Optimal Revenue-Sharing and Network Investment Strategies in Internet Market

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Abstract: In this paper, we consider a revenue-sharing and network investment problem between an Internet service provider (ISP) and a content provider (CP) by applying dynamic agency theory. We formulate the problem as a principal-agent problem where the ISP is the principal and the CP is the agent. The principal-agent problem is then transformed to a stochastic optimal control problem in which the objective of the ISP is to find an optimal revenue-sharing strategy and a network investment strategy, and to advise an incentive-compatible effort level to the CP. The sufficient conditions for the existence of the optimal revenue-sharing strategy, the optimal network investment strategy and the incentive-compatible effort level are obtained. A numerical example is solved to show the existence of such strategies.

Keywords: Internet service provider, content provider, dynamic agency theory, stochastic control problem.

1. INTRODUCTION

Commercial Internet services started nearly 20 years ago. Since then, various network businesses such as application- and content-providing services have been developed. The Internet has now become an integral infrastructure of the society which affects people’s life, and their social and economic activities.

Internet service providers (ISPs) usually adopt a fixed access-charging system, which charges end-users with a fixed monthly fee, without considering traffic volume. In the early years of Internet, as end-users increase, the profit of ISPs has expanded smoothly by adopting such a fixed access-charging system. However, in recent years, the financial situation of ISPs is not as good as before because the growth of end-users has approached a saturable level. Moreover, as the spread of broadband Internet and the wide use of various contents provided by content providers (CPs) such as Google, Yahoo, etc, the traffic volume of an end-user has increased quickly. As a result, the ISPs are required to increase their network investment to meet the demand and to improve the quality of networks. However, it is difficult for the ISPs to collect the network investment by raising the access charge to end-users under the existing fixed access-charging system because it will cause damage to the fairness among end-users.

Taking this situation into account, some ISPs that burden higher and higher network cost have begun to request those mass-content delivering CPs such as Google and Wikipedia to utilize priority services and charge them according to their traffic volume (Economides (2003), Wu (2003), Sidak (2006)). It is possible for an ISP to charge a mass-content delivering CP, which is connected to his network directly, based on the traffic volume of the CP, it seems difficult for an ISP to charge a mass-content delivering CP, which does not connect to his network directly, but delivers contents to his end-users through using his network indirectly, since only the traffic between two end-users can be measured.

In this paper, we consider a revenue-sharing and network investment problem between an ISP and a CP, where the CP does not connect to the ISP’s network directly, but delivers his contents to the ISP’s end-users. We formulate the problem as a dynamic principal-agent problem (Sunnikov (2008)) where the ISP is the principal and the CP is the agent. The principal-agent problem is then transformed to a stochastic optimal control problem in which the ISP tries to find an optimal revenue-sharing strategy and an optimal-network investment strategy, and to advise an incentive compatible effort level to the CP. The sufficient conditions for the existences of the optimal revenue-sharing strategy, the optimal network investment strategy and the incentive compatible effort to the CP are obtained. A numerical example is solved to show the existence of such strategies.

2. PROBLEM FORMULATION

Consider an Internet system which consists of two ISPs and one CP. The ISPs are peered with each other and the CP is only connected to one ISP directly. For simplicity, we call the ISP connected with the CP as the CP, and the ISP without direct connection with the CP as the ISP.
The network capacity \( C_t > 0; t \in [0; 1) \) of the ISP evolves according to
\[
dC_t = (I_t - C_t) dt, \quad (1)
\]
where \( I_t \in [0; 1) \) denotes the network investment rate and \( \delta > 0 \) is the decay rate of the network capacity of the ISP. The network capacity \( C_t \) is known by both the ISP and the CP. Associated with the network investment, adjustment cost (Hayashi (1982)) \( g(I, C) \) arises which satisfies the conditions
\[
g(0, C) = 0, \quad \partial g(I, C)/\partial I = 0, \quad \partial^2 g(I, C)/\partial I^2 > 0.
\]
In this paper, the adjustment cost function is given by
\[
g(I, C) := \frac{\theta I^2}{2C}, \quad (2)
\]
where \( \theta \) is a positive constant. Since the CP is peered with the ISP, the revenue-sharing with the CP is demanded by the ISP. This can be done through signing up a peering contract between the ISP and the CP. The revenue of the CP depends on the amount of contents downloaded by those end-users who have accessing contract with the ISP. Moreover, the amount of contents downloaded by the end-users depends on the CP’s continuous effort to improve the quality of contents, and a stochastically changing element. As the quality of the contents is improved, more end-users will make use of the contents. As the result, the amount of download will increase. However, the download amount is restricted by the ISP’s network capacity. In this paper, it is assumed that the download amount requested by end-users always exceeds the network capacity. Therefore, the download amount is equal to the network capacity of the ISP. The download amount of contents is observable by both the ISP and the CP. But, the CP’s effort level is not observable by the ISP.

