Output Feedback Lagrange Stabilization for Pendulum-like Systems with Multiple Nonlinearities

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Abstract: This paper addresses an output feedback Lagrange stabilization problem for pendulum-like systems with multiple nonlinearities. A method based on output feedback pseudo $H_{\infty}$ control is developed to solve the output feedback Lagrange stabilization problem. This method is based on a new Lagrange stability criterion which does not require the linear part of the pendulum-like system to be minimal. To illustrate the efficacy of the proposed method, an example involving coupled pendulums is considered which can be regarded as a prototype of many practical interconnected oscillating systems.

Keywords: Nonlinear system, Connected Pendulums, Lagrange stability, Output feedback Lagrange stabilization, Pendulum-like system

1. INTRODUCTION

The class of pendulum-like systems is a class of systems which are periodic in the system state (not time) and have an infinite number of equilibria (Leonov et al. (1996)). This system model is used to describe an important class of nonlinear systems arising in electronics, mechanics and power systems. For instance, interconnected oscillators, synchronous electrical machines and electronic phase-locked loop devices are modeled as pendulum-like systems in Shakhgildyan and Belyustina (1982); Wang et al. (2006). This class of nonlinear systems has attracted the interest of researchers for many years. There are many results on the stability analysis of pendulum-like systems reported in the literature, for example, see Leonov et al. (1996); Yakubovich et al. (2004); Liu et al. (2006, 2007); Ouyang et al. (2008).

From a practical viewpoint, for stable operation, a nonlinear pendulum-like system with an infinite number of equilibria must have trajectories which do not escape to infinity. Hence, Lagrange stability is a minimum stability requirement for such systems to be practical. Therefore, an important control objective in relation to controlling this class of nonlinear systems is to ensure that the closed-loop system remains inside the properties of a pendulum-like system and is Lagrange stable. In combination with other analytical tools such as dichotomy, this enables global asymptotic properties of the system to be established. For example, the monograph Leonov et al. (1996) makes extensive use of this approach to study global asymptotic behavior of nonlinear systems with periodic nonlinearities and an infinite number of equilibria.

Previously, Yang and Huang (2003); Li and Zhong (2005); Wang et al. (2006); Gao (2009) have studied the Lagrange stabilizing controller synthesis problem for nonlinear systems with a single periodic (in state) nonlinearity. Since these results utilize the Lagrange stability criterion in Leonov et al. (1996), they inherit the minimal realization requirements on the linear part of resulting closed-loop systems and hence need a check of the minimality of the resulting closed-loop system. Also, these results assume that the nonlinear systems under consideration have a special structure. All these facts restrict the methods developed in Yang and Huang (2003); Wang et al. (2006); Li and Zhong (2005); Gao (2009) from being applied to more general nonlinear systems. This motivates the research on the Lagrange stabilization of more general nonlinear systems.

A step towards developing such a more general Lagrange stabilization theory for pendulum-like systems with multiple nonlinearities was made in the authors’ previous work, Ouyang et al. (2008), where a Lagrange stability criterion was established without assuming the linear part of the nonlinear system under consideration to be a minimal realization. However, the stabilization result in that paper was limited in that it required the system output to have a special structure. This paper will use a different approach to study the output feedback Lagrange stabilization problem for a more general class of nonlinear systems which have multiple nonlinearities. Our result will be based on the Lagrange stability criterion developed in Ouyang et al. (2008) which does not require the linear part of the nonlinear system under consideration to be a minimal realization. At the same time, we dispense with restrictive assumptions of Ouyang et al. (2008) relating to the structure of the nonlinearity. It is worthwhile pointing out that the results in this paper are different from those in the authors’ previous work, Ouyang et al. (2009, 2010), where the state feedback and output feedback Lagrange stabilizing controller synthesis problems for the nonlinear systems with only a single nonlinearity are considered.

The paper is organized as follows: Section 2 formulates the output feedback Lagrange stabilization problem for pendulum-like systems with multiple nonlinearities; Section 3 presents the main results of the paper; Section 4 gives an illustrative exam-
ple. This example involves coupled nonlinear pendulums and can be regarded as a prototype of many practical interconnected nonlinear oscillating systems. Section 5 concludes the paper.

