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# Flows and Limit States in Bidirectional Resource Networks

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**Abstract:** The resource network is a flow model represented by oriented weighted graph, in which every two vertices are either not adjacent or connected by a pair of oppositely directed arcs. Resources are assigned to vertices, which have unlimited volume; the weights of arcs indicate their capacities. The vertices exchange resources, following the definite rules. The flows of resources in the networks with arbitrary capacities of edges are considered. The conditions of their stabilization are analyzed.

***Keywords:*** resource network, flow, attractor, capacity, limit state, Markov processes, stability.

## 1. INTRODUCTION

A number of applied problems related to flows in networks cannot be formalized within the Ford – Fulkerson model (Ford and Fulkerson, 1962; Ahuja, Magnati, and Orlin, 1993) and its dynamic modifications (Fleischer and Skutella, 2003). Among them we can name a management of resource allocation in virtual networks (see e.g. Szeto, Iraqi, and Boutaba, 2003), coordinated resource sharing in the grid computing paradigm (Siddiqui and Fahringer, 2010), modelling of chemical substances and passive biological objects distribution in an aqueous medium, etc.

The resource network represented in Kuznetsov, 2009, is a dynamic model, where the resources are exchanged between vertices under certain rules with the total resource being constant. The network is a bidirectional graph: adjacent vertices are connected by a pair of oppositely directed arcs. The weights of arcs denote their capacities (abilities to transfer resources). As opposed to the classical Ford – Fulkerson flow model, in which resources flow from sources to sinks and are contained in arcs, in the resource network resources are contained in vertices and there are no sources, sinks, and flow direction.

In Kuznetsov, 2009 the complete homogeneous (with identical arc capacities) networks without loops were considered. Analysis of complete heterogeneous networks with loops and their limit states is made in Kuznetsov and Zhilyakova, 2010.

In this paper, we consider the properties of bidirectional resource networks with loops, arbitrary connections, and various capacities.

We introduce the resource flow concept and investigate its stabilization for various amount of total resource. It is proved that in the limit state, flow cannot exceed a certain constant value, regardless of the amount of total resource.

## 2. BASIC DEFINITIONS

1.1. A *resource network* is a directed graph where vertices $v_i$ are assigned with nonnegative numbers $q_i(t)$ (called *resources*) varying in discrete time $t$, and arcs $(v_i, v_j)$ are assigned with time-invariant positive numbers $r_{ij}$ (called *capacities*); $n$ is a number of vertices.

1.2. The *state of the network* at time $t$ is defined by the vector $Q(t) = (q_1(t), \ldots, q_2(t), \ldots, q_n(t))$. At each time, resources are transferred from the vertices along the output arcs with the resource amounts depending on the arc capacities. The rules for resource transfer satisfy the following conditions:

(i) The network is closed; i.e., no resources are supplied from the outside and leak outside.

(ii) A resource amount sent out of a vertex is subtracted from its total resource, while a resource amount arriving at a vertex is added to its total resource; i.e., the total resource $W$ is conserved: $\forall t \sum_{i=1}^{n} q_i(t) = W$.

1.3 A state $Q(t)$ is called *stable* if

$$Q(t) = Q(t+1) = Q(t+2) = Q(t+3) = \ldots$$

A state $Q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ is said to be *asymptotically reachable* from the state $Q(0)$ if for any $\varepsilon > 0$ there exists $t_\varepsilon$ such that $|q_i^* - q_i(t)| < \varepsilon$, $i = 1, 2, \ldots, n$ for all $t > t_\varepsilon$.

A network state is called a *limit state* if it is either stable or asymptotically reachable.

1.4. A network is said to be *homogeneous* if all the capacities are identical (let them be equal to $r$). Otherwise it is called a *heterogeneous* network.

1.5. The homogeneous network functioning rules are as follows: at the time $t$, the resource amount transferred by the vertex $v_i$ along each of the $m_i$ output arcs is:

$r$ if $m_i r \le q_i(t)$ (rule 1);

$\frac{q_i(t)}{m_i}$ otherwise (rule 2).