The CP’s cumulative sales \( X_t \) at time \( t \) evolves according to
\[
x_t = p(a_t)C_t dt + \sigma C_t dZ_t, \quad (3)
\]
where \( C_t \) is the ISP’s network capacity, \( a_t \) is the CP’s choice of effort level, \( p(a_t) \) is the content value (or price) per unit which is the function of the CP’s effort level and \( \sigma \) is a constant. \( Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty \} \) is a standard Brownian motion driving the sales process, and \( \{\mathcal{F}_t; 0 \leq t < \infty \} \) is the filtration determined by \( \{X_t; 0 \leq t < \infty \} \). The CP’s effort range is denoted by \( a_t \in [0, \bar{a}] \) where \( \bar{a} \) is the upper bound of the effort level, \( p(a_t) \) is a continuous, strictly increasing and concave function of \( a_t \) which is known by the ISP. For simplicity, it is assumed that all contents of the CP have the same content value \( p(a_t) \), and the cumulative sales is observable by the ISP.

The CP’s sales \( X_t \) is allocated by the CP and the ISP under the conditions in a revenue-sharing contract. Let \( p(a_t)C_t \in [0, \infty) \) denote the revenue obtained from the download by the ISP’s end users at time \( t \). Then, \( \gamma_t C_t \in [0, \infty) \) is the revenue shared by the CP at time \( t \). \( \gamma_t \) is the revenue shared by the CP for a unit content and is called the revenue-sharing strategy. It is worth noting that the CP’s expected sales can be determined by a unit content value \( p(a_t) \) and the network capacity \( C_t \) since all contents provided by the CP have the same content value. A rational ISP will always demand a nonnegative allocation of sales, there exists an upper bound on the revenue-sharing strategy \( \gamma_t \), that is,
\[
\gamma_t C_t \leq \pi_t C_t = E[X_t] = p(a_t)C_t.
\]

Suppose that the CP obtains the utility \( u(\gamma_t)C_t \) from the revenue-sharing \( \gamma_t C_t \), where \( u(\gamma_t) \) is an increasing, concave and \( C^2 \) function that satisfies \( u(0) = 0 \). On the other hand, the CP incurs cost of effort \( h(a_t)C_t \), measured in the same unit as the utility of revenue-sharing, where, \( h(a_t) \) is a continuous, increasing and convex function of \( a_t \). It is assumed that the ISP knows the CP’s utility function and cost function.

The ISP invests \( \lambda_t \) to procure network equipments and incurs the network maintenance cost \( \beta C_t \), where \( \lambda, \beta \) are positive constants.

For simplicity, it is assumed that both the ISP and the CP discount the flow of profit and utility at a common rate \( r \). If the CP chooses an effort level \( a_t, 0 \leq t < \infty \), the CP’s total expected utility is given by
\[
E\left[\int_{0}^{\infty} e^{-rt} u(\gamma_t) C_t dt \right],
\]
and the ISP’s total expected profit is
\[
E\left[\int_{0}^{\infty} e^{-rt} dX_t - \int_{0}^{\infty} e^{-rt} \gamma_t C_t dt - \int_{0}^{\infty} e^{-rt} \lambda_t dt \right]
\]
\[
- \int_{0}^{\infty} e^{-rt} g(I_t, C_t) dt - \int_{0}^{\infty} e^{-rt} \beta C_t dt \]
\[
= E\left[\int_{0}^{\infty} e^{-rt} (p(a_t) - \gamma_t - \beta) C_t - \lambda_t - g(I_t, C_t) \right] dt \]
where \( \lambda_t, \beta C_t \) denote the investment cost and the network maintenance cost, respectively.