Notation: $\mathcal{D}$ denotes the set of integers, $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$ denote the space of $n \times m$ real matrices and the space of $n \times m$ complex matrices, respectively. $\mathcal{D}$ denotes the set of rational numbers and $\mathbb{Z}^m$ denotes the set of vectors of $m$ rational numbers. $\sigma(A)$ denotes the set of eigenvalues of a matrix $A$. $\sigma_{\text{max}}[\cdot]$ denotes the maximum singular value of a matrix, $\mathbb{R} \mathcal{H}_c$ denotes the space of all proper and real rational stable transfer function matrices. $\mathbb{R}_+$ denotes the set of positive real numbers and $\mathbb{A}_n^+(\mathbb{R}_+)^p$, $\rho(A)$ denotes the spectral radius of the matrix $A$. diag$[a_1, \ldots, a_n]$ is a diagonal matrix with $a_1, \ldots, a_n$ as its diagonal elements. Given a vector $v \in \mathbb{R}^m$, $\text{LCM}(v)$ denotes the least common multiple (LCM) of the denominators of all the elements of $v$.

2. PROBLEM FORMULATION FOR LAGRANGE STABILIZATION OF PENDULUM-LIKE SYSTEMS

This section gives some preliminary results on pendulum-like systems and formulates the output feedback Lagrange stabilization problem for pendulum-like systems.

2.1 Pendulum-like Systems

We consider a class of nonlinear systems defined as follows:

$$
\begin{align*}
\dot{x} &= Ax + Bw, \\
z &= Cx,
\end{align*}
$$

where $x \in \mathbb{R}^m$ is the state, $z \in \mathbb{R}^m$ is the vector of nonlinearity outputs and $w \in \mathbb{R}^m$ is the vector of nonlinearity inputs. Also, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C = [C_1^T, \ldots, C_m^T]^T \in \mathbb{R}^{m \times n}$, $C_i \in \mathbb{R}^{1 \times m}$, $i = 1, \ldots, m$. The components of the vector $w = [w_1, \ldots, w_m]^T$ are determined from the corresponding components of the vector $z = [z_1, \ldots, z_m]^T$ via the nonlinear functions $w_i = \psi_i(t, z_i)$

$$
\psi_i(t, z_i + \Delta_i) = \psi_i(t, z_i), \quad \forall t \in \mathbb{R}_+, \ z_i \in \mathbb{R}.
$$

The transfer function of the linear part of the system is given by $G(s) = C(sI - A)^{-1}B$. The nonlinear functions $\psi_i(t, z_i), i = 1, \ldots, m$ are assumed to satisfy the sector conditions,

$$
-\mu_i \leq \frac{\psi_i(t, z_i)}{z_i} \leq \mu_i, \quad \forall t \in \mathbb{R}_+, \quad z_i \neq 0,
$$

where $\mu_i \in \mathbb{R}_+, i = 1, \ldots, m$.

We define $\Delta \in \mathbb{R}^{m \times m}$ as $\Delta = \text{diag}([\Delta_1, \ldots, \Delta_m])$. Given a vector $d \in \mathbb{R}^n$, let $\Pi(d) = \{kd | k \in \mathcal{D}\}$.

Definition 1. (Pendulum-like System; see Leonov et al. (1996))

The nonlinear system (1), (2), (3) is pendulum-like with respect to $\Pi(d)$ if for any solution $x(t, t_0, x_0)$ of (1), (2) with $x(t_0) = x_0$, we have $x(t, t_0, x_0) + d = x(t, t_0, x_0 + d)$, for all $t \geq t_0$, and all $d \in \Pi(d)$.

Definition 2. (Lagrange Stability; see Leonov et al. (1996))

The nonlinear system (1), (2) is said to be Lagrange stable if all its solutions are bounded.

Fig. 1. Nonlinear Control System with Periodic Nonlinearities.

2.2 Lagrange Stabilization Problem for Pendulum-like Systems

The pendulum-like system to be stabilized will be a controlled version of nonlinear system (1), (2), (3), (4). That is, the linear part of the system is described by the state equations

$$
\begin{align*}
\dot{x} &= Ax + B_2u + B_1w, \\
z &= C_1x + D_21u, \\
y &= C_2x + D_21w,
\end{align*}
$$

where $x \in \mathbb{R}^m$, $w \in \mathbb{R}^m$, $z \in \mathbb{R}^m$ are defined as in (1), $u \in \mathbb{R}^q$ is the control input, and $y \in \mathbb{R}^p$ is the measured output. Here, the matrices are assumed to have compatible dimensions. Also, the components of $w = [w_1, \ldots, w_m]^T$ are related to the components of $z = [z_1, \ldots, z_m]^T$ as in (2), (3). Furthermore, the nonlinearities are assumed to satisfy the sector condition (4). The system block diagram is shown in Figure 1.

