Robust output tracking control of T-S fuzzy systems and its application to DC-DC converters

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Abstract: This paper studies the problem of robust $H_{\infty}$ static output tracking controller for a class of uncertain discrete-time fuzzy systems. The designed controller guarantees asymptotic tracking of prescribed reference outputs while rejecting disturbances. Simulation example, of a DC-DC buck converter, illustrates the applicability of the proposed control scheme.

1. INTRODUCTION

Over the past decades, many advances have been made in the field of control theory which rely on state-space theory. The control design methodology has widely been investigated for the state-feedback control, see for example Chen et al. (1999); Chen (1998); Geromel and Colaneri (2006). The state-feedback control design supposes that all the system states are available, which is not always possible in realistic applications. Instead, one has to deal with the absence of full-state information by using observers Geromel et al. (2008). From the control point of view, observers can be used as part of dynamical controllers. This observer-based design has been extensively studied in the literature Johansson and Robertsson (2002); Zhou and Wu (2008); Geromel et al. (2008). However, it leads to high-order controllers. As a matter of fact, one has to solve a large problem, which increases numerical computations for large scale systems. Other difficulties may arise, if we consider additional performances, such as disturbance rejection, time delays, uncertainties, etc. Hence, it is more suitable to develop methodologies which involve a design with a low dimensionality. In this context, intensive efforts have been devoted to design low-order controllers Syrmos et al. (1997); Ghaoui et al. (1997); Cao and Sun (1998); Geromel et al. (1998). In particular, it has been shown that designing reduced order stabilizing controllers can be cast as a Static Output-Feedback (SOF) stabilization problem. Also, it is recognized that, in general, the SOF control design may not exist for certain systems. Note that an important advantage of these controllers is that they are easy to implement without significant numerical burden.

Recently, there is rapidly growing interest in stability analysis and control design methodologies for T-S fuzzy models Kruszewski et al. (2008); Sala and Arino (2007). Indeed, a T-S fuzzy model is a collection of a set of linear ones blended together by nonlinear membership functions. Moreover, it is well known that an affine nonlinear system can be exactly matched on a compact set of the state space using for instance the sector nonlinearity approach Tanaka and Wang (2001). The main interest of such approach is that it makes possible extending some of the linear control concepts to the case of nonlinear analysis.

However, most of designs of fuzzy control systems deal with stabilization problems. In contrast to stabilizing controllers, tracking controllers are more demanded from many practical dynamical processes. With regard to the literature, the most of the available approaches are based on dynamic output-feedback controller Tseng et al. (2001); Tseng and Chen (2001); Tong et al. (2002); Mansouri et al. (2009) to achieve the tracking performances of T-S systems. Unfortunately these controllers are complex from the point of view implementation, and the observation of T-S systems is difficult, specially if the decision variables contain states, as it is usual the case in practice. To overcome these difficulties, Static Output Tracking (SOT) controllers are studied in this paper, as they are less expensive for controller implementations and more reliable in practice. However, due to the fact that the T-S models aggregate a set of local linear subsystems, blended together through nonlinear scalar functions, the static output control problem is complicated to solve even in stabilization case. We can just cite Lo and Lin (2003) provides a sufficient condition for designing robust $H_{\infty}$ SOF controllers by using diagonal structure Lyapunov matrices for uncertain discrete-time T-S fuzzy systems. Moreover, by inserting an equality constrained condition about Lyapunov matrix, sufficient conditions are given in Crusius and Trofino (1999) for linear systems and Kau et al. (2007) for T-S fuzzy systems. Although these conditions are convex, the requirements for Lyapunov matrix to be with diagonal structure Lo and Lin (2003) or satisfy an equality condition Kau et al. (2007); Crusius and Trofino (1999) are strict, which might result in conservative designs. In order to develop less conservative conditions for the SOF control synthesis, a new method, which is based on parameter-dependent Lyapunov function approach, is given in Dong and Yang (2009). However, a lower triangular structure constraint is also imposed on the introduced parameter-dependent slack variable. Another SOF scheme for nominal linear plants using T-S fuzzy models was introduced in Nachidi et al. (2008). In contrast to the existing approaches with the constraints on parameter-dependent variables Lo and Lin (2003); Kau et al. (2007); Dong and Yang (2009), an iterative algorithm based on the cone complementarity formulation Ghaoui et al. (1997) has been proposed to solve the nonconvexity inherent to SOF control design. In this paper, we extend the controller of Nachidi et al. (2008) to the case of uncertain nonlinear plant described by T-S fuzzy models. Our objective is not only guarantee the system stability but also ensure a
good reference tracking, robustness and disturbance rejection.
The application of the proposed controller to a DC-DC buck
converter is included to show the viability of its practical im-
plementation using SimPower Systems (SPS) toolbox of Matlab.
The rest of this paper is organized as follows. Section 2 states
the problem. Section 3, gives stability conditions in BMI form.
Section 4 solves the $H_\infty$ SOT control problem. Section 5 illus-
trates the controller design procedure and simulations results.
Finally, Section 6 gives some conclusion remarks.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following uncertain T-S fuzzy system:

$$x(k + 1) = \sum_{i=1}^{N} \alpha_i(k)((A_i + \Delta A_i)x(k) + (B_i + \Delta B_i)u(k)$$
$$+ E_i w(k)),$$  \hspace{1cm} (1)

$$y(k) = \sum_{i=1}^{N} \alpha_i(k)C_i x(k),$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^{nu}$ is the input
vector, $w(k) \in \mathbb{R}^{nu}$ is the bounded external distur-
bances and $y(k) \in \mathbb{R}^{mv}$ is the system output. $N$ is the number
of IF-THEN rules. $\alpha_i(k)$ are the normalized membership func-
tions Tanaka and Wang (2001), that fulfill $0 \leq \alpha_i(k) \leq 1$ and

$$\sum_{i=1}^{N} \alpha_i(k) = 1. A_i, B_i, C_i \text{ and } E_i \text{ are known constant matrices of}
$$

appropriate size, $\Delta A_i, \Delta B_i$ are unknown matrices represent-
ing time-varying parameter uncertainties, and are assumed to be as

$$[\Delta A_i, \Delta B_i] = [M_i F(k)N_{i1}, M_2 F(k)N_{i2}], \text{ } i = 1, \ldots, N,$$  \hspace{1cm} (2)

where $M_i, N_{i1}$ and $N_{i2}$ are known real constant matrices. $F(k)$ is
the uncertainty function that satisfies the classical bounded
condition Zhou and Khargonekar (1988):

$$F(k)^T F(k) \leq I, \forall k.$$  \hspace{1cm} (3)

In the purpose of this article, the control law is constructed
based on the classical structure of the Parallel Distributed Com-
ensation (PDC) concept i.e., the designed fuzzy controller
shares the same fuzzy sets with the fuzzy model in the premise
parts (see Tanaka and Wang (2001) for more details). For the
fuzzy model (1), the following SOT controller is given:

$$u(k) = \sum_{i=1}^{N} \alpha_i(k)K_i(y(k) - y_d(k)),$$  \hspace{1cm} (4)

where $K_i$ are the tuning gains and $y_d(k)$ is the desired refer-
ence trajectory for the output $y(k)$. $y_d(k)$ is generated by the
following system

$$\begin{cases} x_d(k + 1) = Ax_d(k) + Br(k), \\ y_d(k) = Cx_d(k), \end{cases}$$  \hspace{1cm} (5)

where, $y_d(k)$ has the same dimension as $y(k)$, $x_d(k)$ and
$r(k) \in \mathbb{R}^{vr}$ are respectively the reference state and the bounded
reference input, $A$, $B$ and $C$ are appropriately dimensional
constant matrices with $A$ is stable.
Thus, the main objective of the next paragraph is to provide
stability conditions that ensure the tracking performance for the
uncertain T-S models (1).

Notation: Throughout the paper the following notation will be used:

$$\alpha_i(k) \equiv \alpha_i(x(k)), \alpha_{ij} (k) \equiv \alpha_i(x(k))\alpha_j(x(k))\alpha_s(x(k)),$$

$A_{ij} \equiv A_i + B_iK_jC_s.$

$I$ is the identity matrix with appropriate dimension.
The star symbol $*$ in a symmetric matrix denotes the transposed
block in the symmetric position.