Before we formulate the optimal control problem for the ISP, let us confirm the CP’s effort level and the network investment when there is no revenue-sharing with the ISP. In this situation, the ISP will have no incentive to invest in network to increase the network capacity. What the ISP needs is the investment to maintain the existing network capacity, respectively.

\[
I_t = \delta C_t \quad (4)
\]
from (1). Moreover, since the network capacity is fixed, the CP will choose an effort to maximize the profit \( p(a_t)C_t - h(a_t)C_0 \), that is,
\[
p'(a) = h'(a). \quad (5)
\]

2.1 ISP’s Problem

Under the condition of a revenue-sharing contract, the CP will choose an effort to maximize its expected utility. Knowing the behavior of the CP, the ISP’s problem is to offer a revenue-sharing contract to the CP, which includes an incentive-compatible advice of effort level \( \{a_t, 0 \leq t < \infty \} \) to the CP and a revenue-sharing strategy \( \{\gamma_t, 0 \leq t < \infty \} \), and a network investment strategy \( \{I_t, 0 \leq t < \infty \} \), such that the ISP’s total expected profit
\[ E \left[ \int_0^1 e^{-rt} \left( (p(a_t) - \gamma_t - \beta) C_t - \lambda_I t - g(I_t, C_t) \right) dt \right] \] (6)

is maximized. We say that an effort level \( \{a_t, 0 \leq t \leq \infty\} \) is incentive compatible with respect to the revenue-sharing strategy \( \{\gamma_t, 0 \leq t \leq \infty\} \) if it satisfies

\[ a_t \in \arg \max_{\tilde{a}_t} E \left[ \int_0^\infty e^{-rt} \left( u(\gamma_t) - h(\tilde{a}_t) \right) C_t dt \right], \] (7)

\[ E \left[ \int_0^\infty e^{-rt} \left( u(\gamma_t) - h(\tilde{a}_t) \right) C_t dt \right] \geq 0. \] (8)

It is obvious that an incentive-compatible effort level is relevant to a revenue-sharing strategy from (7).

### 2.2 The CP’s Continuation Value

In order to make the CP choose a recommended incentive-compatible effort level, the ISP is required to design an revenue-sharing strategy which will change the allocation of sales to the CP according to his effort. Instead of designing a strategy which depends on the sales of the CP, we design a strategy which depends on the CP’s continuation value \( W_t \). The continuation value \( W_t \) is the total utility that the ISP expects the CP to derive from the future after a given moment of time \( t \geq 0 \).

Suppose that a network capacity \( C = \{C_t\} \), a sales allocated to the CP \( \gamma = \{\gamma_t\} \) and an effort level \( a = \{a_t\} \) are given. The CP’s continuation value is

\[ W_t(\gamma, I, a) = \int E_a \left[ \int_t^\infty e^{-r(s-t)} \left( u(\gamma_s) - h(a_s) \right) C_s ds | F_t \right] \] (9)

where \( E_a \) denotes the expectation under the probability measure \( \Omega_a \) induced by the CP’s effort \( a = \{a_t\} \). In the designing of the ISP’s incentive revenue-sharing strategy, \( W_t \) will play the role of the unique state variable that determines how much the CP’s allocation of the sales is, what effort level the CP is advised to choose, and how \( W_t \) itself evolves with the realization of the sales. The ISP is required to use \( W_t \) as state feedback to design a revenue-sharing strategy \( \gamma_t \), a recommended effort level \( a_t \) and a network investment strategy \( I_t \) to achieve two objectives. First, the CP must have sufficient incentive to choose the recommended effort level. Second, the ISP’s profit is maximized.