Problem 1. (Output Feedback Lagrange Stabilization) The output feedback Lagrange stabilization problem for the nonlinear system (5), (2), (3), (4) is to design a linear controller with transfer function $K(s)$ and state-space realization:

$$
\begin{align*}
\dot{x}_e &= A_e x_e + B_2 y, \\
u &= C_e x_e,
\end{align*}
$$

such that the resulting closed-loop nonlinear system is pendulum-like and Lagrange stable.

Note that in some cases, it may be possible to design a controller in the form of (6) to asymptotically stabilize the system (5), (2), (4). It is worthwhile pointing out that these cases are trivial from the point of view of Lagrange stabilization and this paper will not discuss these cases. In order to rule out these trivial cases and to guarantee that the closed-loop system is a pendulum-like system, in the sequel we assume there exist uncontrollable or unobservable modes at the origin of (5).

Using a special nonlinearity structure, Yang and Huang (2003); Li and Zhong (2005); Wang et al. (2006); Gao (2009) have studied the above Problem 1 for the cases where the nonlinear system (5) contains a single nonlinearity and its linear part is minimal. The state feedback and output feedback Lagrange stabilization problems for the nonlinear system (5) with a single nonlinearity are also discussed in the authors’ previous work (Ouyang et al. (2009, 2010)). These results do not require the linear part of the nonlinear system (5) to be minimal. Also, Ouyang et al. (2008) has studied a special state feedback case of Problem 1. This paper addresses Problem 1 for a general class of nonlinear pendulum-like systems.
To solve the above Problem 1, the following two lemmas will be used:

Lemma 1. (Ouyang et al. (2008)) Consider the nonlinear system (1), (2), (3). Suppose detA = 0 and there exists a vector $d \neq 0$ such that $Ad = 0, Cd \neq 0, i = 1, \ldots, m$, and $(\Delta^{-1}) Cd \in \mathbb{R}^m$. Then, the nonlinear system (1), (2), (3) is a pendulum-like system with respect to $P(d)$ where $d = \tilde{p}d$.

Lemma 2. (Ouyang et al. (2008)) (Lagrange Stability Criterion) Suppose the nonlinear system (1), (2), (3), (4) is a pendulum-like system. Also, suppose there exist a constant $\lambda > 0$ and a vector $\tau = [\tau_1, \ldots, \tau_n]^T \in \mathbb{R}^n$ satisfying the following conditions:

i. $A + \lambda I$ has $n - 1$ eigenvalues with negative real parts and one eigenvalue with positive real part;

ii. $G^T (-jo - \lambda) M \sigma G (jo - \lambda) \preceq M_p^{-1} M \sigma M_p^{-1},$ for all $\omega \geq 0$.

Then, the nonlinear system (1), (2), (3), (4) is Lagrange stable.

The proofs of these two results will appear in the journal version of Ouyang et al. (2008).

Lemma 2 is the key result used to establish Lagrange stability of the closed-loop system under consideration. It involves a frequency domain condition, which is similar to the bounded real property in Anderson and Vongpanitlerd (1973), and a system state matrix $A + \lambda I$ which has one unstable eigenvalue. However, it does not require the minimality of the linear part of the system (1). To establish these conditions in the Lagrange stabilization Problems 1 and 2, Ouyang et al. (2009) and Ouyang et al. (2010) develop a pseudo-$H_\infty$ control theory, which is analogous to the standard $H_\infty$ control theory. The following results from Ouyang et al. (2010) will be used to establish the main results of this paper:

Definition 3. A matrix $A \in \mathbb{R}^{n \times n}$ which has $n - 1$ eigenvalues with negative real parts and one eigenvalue with positive real part is said to be pseudo-Hurwitz. A symmetric matrix $P \in \mathbb{R}^{m \times m}$ is said to be pseudo-positive definite if it has $n - 1$ positive eigenvalues and one negative eigenvalue.