1.6. A pair of arcs $<(v_i, v_j), (v_j, v_i)>$ is called a *bidirectional pair*. A network where vertices are connected only by bidirectional pairs is called a *bidirectional network*.

1.7. The capacity matrix of a network is defined as $R = \|r_{ij}\|_{n\times n}$. The properties of $R$ for a bidirectional network with loops follow directly from its definition:

(i) $R$ is a nonnegative matrix: $\forall i,j\ r_{ij} \ge 0$;

(ii) $\forall i\ r_{ii} > 0$;

(iii) $\forall i,j\ (r_{ij} > 0 \Leftrightarrow r_{ji} > 0)$

If the network is complete, $R$ is a positive matrix.

1.8. The total capacity $r_{sum}$ of a network is defined as the sum of the capacities of all its arcs: $r_{sum} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}$ .

1.9. Let the total capacity $\sum_{j=1}^{n} r_{ji}$ of all input arcs of the $i$-th vertex be denoted by $r_i^{in}$ and called *total input capacity*; the total output capacity $\sum_{j=1}^{n} r_{ij}$ be denoted by $r_i^{out}$ and called *total output capacity*. The capacity of a loop is included in both sums.

1.10. The resource outgoing from the vertex $v_i$ along the arc $(v_i, v_j)$ at the time $t$, comes to $v_j$ at the time $t$+1. Respectively, this resource is assumed to be at the arc $(v_i, v_j)$ within the time interval $(t, t+1)$. The amount of resource at the arc is called *flow* $s_{ij}(t)$.

1.11. The flow matrix is defined as $S = \|s_{ij}\|_{n\times n}$.

1.12. $\sum_{j=1}^{n} s_{ij}(t) = s_i^{out}(t)$ – *output flow* of the vertex $v_i$ at the time $t$ (the sum of $i$-th row of matrix $S(t)$); $s_i^{out}(t) \le r_i^{out}$ .

1.13. The sum of $i$-th column of matrix $S(t)$: $s_i^{in}(t+1) = \sum_{j=1}^{n} s_{ji}(t)$ is called *input flow* $s_i^{in}(t+1)$ of the vertex $v_i$ at the time $t$+1; it is assumed that $s_i^{in}(0) = 0$ by definition.

## 3. HOMOGENEOUS COMPLETE BIDIRECTIONAL NETWORKS WITH LOOPS

Complete homogeneous networks with loops have three properties, which are derived from properties 1-3 formulated in Kuznetsov, 2009 for loopless networks, by virtue of substitution of $rn$ for $r(n–1)$.

**Property 1.** If $q_i(t') = q_j(t')$ for some $t'$, then $q_i(t) = q_j(t)$ for all $t > t'$.

**Property 2.** If $q_i(t') \le rn$ for some $t'$, then $q_i(t) \le rn$ for all $t > t'$.

**Property 3.** If $q_i(t) \ge rn$ for all $i$, then the state $Q(t)$ is stable.

The set of vertices for which $q_i(t) > rn$ is called the zone $Z^+(t)$, while the set of vertices for which $q_i(t) \le rn$ is called the zone $Z^-(t)$. Let zone $Z^+(0)$ consist of $k$ vertices, and an initial ordering of the vertices be represented as: $q_i(0) \ge q_{i+1}(0)$, $i = 1, \ldots, n-1$. Then zone $Z^+(0)$ consists of the vertices with numbers from 1 to $k$; and $Z^-(0)$ consists of the remaining vertices.

Let the segment of the vector $Q(t)$ which includes $q_i(t) \in Z^+(t)$ be denoted as $Q^+(t)$, and the segment corresponding to $Z^-(t)$ as $Q^-(t)$.

Let represent them as:

$Q^+(t) = (rn + c_1(t), \ldots, rn + c_k(t))$,

$Q^-(t) = (rn - d_{k+1}(t), \ldots, rn - d_n(t)$,
where $\forall i, j\ c_i(t) > 0$, $d_j(t) \ge 0$.