It is worth noting that, no matter how much the continuation value \( W_t \) is, the ISP has the option to stop the Internet peering contract with the CP suppose that the ISP is willing to pay the cancelation cost to the CP. The cancelation cost is determined by the continuation value \( W_t \) and the network capacity \( C_t \) at the time of cancelation. The ISP’s profit function at the time of cancelation is \( \Psi(W_t, C_t) = -\gamma_t C_t \), where \( \Psi(0, C_t) = 0 \). Since the CP can choose zero effort after contract cancelation, the CP’s continuation value at time \( t \) is \( W_t = u(\gamma_t) C_t \).

If the CP’s continuation value \( W_t \) is extremely high, the ISP will cancel the contract with the CP. The reason is that the continuation value \( W_t \) increases as the increase of the allocation of the sales to the CP. However, if the allocation of the sales to the CP is too high, the allocation of the sales to the ISP will be less than the network cost incurred by the ISP. Therefore, there exists a continuation value \( W^* > 0 \) such that the ISP is willing to pay the cancelation cost \( \Psi(W^*, C_t) \) to the CP to stop the contract.

### 3. Optimal Revenue-Sharing and Investment Strategies

In this section, we derive the optimal solutions to the problem formulated in the above section. First, as a preliminary result, we give the following Proposition, which is proved formally in Appendix A, to describe the evolution of the CP’s continuation value \( W_t \).

**Proposition 1.** Suppose that a revenue-sharing strategy \( \gamma = \{\gamma_t\} \), a network investment strategy \( I = \{I_t\} \) and an effort level \( a = \{a_t\} \) after time \( t > 0 \) are given. There exists a \( F_t \)-progressively measurable process \( Y_t \) such that the CP’s continuation value \( W_t(\gamma, I, a) \) defined by (9) can be described by the stochastic differential equation

\[ dW_t(\gamma, I, a) = \left( r W_t(\gamma, a) - (u(\gamma_t) - h(a_t)) C_t \right) dt + \sigma Y_t C_t dZ_t. \] (10)

Second, we give the following Proposition, which is proved formally in Appendix B, to describe the incentive-compatibility condition on the CP’s effort.

**Proposition 2.** Suppose that \( Y_t \) is a progressively measurable process defined by Proposition 1. Then the CP’s effort \( a_t \) is optimal if and only if

\[ a_t \in \arg \max_{\tilde{a}_t \in [0, a]} \left( Y_t p(\tilde{a}_t) - h(\tilde{a}_t) \right) C_t, \] (11)

almost everywhere.

From Proposition 2, it is found that \( Y_t \) is the function of the CP’s incentive compatible effort \( a_t \), that is,

\[ Y_t = \frac{h'(a_t)}{p'(a_t)} = g(a_t) > 0. \] (12)

\( g(a_t) \) is an increasing function of \( a_t \). Since \( Y_t \) of (10) represents the volatility of the CP’s continuation value \( W_t(\gamma, I, a) \), the CP’s risk will increase as the increase of the effort.

Suppose that the evolution of the CP’s continuation value \( W_t \) is known. The ISP’s optimal control problem to find an optimal revenue-sharing strategy \( \gamma_t \), an optimal network investment strategy \( I_t \) and a recommended effort \( a_t \), which satisfies the incentive compatibility condition, can be formulated as a stochastic optimal control problem:

\[ \Pi(W, C) = \max_{a, \gamma, I} E \left[ \int_0^\infty e^{-r(s-t)} \left( (p(a_s) - \gamma_s - \beta) C_s - \lambda I_s - g(I_s, C_s) \right) ds \right] \] (13)

subject to
Furthermore, the initial condition (17) and the smooth-pasting conditions (18) are denoted by
\[
\pi(0) = 0, \quad \pi(y^*(\tau)) = -\psi(w^*(\tau)), \quad \pi'(y^*(\tau)) = -\psi'(w^*(\tau)),
\]
where \( W^2(\tau) = Cw^2(\tau) \), and \( \psi(\cdot) \) is the ISP’s value function (21) when \( a = 0, I = 0, \delta = 0, \beta = 0 \).