Definition 4. A linear time-invariant (LTI) system (1) is called pseudo strict bounded real if the following conditions hold:

i. $A$ is pseudo Hurwitz;

ii. $\max_{\omega \in \Omega} \{\sigma_{max}\{G(-jo I)G(jo I)\}\} < 1. \quad (7)$

The following assumption is also made on the system (5):

Assumption 1. $E_1 = D_{12}^T D_{12} > 0$ and $E_2 = D_{21}^T D_{21} > 0$.

Analogous to the standard output feedback $H_\infty$ control problem, the output feedback pseudo $H_\infty$ control problem for the system (5) involves designing a compensator of the form (6) to make the corresponding closed-loop system pseudo strict bounded real. The following two lemmas each give a sufficient condition for the existence of a solution to the output feedback pseudo-$H_\infty$ control problem for a system of the form (5).

Lemma 3. Suppose the system (5) satisfies Assumption 1 and the following conditions are satisfied:

i. The Riccati equation

$$(A - B_2 E_1^{-1} D_{12}^T C_1) X + X (A - B_2 E_1^{-1} D_{12}^T C_1)^T + X (B_1 B_1^T - B_2 E_1^{-1} B_1^T) X + C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0 \quad (8)$$

has a stabilizing solution $X = X^T$ which is pseudo-positive definite;

ii. The Riccati equation

$$(A - B_1 D_{21}^T E_2^{-1} C_2) Y + Y (A - B_1 D_{21}^T E_2^{-1} C_2)^T + Y (C_2^T I - C_2^T E_2^{-1} C_2) Y + B_1 (I - D_{21} E_2^{-1} D_{21}) B_1^T = 0 \quad (9)$$

has a stabilizing solution $Y = Y^T$ which is positive definite;

iii. The matrix $XY$ has a spectral radius strictly less than one, moreover, $\rho(XY) < 1. \quad (10)$

Then, there exists a dynamic output feedback compensator of the form (6) such that the resulting closed-loop system is pseudo strict bounded real. Furthermore, the matrices defining the required dynamic feedback controller (6) can be constructed as follows:

$$\begin{align*}
A_c &= A + B_2 C_2 - B_2 C_2 + (B_1 - B_1 D_{21}) B_1^T X,
B_c &= (I - Y X)^{-1} (Y C_2^T + B_1 D_{21}) E_2^{-1},
C_c &= -E_1^{-1} (B_1^T X + D_{12}^T C_1).
\end{align*} \quad (10)$$

3. OUTPUT FEEDBACK LAGRANGE STABILIZATION FOR PENDULUM-LIKE SYSTEM WITH MULTIPLE NONLINEARITIES

In this section, the systems under consideration are assumed to be either unobservable or uncontrollable. These assumptions are used to ensure that the closed-loop system is pendulum-like and to rule out trivial cases in which the nonlinear system can be asymptotically stabilized. Accordingly, this section will give sufficient conditions for output feedback Lagrange stabilization for these two cases.

3.1 Solution to Problem 1 for Unobservable Systems

Assumption 2. There exists a non-zero vector $x$ such that $A x = 0$ and $C_2 x = 0$.

Assumption 2 implies that $(A, C_2)$ is unobservable. Using the Kalman decomposition in the unobservable form (e.g., see Antsaklis and Michel (2006)), it follows that there exists a non-singular state-space transformation matrix $T$ such that the system matrices of the system (5) are transformed to the form

$$\begin{align*}
\begin{bmatrix}
A_1 & B_1 \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
X_1 & X_2 \\
0 & X_3
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_1 & Y_2 \\
0 & Y_3
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\end{align*}$$

where $A_1$ and $A_2$ are matrices of suitable dimensions.
\[ \tilde{A} = T^{-1}AT = \begin{bmatrix} \tilde{A}_1 & 0 \\ \tilde{A}_2 & 0 \end{bmatrix} \]
\[ \tilde{B}_2 = T^{-1}B_2 = \begin{bmatrix} \tilde{B}_{2a} \\ \tilde{B}_{2b} \end{bmatrix} \]
\[ \tilde{C}_1 = C_1T = \begin{bmatrix} \tilde{C}_{1a} \\ \tilde{C}_{1b} \end{bmatrix} \]
\[ D_{21} = D_{12}, \quad \tilde{C}_2 = C_2T = \begin{bmatrix} \tilde{C}_{2a} \\ 0 \end{bmatrix} \]
where \(\tilde{A}_1 \in \mathbb{R}^{(n-l) \times (n-l)}, \quad \tilde{A}_2 \in \mathbb{R}^{(n-l) \times l}, \quad \tilde{B}_{2a} \in \mathbb{R}^{(n-l) \times q}, \quad \tilde{B}_{1a} \in \mathbb{R}^{(n-l) \times m}, \quad \tilde{C}_{1a}, \tilde{C}_{2a} \in \mathbb{R}^{m \times (n-l)}.\)