Let $C(t)$ and $D(t)$ denote the sums of all $c_i(t)$ and all $d_j(t)$, respectively. In general case $k$ could be variable: $k = k(t)$. Then $C(t) = \sum_{i=1}^{k(t)} c_i(t)$; $D(t) = \sum_{i=k(t)+1}^{n} d_i(t)$ .

The value $D(t)$ specifies the resource that the $Z^-(t)$ zone lacks for to reach the value $rn(n-k)$.

Let us express the sum of all the coordinates $Q(0)$ in terms of $D(0)$ and $C(0)$:

$$\sum_{i=1}^{n} q_i(0) = W = rn^2 + C(0) - D(0),$$

hence $C(t) - D(t) = \text{const} = p$ and $W = rn^2 + p$.

**Example.** Let's consider the complete graph with following parameters: $n = 5$, $r = 2$, $rn = 10$, $W = 55$. $Q(0) = (22, 15, 8, 7, 3)$. Then $k = 2$, the zone $Z^+(0)$ consists of the two first vertices; $c_1(0) = 12$, $c_2(0) = 5$, $C(0) = 17$, $d_3(0) = 2$, $d_4(0) = 3$, $d_5(0) = 7$, $D(0)=12$, $p=5$,

$$S^{out}(0) = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 1{,}6 & 1{,}6 & 1{,}6 & 1{,}6 & 1{,}6 \\ 1{,}4 & 1{,}4 & 1{,}4 & 1{,}4 & 1{,}4 \\ 0{,}6 & 0{,}6 & 0{,}6 & 0{,}6 & 0{,}6 \end{pmatrix};\ S^{in}(0) = 0.$$

The homogeneous networks with loops unlike the homogeneous networks without loops have one more important property:

**Property 4.** $s_i^{in}(t) = s_j^{in}(t)$ for all $t$, $i, j = 1, 2, \ldots, n$.

This follows from the fact that all the vertices send out identical amounts of resources into output arcs (see the example).

**Lemma 1.** If at the time $t$ the vertices $v_{i1}, \ldots, v_{im}$ $(m \le n)$ are in the zone $Z^-(t)$, then $q_{i1}(t+1) = \ldots = q_{im}(t+1)$.

Since all the vertices of the zone $Z^-(t)$ send out all their resource, amount of their resource at the time $t+1$ is equal to their input flow. Then the lemma is true owing to property 4.

**Theorem 1.** For complete homogeneous bidirectional network with loops and with the number of vertices $n > 2$, if the total network resource $W$ does not exceed $T = rn^2$, then for any initial state of the network its limit state is the vector $(\frac{W}{n}, \frac{W}{n}, \ldots \frac{W}{n})$.

**Theorem 2.** If $W > rn^2$, then the limit state is the vector

$Q^* = (q_1(0) - w^*, \ldots, q_l(0) - w^*, rn, \ldots, rn)$,

where $l = k$ and $w^* = \frac{D(0)}{k}$, if $c_k(0) \ge \frac{D(0)}{k}$;
otherwise, $l \le k$ is the largest integer such that:

$c_l(0) \ge w^*$, $w^* = \frac{C_l(0) - p}{l}$, where $C_l(0) = \sum_{i=1}^{l} c_i(0)$.

## 4. NONSYMMETRIC BIDIRECTIONAL NETWORKS

In this section, we consider *connected nonsymmetric bidirectional networks with loops*. The analysis of these networks with small resources relies on the theory of matrices and discrete Markov chains: Gantmacher 1959, Kemeny and Snell, 1959. When the total resource exceeds a certain threshold value, the processes of resource reallocation loose the Markov property, and it becomes necessary to introduce the concept of flow.

Let us introduce the additional definitions.

4.1. A resource network is said to be *symmetric* if its capacity matrix $R$ is symmetric. The input and output capacities of every vertex in symmetric network are identical.

However, the symmetry of the matrix is not a necessary condition for the equality of input and output capacities.

4.2. A resource network with a nonsymmetric matrix $R$ is called quasisymmetric if

$$\forall i:\ r_i^{in} = r_i^{out}. \qquad (1)$$

4.3. If the network is not quasisymmetric there exists at least a pair of vertices, for which $r_i^{in} - r_i^{out} \ne 0$. For arbitrary vertex $v_i$ this difference is denoted as $\Delta r_i$: $\Delta r_i = r_i^{in} - r_i^{out}$.