3.2 The Optimal Strategies

Suppose that the solution \( \pi(w) \) of (21) exists, we have the following Propostion, which is proved formally in Appendix C.

**Proposition 3.** Suppose that \( \pi(w) \) satisfies the HJB equation (21) with respect to \( w_t \in [0, w^2] \) in \( t \in [0, \tau] \), the initial condition (23) and the final condition (24) at \( t = \tau \). If \( a_t \) and \( \gamma_t \) are the effort level and the revenue-sharing strategy maximizing the right-hand side of (21) then \( a_t \) and \( \gamma_t \) are the optimal recommended effort level and the optimal revenue-sharing strategy.

From Proposition 3, the optimal recommended effort \( a(w_t) \) is obtained as the function of \( w_t \) by maximizing
\[
p(a) + h(a)\pi^*(w) + \frac{1}{2}\sigma^2 y(a)^2 \pi''(w),
\]
where \( p(a) \) is the revenue flow, \( -h(a)\pi(w) \) is the effort compensation to the CP, and \( \frac{1}{2}\sigma^2 y(a)^2 \pi''(w) \) is the risk premium paid to the CP for the uncertain business environment.

Similarly, the optimal revenue-sharing strategy is obtained by maximizing
\[
-\gamma - u(\gamma)\pi^*(w).
\]

From the first-order condition \( \pi'(W) = -\frac{1}{w(\gamma)} \), \( \gamma(w_t) \) is obtained as the function of continuation value \( w_t \). \(-\pi'(w)\) represents the ISP’s marginal decreasing of the value function with respect to the continuation value. \( \frac{1}{w(\gamma)} \) represents the ISP’s marginal revenue share with respect to the CP’s utility. Moreover, when \( w \leq w^* \) where \( w^* \) is a point such that \( \pi(w^*) = 0 \), since \( u(\gamma) \geq 0 \) and \( \pi'(w > 0) = 0, \gamma = 0 \) from (26).

The solution \( \pi(w) \) of the HJB equation (21) can be obtained through a numerical computation. In order to show the existence of the solution of (21) and the existence of the corresponding optimal revenue-sharing strategy, the optimal recommended effort and the optimal network investment strategy, an illustrative example is provided. The functions and parameters appeared in the problem formulation are defined as follows.

\[
p(a) = a, \quad u(\gamma) = \sqrt{c}, \quad h(a) = 0.5\sigma^2 + 0.4a, \quad r = 0.1, \quad \delta = 0.05, \quad \theta = 20, \quad \beta = 0.1, \quad \lambda = 0.75, \quad \sigma = 1.
\]

The numerical results of the ISP’s value function, the optimal recommended effort, the optimal revenue-sharing strategy and the network investment strategy are shown in Fig. 1. Moreover, \( w^* = 1.132, \pi(w^*) = 0.8055 \).

Furthermore, it is found from (5) and (4) that the CP’s effort level \( (a = 0.6 \text{ in the example}) \) and the ISP’s network...
investment ratio \((i = 0.05\) in this example) when the revenue-sharing does not exist are lower than the effort level and the network investment ratio when the revenue-sharing exists.

Consider an arbitrary alternative strategy \(\tilde{a} = (\tilde{a}_t; \tilde{a}_t \in [0; a])\). Define by

\[
V_t = V_0 + \int_0^t e^{-r s} \sigma Y_s C_t dZ_s, \quad 0 \leq t < \infty
\]

from the Martingale Representation Theorem. Differentiating (A.1), (A.2) with respect to \(t\) gives

\[
dV_t = e^{-rt} \left( u(\gamma_t) - h(a_t) \right) C_t dt + d(e^{-rt} W_t(\gamma, I, a)).
\]

It is easy to confirm that \(V_t\) is an \(E_a\)-martingale. Since the filtration \(\{F_t\}\) is the same as \(\sigma\)-additive class induced by the stochastic process \(dZ_t = \frac{1}{\sigma} \left[ dX_t - p(a_t) dt \right]\), there exists a measurable process \(\{Y_t, F_t; 0 \leq t < \infty\}\) such that

\[
V_t = V_0 + \int_0^t e^{-r s} \sigma Y_s C_t dZ_s, \quad 0 \leq t < \infty
\]