\[ \tilde{A}_1 \in \mathbb{R}^{(n-l) \times (n-l)}, \quad \tilde{A}_2 \in \mathbb{R}^{(n-l) \times l}, \quad \tilde{B}_{2a} \in \mathbb{R}^{(n-l) \times q}, \quad \tilde{B}_{1a} \in \mathbb{R}^{(n-l) \times m}, \quad \tilde{C}_{1a}, \tilde{C}_{2a} \in \mathbb{R}^{m \times (n-l)}.\]

Also, let \(e_n = [0_{1 \times (n-1)}] \in \mathbb{R}^{n-1}\) and \(d = [0_{1 \times n}e_n^T T^T] \in \mathbb{R}^{n-1}\).

Assumption 3. There exists a constant \(t_0 > 0\) such that all the elements of the vector \(v = v_0\tilde{A}^{-1}X\) are non-zero rational numbers.

Remark 1. In the case that the coefficients in the system (5) are all rational numbers, Assumption 3 amounts to an assumption that the periods of the nonlinearities are commensurate.

The main result of this section involves the following Riccati equations dependent on parameters \(\lambda > 0\) and \(\tau > 0\), \(i = 1, \ldots, m;\)

\[ (\lambda I + A - B_2 \tilde{E}_1^{-1} D_{12}^T M_2 C_1) X + X (\lambda I + A - B_2 \tilde{E}_1^{-1} D_{12}^T M_2 C_1)^T 
+ X (B_1 (M_1 \tilde{M}_2^{-1} M_2 B_2 - B_2 \tilde{E}_1^{-1} B_2^T) X 
+ C_2^T (M_2 - M_2 D_2 \tilde{E}_1^{-1} D_{12}^T M_2) C_1 = 0, \]

where \(\tilde{E}_1 = D_{12}^T M_2 D_{12}^T\) and \(\tilde{E}_2 = D_{21} M_2^{-1} M_2 D_{12}^T\). If these Riccati equations have suitable solutions, we will define the parameter matrices of the controller (6) as follows:

\[ A_c = A + B_2 C_2 - B_2 C_1 \quad Y C_2^T (M_2 C_1 - M_2 D_2 C_1), \]
\[ B_c = - (Y C_2^T + B_1 M_2^{-1} M_2 D_{12}^T) \tilde{E}_2^{-1}, \]
\[ C_c = \tilde{E}_1^{-1} (B_2^T X + D_{12}^T M_2 C_2) (I - Y X)^{-1}. \]

The following theorem, which is one of the main results of this paper, gives a sufficient condition for the existence of a Lagrange stabilizing controller for the nonlinear system (5), (2), (3), (4):

**Theorem 1.** Suppose Assumptions 1, 2 and 3 hold for the nonlinear system (5), (2), (3), (4). Also, suppose there exist constants \(\lambda > 0\) and \(\tau > 0\), \(i = 0, \ldots, m\) such that the following conditions are satisfied:

**I.** The Riccati equation (13) has a stabilizing solution \(X = X^T\) which is positive definite;

**II.** The Riccati equation (14) has a stabilizing solution \(Y = Y^T\) which is positive pseudo definite;

**III.** The matrix \(XY\) has a spectral radius strictly less than one, \(\rho(XY) < 1\).

Then, the resulting closed-loop system corresponding to the controller (6), (15) is a pendulum-like system with respect to \(\Pi(n,pd)\) and is Lagrange stable. Here \(\tilde{p} = \text{LCMD}(v)\).

