frac{\hat{\phi}_i(t) - \hat{\phi}_{i-1}(t)}{2} > C_{max}(t,p), \\ \\ \sum_{j=1}^{n}(1-r)\bar{x}_j(t) + \frac{\hat{\phi}_i(t) - \hat{\phi}_{i-1}(t)}{2}, \\ \text{if} \quad C_{max}(t,p) \geq \sum_{j=1}^{n}(1-r)\bar{x}_j(t) + \\ \frac{\hat{\phi}_i(t) - \hat{\phi}_{i-1}(t)}{2} \geq C_{min}(t,p), \\ \\ C_{min}(t,p), \quad \text{if} \quad \sum_{j=1}^{n}(1-r)\bar{x}_j(t) + \\ \frac{\hat{\phi}_i(t) - \hat{\phi}_{i-1}(t)}{2} < C_{min}(t,p). \end{cases}$$

With these minimizing equations, the system (15) is now a system of ordinary differential equations, which can be solved numerically.

## 7. NUMERICAL EXAMPLE

For this example, Mathematica 7 was used to solve the system of ODE's and provide plots which demonstrate the effect of the optimal price control on consumption. The following functions and parameters were used as the basis of the model:

- $N = 21$;
- $M = 11$;
- $T = 1$;
- maximum price, $p_{max}(t) = 2.5$, and minimum price, $p_{min}(t) = 2$;
- inflow function, $\lambda(t) = \sin(2\pi t) + 10$;
- natural loss function at the maximum level, $\mu_L(t) = -\sin(2\pi t) + 2.5$;
- natural loss function at lower levels, $\mu_i(t) = \frac{L-i}{L}\mu_L(t)$ for $i = 1, ..., L-1$;
- demand functions, $\bar{x}_1(t) = \cos(2\pi t) + 4.5$, $\bar{x}_2(t) = 0.3\cos(2\pi t) + 3.5$ and $\bar{x}_3(t) = 0.5\cos(2\pi t) + 5$;
- $r = 0.25$, $\alpha = 2.31$; and
- $K = 100$.

The above parameters give us a one year control period for a dam with twenty levels. The regulator has stipulated that the maximum price to be charged is 2.5 (dollars per kiloliter, say) and the minimum is 2. We have a minimum reduction target of 25% off uncontrolled demand and have limited the water that can be consumed above net natural flows to 20%. Natural losses are mostly due to evaporation and this largely depends on the surface area of the dam. For this simple model we have assumed that the losses decrease linearly, however, for any real dam this would require significant modelling in itself. We also have a terminal cost penalty of 100 if the dam level is at or below level 11 at time $T = 1$. This penalty would be paid by the dam manager to the regulator. Recall that the $J_2$ criterion added a unit cost to the running cost of each level. This was found to be too low a cost and the solution was insensitive to it, so for each state where such a cost applied, it was multiplied by $K = 100$.

Figure 2 shows the demand functions and the unweighted mean natural losses along with the inflow function. Clearly total demand and loss exceeds inflows and so the necessity of controlling the demand is well demonstrated, particularly if the dam starts in a low level. Figure 3 gives the maximum and minimum consumption curves used to decide the optimal consumption function for each level in this model. Again, it is clear that the uncontrolled demand is well above the maximum consumption.

After the solution of the system of ODE's, the solutions were substituted back into the consumption equations, $C_i(t,p)$, $i = 1, ..., 20$. From these equations the optimal price functions were easily found. Figure 4 shows the weighted average of controlled demand and the original demand over the control period, assuming that the dam started in level 11. The weighting is given by the probability that the dam was in state $i$ at time $t$ and was found by solving the forward Kolmogorov equation,

$$\frac{dP(t)}{dt} = A(t, p(t))P(t)^T \tag{20}$$

with initial condition,

$$P(0) = e_{11}.$$

So, the weighted average of controlled demand is

$$\sum_{i=0}^{N} p_i(t)C_i(t,p),$$

where $p_i(t)$ is the probability of the dam being in state $i$ at time $t$. Figure 5 shows four of the controlled demand functions and the shape of these functions explains why the weighted average of controlled demand is not a smooth function. The probabilities are smooth but the multiplica-

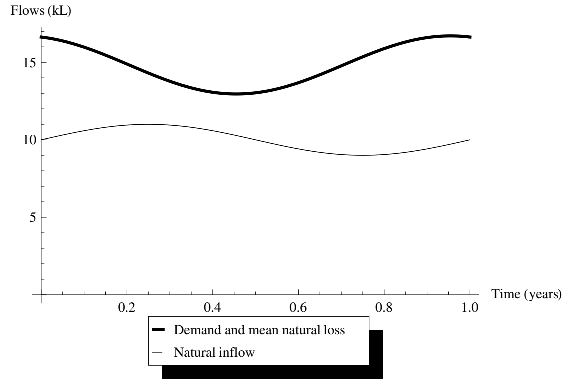


Fig. 2. Demands, mean natural flows and losses.

tion by non-smooth functions leads to a rather irregular curve. Even so, it is clear that the control is effective on the control period.