 REFERENCES


Appendix A. PROOF OF PROPOSITION 1

Suppose that the information before time \(t\) is given, and the strategies \((\gamma, I, a)\) are applied after time \(t\). The CP’s total expected payoff is

\[
V_t = \int_0^t e^{-r s} \left( u(\gamma_s) - h(a_s) \right) C_t ds + e^{-r t} W_t(\gamma, I, a).
\]

\[
(A.1)
\]

4. CONCLUSION

In this paper, we consider a revenue-sharing and network investment problem between the ISP and the CP by applying the dynamic agency theory. We have formulated the problem as the dynamic principal-agent problem. The principal-agent problem is then transformed to the stochastic optimal control problem in which the ISP tries to find an optimal revenue sharing strategy and an optimal network investment strategy, and to advise an incentive compatible effort level to the CP. The sufficient conditions for the existence of the optimal revenue-sharing strategy, the optimal network investment strategy and the incentive compatible effort to the CP are obtained. A numerical example is solved to show the existence of such strategies.
\[ \dot{V}_t = \int_0^t e^{-rs} \left( u(\gamma_s) - h(\tilde{a}_s) \right) C_t ds \]
\[ + e^{-rt} W_t(\gamma, I, a) \]  
the time-t expectation of the CP’s total payoff if the CP experienced the cost of effort from the strategy \( \bar{a} \) before time \( t \), and plans to follow the strategy \( a \) after time \( t \). \( \bar{F}_t \) measurable stochastic process \( \bar{V}_t \) becomes

\[ d\bar{V}_t = e^{-rt} \left( u(\gamma_t) - h(\tilde{a}_t) \right) C_t dt + d(e^{-rt} W_t(\gamma, I, a)) \]
\[ = e^{-rt} \left( u(\gamma_t) - h(\tilde{a}_t) \right) C_t dt \]
\[ - e^{-rt} \left( u(\gamma_t) - h(a_t) \right) C_t dt + e^{-rt} \sigma Y_tC_t dZ_t \]
\[ = e^{-rt} \left( (h(a_t) - h(\tilde{a}_t)) C_t + Y_t (p(\tilde{a}_t) - p(a_t)) C_t \right) dt \]
\[ + e^{-rt} \sigma Y_tC_t \tilde{Z}_t, \]  
(B.2)

where \( \sigma C_t \tilde{Z}_t = \sigma C_t \tilde{Z}_t + \int_0^t (p(\tilde{a}_s) - p(a_s)) C_s ds \).

If \( \{a_t^*\} \) does not meet (11) on a set of positive measure, choose \( a_t^* \) that maximizes \( (Y_t^s p(\tilde{a}_t) - h(\tilde{a}_t)) C_t \). Then, the drift of \( \bar{V}_t \) is nonnegative and positive on a set of positive measure. Therefore, there exists a time \( t > 0 \) such that

\[ E_{\{a_t^*\}} [\bar{V}_t] \geq \bar{V}_0 = W_0(\gamma, I, a) \]  
(B.3)

\[ E_{\{a_t\}} [\bar{V}_t] \] is the CP’s utility when the CP follows \( \{a_t\} \) until \( t \) and then switches to \( \{a_t^*\} \). Since \( E_{\{a_t\}} [\bar{V}_t] \) is larger than \( W_0(\gamma, I, a) \) which is the CP’s utility when \( \{a_t\} \) is followed from time \( t = 0 \), \( \{a_t\} \) is not optimal.

Suppose (11) holds for the strategy \( a \). Then \( \bar{V}_t \) is a supermartingale for any alternative strategy \( \tilde{a} \). Moreover, since the stochastic process \( W(\gamma, I, a) \) is bounded from below, we can take

\[ \bar{V}_\infty = \int_0^\infty e^{-rs} \left( u(\gamma_s) - h(\tilde{a}_s) \right) C_s ds. \]  
(B.4)

as the limit of \( \bar{V}_t \) (Karatzas and Shreve (1991)). Therefore,

\[ W_0(\gamma, I, a) = \bar{V}_0 \geq E_{\{a_t\}} [\bar{V}_\infty] = W_0(\gamma, I, a), \]  
(B.5)

so the strategy \( a \) is at least as good as any alternative strategy \( \tilde{a} \).