**Proof:** Substitute the controller (6), (15) into the system (5):

\[ \begin{bmatrix} \dot{x}_c \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_c & B_c C_2 & A \end{bmatrix} \begin{bmatrix} x_c \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{2c} & \frac{B_{2c} \tilde{D}_c}{\tilde{C}_{2a}} \end{bmatrix} \begin{bmatrix} \tilde{D}_{12} C_1 \end{bmatrix} \begin{bmatrix} x_c \end{bmatrix}, \]

\[ z = \begin{bmatrix} D_{12} C_1 \end{bmatrix} \begin{bmatrix} x_c \end{bmatrix}, \quad (16) \]

We first prove that the resulting closed-loop system is pendulum-like. Let \(\bar{d} = [0_{1 \times n} e_n^T T^T] \in \mathbb{R}^{n-1}\). Note that the identity

\[ \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}^{-1} \begin{bmatrix} A_c & B_c C_2 \\ B_{2c} & A \end{bmatrix} \begin{bmatrix} 0_{(2n-1) \times 1} \end{bmatrix} \quad (17) \]

\[ \begin{bmatrix} 0_{(2n-1) \times 1} \end{bmatrix} = 0. \]

Using this fact and Assumption 2, it follows from Lemma 1 that the resulting closed-loop system (16) is pendulum-like system with respect to the set \(\Pi(n, p_d)\).

From the output feedback pseudo-Hamilton control theory in Section III, Conditions I, II, III of the theorem imply that the matrix \(\begin{bmatrix} \lambda I + A & B_2 C_2 \\ B_2 C_2 & \lambda I + A_c \end{bmatrix}\) is pseudo-Hurwitz and the frequency-domain matrix \(G_\sigma(\tau) \in \mathbb{R}^{n \times m}\) holds, where

\[ \tilde{G}(\omega) = \begin{bmatrix} M_2 G(\omega) M_2^{-1} \end{bmatrix} \begin{bmatrix} s \end{bmatrix} < 1 \text{ for all } \omega \in \mathbb{R}. \]

Then, it follows that \(G_\sigma(\tau) \in \mathbb{R}^{n \times m}\) is Lagrange stable. This completes the proof.

**Remark 2.** It is straightforward to verify that the matrix A in (5) must have a single observable mode at the origin if the conditions of Theorem 1 are satisfied.

**Remark 3.** The rationality condition for Theorem 1 is a property of system (5) and does not relate to the solution to the Riccati equations (13) and (14).

### 3.2 Solution to Problem 1 for Uncontrollable Systems

**Assumption 4.** There exists a non-zero vector \(x\) such that \(x^T A = 0\) and \(x^T B_2 = 0\).

In a similar way to Assumption 2, this assumption is used to ensure that the closed-loop system is pendulum-like and to rule out trivial cases in which the system can be asymptotically stabilized. Also, this assumption implies that the pair \((A, B_2)\) is uncontrollable. Using the Kalman Decomposition (e.g., see Antsaklis and Michel (2006)), it follows from Assumption 4 that there exists a non-singular state-space transformation matrix \(T\) such that the matrices of the system (5) are transformed to the form

\[ \tilde{A} = T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \tilde{B}_2 = T^{-1}B_2 = \begin{bmatrix} \tilde{B}_{2a} \\ \tilde{B}_{2b} \end{bmatrix}, \]

\[ \tilde{C}_1 = C_1T = \begin{bmatrix} \tilde{C}_{1a} \\ \tilde{C}_{1b} \end{bmatrix}, \quad \tilde{C}_2 = C_2T = \begin{bmatrix} \tilde{C}_{2a} \\ \tilde{C}_{2b} \end{bmatrix}, \quad \tilde{D}_{12} = D_{12}, \quad \tilde{D}_{21} = D_{21}, \]

where \(A_1 \in \mathbb{R}^{(n-l) \times (n-l)}, \quad A_2 \in \mathbb{R}^{(n-l) \times l}, \quad \tilde{B}_{2a} \in \mathbb{R}^{(n-l) \times q}, \quad \tilde{B}_{1a} \in \mathbb{R}^{(n-l) \times m}, \quad \tilde{C}_{1a}, \tilde{C}_{2a} \in \mathbb{R}^{m \times (n-l)}.