Figure 6 gives an indication of the nature of the price functions produced by the optimization. Each function is piecewise continuous with few jumps on the control period. As the state changes one need simply change to the price function for that state. Clearly price does not increase monotonically as the dam level gets lower. The price depends on the state and time in a very complex way due to the different criteria we want dam performance to meet. Figure 11 gives the solution curves of the ODE system. Since the solutions depend on the difference between the current state, the state below and the state above, we can see that there will be frequent sign changes because the solution curves are so close together. Figures 7,8 and 9 show the price structure at various times on the control period. It is clear that the prices reflect this behavior of the ODE system solutions.

Figure 10 shows the cumulative controlled consumption in states 1,5,10,15 and 20. They are monotonically increasing so our prices have not altered the general nature of outflows. It is likely that the differing demands of each customer in the dam is causing the price behavior we see. This may not be a practical strategy to control the dam level, due to the non-monotonic changes in price, but does demonstrate emphatically the difficulty of finding a practical strategy to control the dam level primarily through price. We must also consider other types of control.

## 8. CONCLUSION

At this stage we have shown that it is possible to find optimal price functions for a dam with twenty states while taking into account a number of important performance criteria. This provides a solid framework to model larger, more realistic systems. The next step is to increase the number of states so as to improve the smoothness of the resulting price function as it moves from state to state. This would make such a strategy more attractive to implement, however, numerically this will be far more computationally intensive and so we must find efficient means of simultaneously solving large systems of differential equations. Another step will be to increase the number of dams in the system and have them coupled together such that water can be moved under control between dams. If there were $N$ states in each dam and $M$ dams, this would lead to a system of $N^M$ differential equations which need to be solved simultaneously. To this end we will try to implement this with high performance computing techniques (HPC), such as parallel computing, and further papers will report on the progress of these efforts.

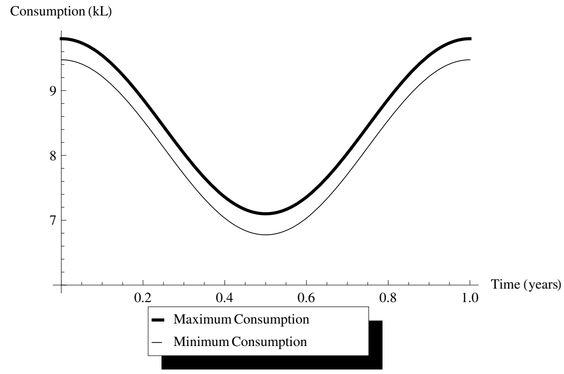


Fig. 3. Maximum and minimum consumption curves.

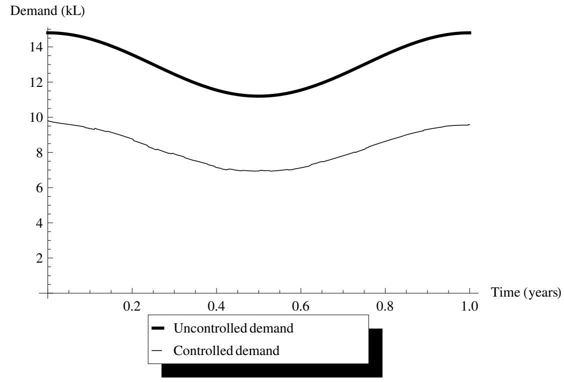


Fig. 4. Uncontrolled consumption and average controlled consumption.

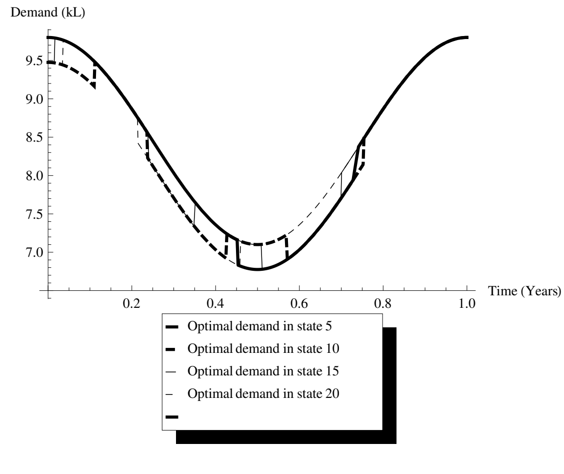


Fig. 5. Optimal demand in states 5,10,15 and 20.