The vertices in a nonsymmetric network are divided into three classes:

(i) *vertices-receivers*, for which $\Delta r_i > 0$;

(ii) *vertices-sources*, $\Delta r_i < 0$;

(iii) *neutral vertices*, $\Delta r_i = 0$.

In symmetric and quasisymmetric networks, all the vertices are neutral.

4.4. The network is called nonsymmetric if (1) does not hold for some $i$. Nonsymmetric network has at least one source and one receiver.

4.5. The rules of resource distribution in a nonsymmetric network are similar to 1.5. Specifically, at the time $t$, the resource amount transferred by the vertex $v_i$ into vertex $v_m$ is:

$r_{im}$ if $q_i(t) > r_i^{out}$ (rule 1);

$\frac{r_{im}}{r_i^{out}} q_i(t)$ otherwise (rule 2).

Suppose that $n$ vertices of a network contain $l$ receivers, $k$ sources, and $n - l - k$ neutral vertices. Assume that the receivers are indexed from 1 to $l$, the sources, from $l + 1$ to $l + k$, and the neutral vertices, from $l + k + 1$ to $n$.

4.6. The path from a neutral vertex to a source which contains no receivers is called a *nonpositive path*.

### *4.1. Properties of Nonsymmetric Networks*

The properties of homogeneous networks in general case are not preserved for heterogeneous ones. Properties 1 and 2 are extended to sources and neutral vertices with some restrictions.

**Property 1a.** If for two vertices $v_i$, $v_j$ $q_i(t') = q_j(t')$ for some $t'$ and $i, j > l$, while: (i) $q_i(t') \le r_i^{out}$, $q_j(t') \le r_j^{out}$, (ii) the sets of vertices adjacent to $v_i$ and $v_j$ are identical and (iii) $r_{mi} = r_{mj}$ for every $m$, then $q_i(t) = q_j(t)$ for all $t > t'$.

**Property 2a.** If $v_i$ is a source or a neutral vertex with a nonpositive path and $q_i(t') \le r_i^{in}$ for some $t'$, then $q_i(t) \le r_i^{in}$ for all $t > t'$.

**Property 5.** In the process of resource exchanging in a nonsymmetric network the amount of resource in neutral vertices can temporarily stabilize and then vary again.

The definitions of the zones $Z^-(t)$ and $Z^+(t)$ for nonsymmetric networks are following. The set of vertices for which $q_i(t) \le r_i^{out}$ is called a zone $Z^-(t)$, while the set of vertices for which $q_i(t) > r_i^{out}$ is called $Z^+(t)$. According to the property 2a, the sources and the neutral vertices with nonpositive paths fallen in $Z^-$(t) will not be able to leave it, because $r_i^{in} \le r_i^{out}$.

**Theorem 3.** In a connected bidirectional nonsymmetric network with any initial state $Q(0) = (q_1(0), q_2(0), \ldots q_n(0))$ and any total resource $W$, for all sources or neutral vertices with nonpositive paths there exists a time $t'$ such that

$$\forall t > t' \;\; q_i(t) < r_i^{in}. \quad (2)$$

Theorem 3 implies that all sources or neutral vertices with nonpositive paths will move into $Z^-(t)$, but for sources condition (2) is stronger.

*4.2. The Limit State of the Network for $W = 1$*

The process of resource exchange in connected nonsymmetric network with the unit total resource ($W = 1$) while the total capacity is more than unit, corresponds to regular Markov chain; a vector of state $Q^1(t)$ corresponds to a probabilistic vector.

The resource in such a network will be reallocated in accordance with the rule 2. So, the vector of state is determined by the recurrent formula:

$$Q^1(t+1) = Q^1(t) \cdot R', \quad (3)$$

$$R' = \begin{pmatrix} \frac{r_{11}}{r_1^{out}} & \frac{r_{12}}{r_1^{out}} & \cdots & \frac{r_{1n}}{r_1^{out}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{r_{n1}}{r_n^{out}} & \frac{r_{n2}}{r_n^{out}} & \cdots & \frac{r_{nn}}{r_n^{out}} \end{pmatrix}. \quad (4)$$

$R'$ is a regular stochastic matrix describing a regular Markov chain. $R'$ is obtained from the nonnegative capacity matrix $R$ by normalizing rows.