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The main result of this section involves the following Riccati equations dependent on parameters $\lambda > 0$ and $\tau_i > 0$, $i = 1, \ldots, m$:

$$
(\lambda I + A - B_2 \tilde{E}_2^{-1} D_{12}^2 M_2 C_1) X + X (\lambda I + A - B_2 \tilde{E}_2^{-1} D_{12}^2 M_2 C_1)
+ X (B_1 M_\mu M_\tau^{-1} M_\mu B_1^T - B_2 \tilde{E}_2^{-1} B_2^T) X
+ C_1^T (M_\tau - M_\tau D_1^2 \tilde{E}_2^{-1} D_{12}^2 M_2 C_1) = 0,
$$

(18)

$$
(\lambda I + A - B_1 M_\mu M_\tau^{-1} M_\mu D_{12}^2 \tilde{E}_2^{-1} C_2^T) Y
+ Y (\lambda I + A - B_1 M_\mu M_\tau^{-1} M_\mu D_{12}^2 \tilde{E}_2^{-1} C_2^T)^T
+ B_1 (M_\mu M_\tau^{-1} M_\mu B_1^T - M_\mu M_\tau^{-1} M_\mu D_{12}^2 \tilde{E}_2^{-1} D_{12} M_\mu M_\tau^{-1} M_\mu) B_1^T
+ Y (C_1^T M_\tau C_1 - C_2^T \tilde{E}_2^{-1} C_2) Y = 0,
$$

(19)

where $\tilde{E}_1 = D_{12}^2 M_2 D_{12}$ and $\tilde{E}_2 = D_{12} M_\mu M_\tau^{-1} M_\mu D_{12}^2$.

If these Riccati equations have suitable solutions, then we can construct the following matrices for the controller (6):

$$
A_c = A + B_2 C_c - B_1 C_2
+ (B_1 M_\mu M_\tau^{-1} M_\mu - B_2 D_{12} M_\mu M_\tau^{-1} M_\mu) B_1^T X,
B_c = (I - Y X)^{-1} (Y C_1^T + B_1 M_\mu M_\tau^{-1} M_\mu D_{12}^2 \tilde{E}_2^{-1}),
C_c = - \tilde{E}_1^{-1} (B_1^T X + D_{12}^2 M_\tau C_1).
$$

(20)

Also, we define two vectors of constants, $d_0 \in \mathbb{R}^{2m-1}$ and $\chi \in \mathbb{R}^m$, as follows:

$$
d_0 = \begin{bmatrix} A_c & B_c C \end{bmatrix}^{-1} \begin{bmatrix} B_c \end{bmatrix},
\chi = \begin{bmatrix} - D_{12} \tilde{E}_1^{-1} (B_1^T X + D_{12}^2 M_\tau C_1) C_1 \end{bmatrix} d,
$$

(21)

where $d = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} d_0 \\ 1 \end{bmatrix}$, with $T$ defined in the Kalman decomposition (17). It will be assumed later that the matrix involved in the definition of $d_0$ is nonsingular. Using this notation, a sufficient condition for the solution to the output feedback Lagrange stabilization Problem 1 can now be presented:

**Theorem 2.** Suppose Assumptions 1 and 4 hold for the system (5), (2), (3), (4). Also, suppose there exist constants $\lambda > 0$ and $\tau_i > 0$, $i = 1, \ldots, m$ such that the following conditions are satisfied for the nonlinear system (5), (2), (3), (4):

I. The Riccati equation (18) has a stabilizing pseudo-positive definite solution $X = X^T$;

II. The Riccati equation (19) has a stabilizing solution $Y = Y^T$ which is positive definite;

III. The matrix $XY$ has a spectral radius strictly less than one, $\rho(XY) < 1$;

IV. The matrix $\begin{bmatrix} A_c & B_c C \end{bmatrix}$ is non-singular and all the elements of the vector $v = \tau_0 \Delta^{-1} \chi$ are non-zero rational numbers.

Then, the closed-loop system consisting of (5), (2), (3), (4) and (6), (20) is a pendulum-like system with respect to $\Pi \lambda = \{ \tilde{d}_0 \}$ and is Lagrange stable. Here $\tilde{d} = \text{LCMD}(v)$.