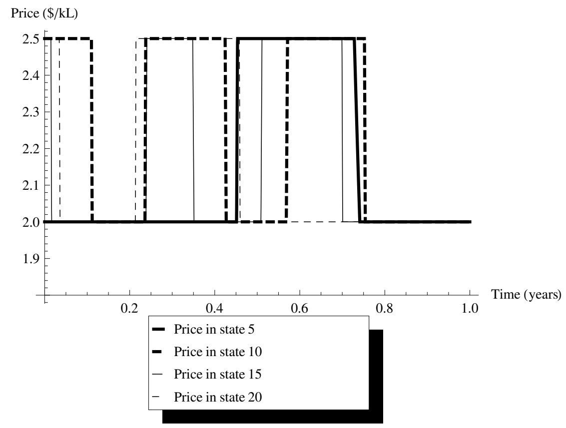


Fig. 6. Optimal price functions for states 5,10,15,and 20.

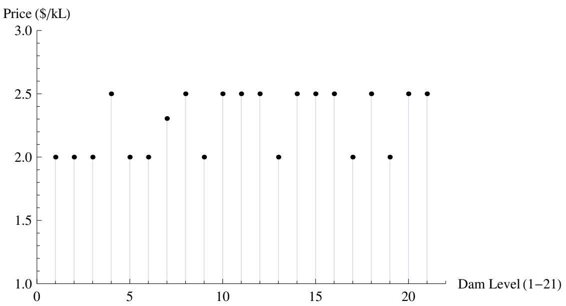


Fig. 7. Price against dam level at t=0.25.

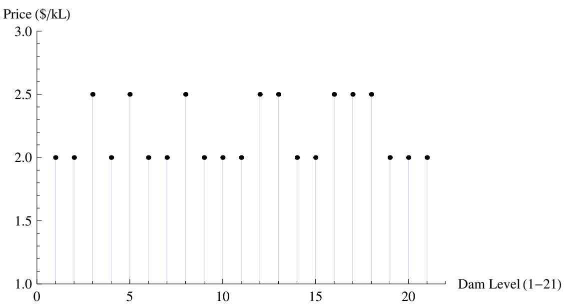


Fig. 8. Price against dam level at t=0.5.

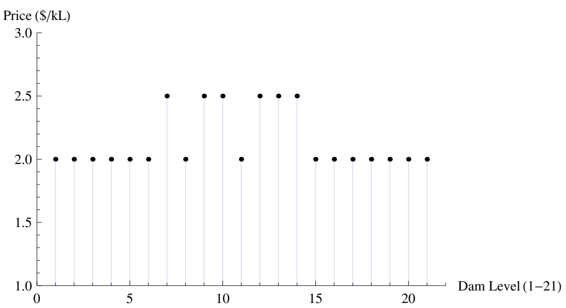


Fig. 9. Price against dam level at t=0.75.

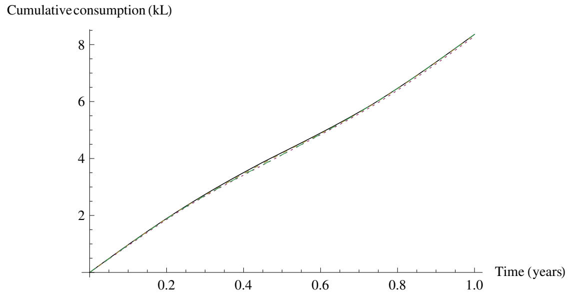


Fig. 10. Cumulative consumption for states 1,5,10,15,20.

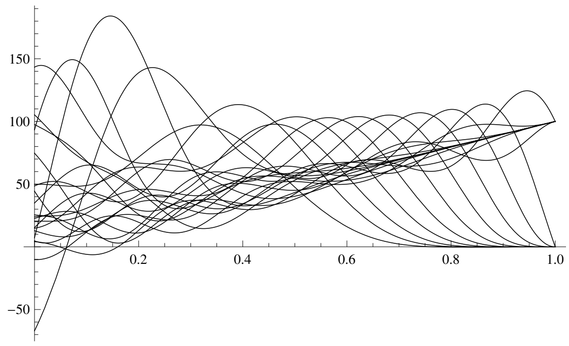

Fig. 11. Dynamic programming ODE solution curves.

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