Appendix C. PROOF OF PROPOSITION 3

Let \( \tilde{a}_t \) and \( \tilde{\gamma}_t \) be arbitrary feasible effort path and revenue-sharing path with corresponding continuation value trajectory \( \tilde{w}_t \) and \( \tilde{i}_t \) be the optimal network investment ratio which satisfies (20). Using Itô's lemma, we extend

\[ -e^{-(r + \delta - i)t} \pi(\tilde{w}) \]
\[ -d(e^{-(r + \delta - i)t} \pi(\tilde{w})) \]
\[ = e^{-(r + \delta - i)t} \left[ (r + \delta - i) \pi(\tilde{w}) - \frac{\sigma^2}{2} \hat{y}(\tilde{a})^2 \pi''(\tilde{w}) \right] dt - e^{-(r + \delta - i)t} \sigma \hat{y}(\tilde{a}) \pi'(\tilde{w}) d\tilde{Z}_t. \]

From the HJB equation (21), since

\[ e^{-r(t + d - i)} \left( p(\tilde{a}) - \tilde{\gamma} - \tilde{\beta} - i - \frac{\theta}{2} \right) \]
\[ \leq e^{-r(t + d - i)} \left[ (r + \delta - i) \tilde{\pi}(\tilde{w}) - \left( (r + \delta - i) \tilde{\pi}(\tilde{w}) - u(\tilde{\gamma}) + h(\tilde{a}) \right) \pi'(\tilde{w}) - \frac{1}{2} \sigma^2 \hat{y}(\tilde{a})^2 \pi''(\tilde{w}) \right], \]

we have

\[ -e^{-r(t + d - i)} \pi(\tilde{w}) \]
\[ \geq -\pi(w_0) + \int_0^t e^{-r(t + d - i)} \left( p(\tilde{a}) - \tilde{\gamma} - \tilde{\beta} - i - \frac{\theta}{2} \right) dt \]
\[ - \int_0^t e^{-r(t + d - i)} \sigma \hat{y}(\tilde{a}) \pi'(\tilde{w}) d\tilde{Z}_t. \]

Therefore,

\[ E_{\tilde{a}} \left[ \int_0^t e^{-r(t + d - i)} \left( p(\tilde{a}) - \tilde{\gamma} - \tilde{\beta} - i - \frac{\theta}{2} \right) dt \right] \]
\[ \leq \pi(w_0) - e^{-r(t + d - i)} \pi(\tilde{w}), \]

On the other hand, for the effort path \( \bar{a}_t \) and the revenue-sharing path \( \bar{\gamma}_t \), which maximize the right-hand side of (21), with corresponding continuation value trajectory \( w_t \), we get

\[ E_{\bar{a}} \left[ \int_0^t e^{-r(t + d - i)} \left( p(\bar{a}) - \gamma - \beta - i - \frac{\theta}{2} \right) dt \right] \]
\[ = \pi(w_0) - e^{-r(t + d - i)} \pi(w), \]

from the HJB equation (21) by making a similar calculation. Comparing two expressions arrive at

\[ E_{\tilde{a}} \left[ \int_0^t e^{-r(t + d - i)} \left( p(\bar{a}) - \tilde{\gamma} - \tilde{\beta} - i - \frac{\theta}{2} \right) dt \right] + e^{-r(t + d - i)} \pi(\tilde{w}) \]
\[ \geq E_{\bar{a}} \left[ \int_0^t e^{-r(t + d - i)} \left( p(\tilde{a}) - \tilde{\gamma} - \tilde{\beta} - i - \frac{\theta}{2} \right) dt \right] + e^{-r(t + d - i)} \pi(\tilde{w}), \]

Therefore, \( a_t \) is the ISP’s optimal recommended effort strategy and \( \gamma_t \) is the ISP’s optimal revenue-sharing strategy.