**Proof:** We first observe that the closed-loop system (16), obtained by applying the compensator (6), (20) to the system (5), is a pendulum-like system. Indeed, since

$$
I \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_c & B_c C \end{bmatrix} \begin{bmatrix} B_2 \end{bmatrix} \begin{bmatrix} A_1 \end{bmatrix} \begin{bmatrix} \tilde{d}_0 \\ 0 \end{bmatrix} = 0
$$

and $\begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$ is a non-singular matrix, it follows that $\begin{bmatrix} A_c & B_c C \end{bmatrix} \begin{bmatrix} \tilde{d}_0 \\ 0 \end{bmatrix} = 0$. Using this fact and Condition IV of the theorem, it follows from Lemma 1 that the augmented closed-loop system (16), (2), (3), (4) is a pendulum-like system with respect to $\Pi(\tilde{d}_0 \tilde{d})$.

Using the output feedback pseudo-$H_\infty$ control theory given in Section III, it follows from Conditions I, II and III of the theorem that the closed-loop system (16) is pseudo strict bounded real. In a similar way to the proof of Theorem 1, we have $G_0^T (- j \omega - \lambda) M_\tau G_i (- j \omega - \lambda) < M_\mu^{-1} M_\tau M_\mu^{-1}$. Now, using Lemma 2, it follows that the closed-loop system (16), (2), (3), (4) is Lagrange stable. \( \square \)

4. ILLUSTRATIVE EXAMPLE

We consider a system consisting of four pendulums connected in a ring using torsional springs, as shown in Figure 2, and both pendulums and springs are supported by a rigid ring. The pendulums oscillate in planes perpendicular to the ring and the torque on the springs obeys the angular form of the Hook’s law $\tau_q = \tau_q \Delta \Phi$ where $\Delta \Phi$ is the angular displacement, $\tau_q$ is the storing torque and $k$ is the torque constant. Suppose that the measurements consist of the angular velocity of a pendulum (e.g., pendulum 4 in the figure) and the angular differences between two neighboring pendulums (e.g., pendulums 1 and 2, 2 and 3, 3 and 4, and 4 and 1 in the figure). This system can be described by equation (5) with the following matrices:

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_4 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 \\
-k_1 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 \\
-k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
C_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

(22)
The vector of nonlinearity inputs is given by
\[ w = [\sin z_1 \sin z_2 \sin z_3 \sin z_4]^T. \]

Hence, each nonlinearity satisfies the sector constraint (4) with \( \mu = 1 \). The parameters of the system are \( \alpha_1 = 0.1, \alpha_2 = 0.05, \alpha_3 = 0.08, \alpha_4 = 0.06, k_1 = 0.02, k_2 = 0.03, k_3 = 0.05, k_4 = 0.04, \epsilon_1 = 0.05, \epsilon_2 = 0.05, \gamma = 0.3, \beta = 0.1 \). It is easy to verify that the matrix \( A \) of the system (22) is singular and the pair of the system (22) is unobservable. It is easy to verify that the system (22) satisfies Assumption 2. Also, as all the coefficients of the system (22) are rational, we can choose \( \tau_0 = 2\pi \) to ensure that Assumption 3 is satisfied (since \( T \) will have rational elements). Therefore, Theorem 1 can be used to solve the output feedback Lagrange stabilizing controller synthesis problem for the nonlinear system (22). Select \( \tau_1 = 0.4, \tau_2 = 0.3, \tau_3 = 0.3, \tau_4 = 0.5 \) and \( \lambda = 0.2 \). The solutions to the Riccati equation (13), (14) satisfy the conditions of Theorem 1.

From Theorem 1, we can verify that the corresponding closed-loop system is Lagrange stable. Now, we use simulations to validate this result. A series of simulations have shown that the trajectories of the closed-loop system are bounded. Figure 3 and 4 show the state responses of the system when the output feedback controller is applied and the corresponding controller state responses. As shown in Figure 3 and 4, all of the trajectories of the augmented closed-loop system are bounded. That is, the resulting closed-loop system is Lagrange stable.

The vector of nonlinearity inputs is given by
\[ w = [\sin z_1 \sin z_2 \sin z_3 \sin z_4]^T. \]

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5. CONCLUSIONS

This paper has studied the output feedback Lagrange stabilization problem for pendulum-like systems with multiple nonlinearities. A method based on an output feedback pseudo \( H_\infty \) control theory is developed to address this output feedback Lagrange stabilization problem. This method is established using a Lagrange stability criterion which does not require the linear part of the closed loop nonlinear system to be a minimal realization. In addition, the results are illustrated by an example which can be regarded as a prototype of many nonlinear oscillatory systems.

REFERENCES