3. MAIN RESULT

This section gives sufficient stability conditions which ensure
an $H_\infty$ output tracking performance of the uncertain system
(1) using a fuzzy Lyapunov function. We recall the following
lemma which will be used in this section.

real matrices of appropriate dimension such that $W^T > 0$ and $FF^T < I.$ Then, for any scalar $\epsilon > 0$ such that $W - \epsilon DD^T > 0$, we have $(A + DFS)^TW^{-1}(A + DFS) \leq A^T(W - \epsilon DD^T)^{-1}A + \epsilon^{-1}S^TS.$

Combining (1), (4) and (5) the following augmented closed-
loop system is obtained

$$\dot{z}(k + 1) = \sum_{i,j,s}^{N} \alpha_{ij}(k)((G_{ij} + G_{j} \alpha)(k) + W_i \dot{w}(k)),$$  \hspace{1cm} (6)

where

$$G_{ij} = \begin{bmatrix} A_i + B_iK_jC_s - B_iK_jC \alpha \equiv \hat{A} \\ 0 \end{bmatrix},$$

$$G_{j} \equiv \begin{bmatrix} \Delta A_i + \Delta B_iK_jC_s - \Delta B_iK_jC \equiv \Delta \hat{A} \\ 0 \end{bmatrix},$$

$$W_i = \begin{bmatrix} E_i & 0 \end{bmatrix}, \tilde{x} = \begin{bmatrix} x(k) \\ x_d(k) \end{bmatrix}, \tilde{w} = \begin{bmatrix} w(k) \\ r(k) \end{bmatrix},$$

$$\alpha_{ij}(k) = \alpha_i(k)\alpha_j(k)\alpha_s(k),$$

$$\sum_{i}^{N} \alpha_i(k) = \sum_{j}^{N} \alpha_j(k) = \sum_{s}^{N} \alpha_s(k) = 1.$$  \hspace{1cm} (7)

Hence, to meet the required tracking performance, the effect of
$\tilde{w}(k)$ on the tracking error $y(k) - y_d(k)$ should be attenuated
below a desired level in the sense of Chen et al. (1999); Mansouri et al. (2009):

$$\sum_{k=0}^{t_i} (y(k) - y_d(k))^T\sum_{k=0}^{t_i} (y(k) - y_d(k)) \leq \gamma_k \sum_{k=0}^{t_i} \tilde{w}(k)^T \tilde{w}(k),$$  \hspace{1cm} (8)

for any desired level $\gamma$ and $k_i \neq 0$, and $\tilde{w}(k) \in l_2$, $k_f$ is the
control final time.

The following theorem shows that the robust $H_\infty$ output track-
ing performance can be guaranteed if there exist some matrices
satisfying certain conditions.

Theorem 3.1. The augmented closed-loop system in (6) achieves
the $H_\infty$ output tracking performance $\gamma$, if there exists matrices
$P_1 > 0, \ldots, P_N > 0$ and controller gains $K_1, \ldots, K_N$ such
that the following conditions hold:

$$\begin{bmatrix} -P_r^{-1} & 0 & 0 & G_{ij} & W_i & \hat{M} \\ 0 & -\epsilon & 0 & N_{ij} & 0 & 0 \\ 0 & 0 & -I & H_i & 0 & 0 \\ * & * & * & -P_s & 0 & 0 \\ * & 0 & 0 & -\gamma^2I & 0 & 0 \\ * & 0 & 0 & 0 & 0 & -\epsilon^{-1}I \end{bmatrix} < 0,$$  \hspace{1cm} (9)

$1 \leq i, j, s, r \leq N,$
where $G_{ijs}$ and $W_i$ are defined in (7) and $H_i = \begin{bmatrix} C_i \end{bmatrix}$, 
\[ \tilde{M} = \begin{bmatrix} M_1 & M_2 \\ 0 & 0 \end{bmatrix}, \tilde{N}_{ijs} = \begin{bmatrix} N_{ijs} \\ N_{2i} K_j C_{s} - N_{2i} K_j C \end{bmatrix}. \]

**Proof 3.1.** Consider the following fuzzy Lyapunov function $V(\tilde{x}, k)$ given by
\[ V(\tilde{x}, k) = \tilde{x}(k)^T \sum_{i=1}^{N} \alpha_i(k) P_i \tilde{x}(k). \]

The stability of (6) is ensured, under zero initial condition, with guaranteed $H_\infty$ performance (8) if Chen et al. (1999); Mansouir et al. (2009):
\[ \Delta V(\tilde{x}, k) + (y(k) - y_d(k))^T (y(k) - y_d(k)) - \gamma^2 \tilde{w}(k)^T \tilde{w}(k) < 0 \]
where $\Delta V(\tilde{x}, k) = V(\tilde{x}(k+1)) - V(\tilde{x}(k))$.