The results obtained for regular Markov chains imply the following properties of the matrix $R'$:

(i) for every connected bidirectional network with loops there exists the limit of powers of the stochastic matrix $R'$: $\lim_{h\to\infty} (R')^h = (R')^\infty$ – the matrix of limit transition probabilities of the Markov chain;

(ii) for every initial distribution of the unit resource there exists a unique limit state vector $Q^{1*}$ which can be found as:

$$Q^{1*} = Q^1(0) \cdot (R')^\infty;$$

(iii) additionally, for any $t > 0$ the equality is true:

$$Q^{1*} = Q^1(t) \cdot (R')^\infty;$$

(iv) the matrix $(R')^\infty$ consists of $n$ rows $Q^{1*}$: $(R')^\infty = \xi\, Q^{1*}$, where $\xi$ is a column vector consisting of $n$ units.

$$(R')^\infty = \begin{pmatrix} q_1^{1*} & q_2^{1*} & \cdots & q_n^{1*} \\ q_1^{1*} & q_2^{1*} & \cdots & q_n^{1*} \\ \cdots & \cdots & \cdots & \cdots \\ q_1^{1*} & q_2^{1*} & \cdots & q_n^{1*} \end{pmatrix}. \quad (5)$$

(v) the vector $Q^{1*}$ is the left eigenvector of $R'$ with the eigenvalue $\lambda_1 = 1$, i.e.,

$Q^{1*} \cdot R' = Q^{1*}$;

(vi) the vector consisting of any coordinate of $Q^{1*}$ (any column of matrix (5)) is the right eigenvector of $R'$ with the eigenvalue $\lambda_1 = 1$;

(vii) $Q^{1*}$ is the left eigenvector of $(R')^\infty$ with the eigenvalue $\lambda_1 = 1$: $Q^{1*} \cdot (R')^\infty = Q^{1*}$. This equality can be derived by proceeding to the limit in (3).

**Remark.** According to the point (ii) a network with unit resource is an *ergodic system*: its limit state is independent of initial resource distribution.

*4.3. Functioning of the Network According to the Rule 2. The threshold value of resource*

Let us consider the behaviour of the network with $W \neq 1$. By Theorem 3, there exists time $t'$, when all the sources and neutral vertices with nonpositive paths function in accordance with the rule 2. Let the value of $W$ be such that all the remaining neutral vertices and receivers are also functioning according to rule 2, i.e. all the vertices are in the zone $Z^-(t)$ $\forall t > t'$. Such a value ($T$) always exists, e.g. if $W < \min_i r_i^{out}$ then no vertices a priori can be in $Z^+(t)$.

**Theorem 4.** In a connected bidirectional nonsymmetric network with any total resource value $W$, for which after a certain time $t'$ all the vertices function in accordance with the rule 2, for any initial distribution $Q(0)$, the limit state vector $Q^*$: (i) exists, (ii) is unique, and (iii) is the left eigenvector of stochastic matrix $R'$ (4) and of the matrix of limit transition probabilities (5) with the eigenvalue $\lambda_1 = 1$: $Q^* = Q^* \cdot R'$ and $Q^* = Q^* \cdot (R')^\infty$.

If the network functions in accordance with the rule 2, the limit state vector is an eigenvector of $R'$ (Theorem 4). On the other hand the positive eigenvector of $R'$ is unique (Gantmacher, 1959), therefore all the limit state vectors are linearly dependent in pairs: their coordinates are proportional. For two resource values $W_1$ and $W_2$ is true:

$$\frac{Q_1^*}{W_1} = \frac{Q_2^*}{W_2}.$$

Then for all $W$, such that the network function in accordance with the rule 2, the coordinates of the limit state vector can be expressed via the vector $Q^{1*}$:

$$Q^* = Q^{1*} \cdot W.$$

**Theorem 5.** In a connected bidirectional nonsymmetric network there exists a threshold value of the total resource $T$, such that if $W \le T$ all the vertices after a certain time $t'$ get into the zone $Z^-(t)$; if $W > T$ the zone $Z^+(t)$ is nonempty for any $t$. For every network $T$ is unique and does not depend on $W$ and its initial distribution $Q(0)$.

*4.4. The Flow of the Resource*

**4.4.1. $W \le T$**

If $W \leq T$ the network function in accordance with the rule 2 after the time moment $t'$. The vertices contain only input resource, i.e. $Q(t) = S^{in}(t)$. On the other hand they send all their resources into output arcs, so $S^{out}(t) = Q(t)$. By theorem 4 the limit of $Q(t)$ exists and is equal to $Q^*$. Then $\lim_{t\to\infty} S^{in}(t)$ and $\lim_{t\to\infty} S^{out}(t)$ exist too. Thus if $W \leq T$ then

$S^{in*} = S^{out*} = Q^*$.

### 4.4.2. *W > T*

Let the limit state vector for $W = T$ be denoted as $\widetilde{Q} = (\widetilde{q}_1, \widetilde{q}_2, ..., \widetilde{q}_n)$. If $W = T$ at least one vertex has in the limit state the resource equal to its output capacity. This follows directly from theorem 5.

Vertices contained in $Z^{+*}$ at the limit state, are called *attractors*. Not every vertex-receiver is able to become an attractor. Attractors belong to the set of vertices for which the equality holds: $\widetilde{q}_i = r_i^{out}$. Let us call these vertices *potential attractors*.

**Theorem 6.** In a connected bidirectional nonsymmetric network with any initial distribution of total resource $W > T$: $Q(0) = (q_1(0), q_2(0), \dots q_n(0))$:

(i) the limit flow exists and is equal to $T$: $S^{in*} = S^{out*} = T$;

(ii) the limit state $Q^*$ exists;

(iii) the coordinates of $Q^*$ are determined as follows: for the vertices with $\widetilde{q}_j < r_j^{out}$, holds: $q_j^* = \widetilde{q}_j$ for any $W > T$. The remaining resource is distributed among potential attractors.

If the network has the unique attractor, it is an ergodic system for any $W$. If there are several potential attractors, then for $W > T$ the distribution of resource among these vertices at limit state depends on its initial distribution. This network becomes a partially ergodic system.

## 5. PRACTICAL APPLICATIONS

The resource network underlies a model simulating the distribution of pollutants, other chemical substances, and passive hydrobionts in the aqueous medium. The network topology is a regular two- or three-dimensional grid in dependence on a simulated process. In order to simulate the distribution of a matter on water surface the two-dimensional grid is used. If the problem is to model the dissemination of a substance through the water mass and the three-dimensional grid has to be built to take into account the depths.

The vertices in the model contain the amount of a matter dissolved under a specified water surface (or in a specified water volume). The capacities of arcs correspond to crossflows between every two adjacent areas. They depend on the existing water flows in the simulated area of water, wind velocity, and the number of other hydrological parameters.

The modelled area can be both open and closed. In the closed area the preservation law must be held. If the area is open its boundaries are permeable and the matter is permitted to go outside till its complete disappearance.

The openness of area is achieved by means of creation of arcs incident to the boundary vertices, directed outside the described area. The vertex sends a resource into such an arc, and this amount of resource leaves the area forever. Thereby this resource is subtracted from a total resource. It means that a modelled matter leaves the described area, and its average concentration is getting lower.

The model simulates a distribution of resource in a specified area of water within a specified time interval. The results include (i) the Excel spreadsheet with the values of substance concentration in every grid node at every step; (ii) the text file with the same values; (iii) the distribution maps and isolines drawn at every time step; (iv) the animated movement of isolines to demonstrate the step-by-step dynamics of distribution of substance in aqueous medium.

These results are aimed to an operational forecasting of the spatio-temporal distribution of matter as the effect of the discharge of pollutants or other substances (e.g. with ballast water).

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