The condition (10) leads to
\[ \tilde{x}(k+1)^T \tilde{P} \tilde{x}(k+1) - \tilde{x}(k)^T \tilde{P} \tilde{x}(k) + (y(k) - y_d(k))^T (y(k) - y_d(k)) - \gamma^2 \tilde{w}(k)^T \tilde{w}(k) < 0, \]
where $P_2 = \sum_{i=1}^{N} \alpha_i(k) P_i$ and $P^+ = \sum_{i=1}^{N} \alpha_i(k+1) P_i$.

(11) can be rewritten as
\[ \begin{bmatrix} G_{ijs} \tilde{x}(k) + W_s \tilde{w}(k) \\ \tilde{x}(k)^T \hat{G}_{ijs} \tilde{x}(k) + W_s \tilde{w}(k) \\ (y(k) - y_d(k))^T (y(k) - y_d(k)) \end{bmatrix} - \tilde{x}(k)^T \tilde{P} \tilde{x}(k) - \gamma^2 \tilde{w}(k)^T \tilde{w}(k) < 0, \]
where
\[ W_s = \sum_{i=1}^{N} \alpha_i(k) W_i \text{ and } G_{i1s} = G_{1s} + G_{2s}, \]
\[ G_{1s} = \sum_{i,j,s=1}^{N} \alpha_{ijs}(k) G_{ijs}, G_{2s} = \sum_{i,j,s=1}^{N} \alpha_{ijs}(k) G_{2ijs}. \]

From (12) we obtain:
\[ \begin{bmatrix} \tilde{x}(k)^T \\ \tilde{w}(k)^T \end{bmatrix} (M_1 - M_2) \begin{bmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{bmatrix} < 0, \]
where
\[ M_1 = \begin{bmatrix} G_{ijs}^T \tilde{P} G_{ijs} + W_s^2 \tilde{P} W_s \\ \frac{W_s^T \tilde{P} G_{ijs}}{0} \end{bmatrix}, M_2 = \begin{bmatrix} P_2 - H_s H_s & 0 \\ 0 & \gamma^2 \end{bmatrix}. \]

Thus, to proof (10), it is sufficient to show that
\[ M_1 - M_2 < 0. \]

Note that, the first part of (15) can also be rewritten as
\[ M_1 - M_2 = (\tilde{G}_{z} + \tilde{M} F(k) N_{z})^T \tilde{P} (\tilde{G}_{z} + \tilde{M} F(k) N_{z}), \]
where $\tilde{G}_{z} = \begin{bmatrix} G_{i1s} W_s \end{bmatrix}$, $\tilde{N}_{z} = \begin{bmatrix} \tilde{N}_{ijs} \end{bmatrix}, \tilde{N}_{s} = \begin{bmatrix} \tilde{N}_{ijs} \end{bmatrix}.$

On the other hand, pre- and post-multiplying (9) by $\text{diag} \{ P_1, I, I, 1, I \}$ gives
\[ \Gamma^r_{ijs} \triangleq \begin{bmatrix} -P_r & 0 & 0 & P_r G_{ijs} & P_r W_i & P_r \tilde{M} \\ -I & 0 & 0 & -I & H_i & 0 \\ 0 & 0 & 0 & -I & P_i \tilde{M} \\ 0 & 0 & 0 & 0 & -\gamma^2 I \\ 0 & 0 & 0 & 0 & -\epsilon^2 I \end{bmatrix} < 0, \]
\[ 1 \leq i, j, s, r, \leq N. \]

Since $\sum_{i=1}^{N} \alpha_i(k) = \sum_{i=1}^{N} \alpha_i(k+1) = 1$, (17) can be written as
\[ \sum_{i=1}^{N} \alpha_i(k+1) \sum_{i,j,s=1}^{N} \alpha_{ijs}(k) \Gamma^r_{ijs} \triangleq \begin{bmatrix} -P^+ & 0 & 0 & P^+ \tilde{G}_{z} & P^+ W_z & P^+ \tilde{M} \\ 0 & -I & 0 & \tilde{N}_{z} & 0 & 0 \\ 0 & 0 & -I & H_z & 0 & 0 \\ 0 & 0 & 0 & -P_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon^2 I \end{bmatrix} < 0, \]
\[ 1 \leq i, j, s, \leq N. \]

Applying Schur complement on (18), it is straightforward to verify that the condition (18) is equivalent to the following inequalities:
\[ (\tilde{P} \tilde{G}_{z})^T (\tilde{P} - \epsilon \tilde{P} \tilde{M} \tilde{M}^T \tilde{P})^{-1} \tilde{P} \tilde{G}_{z} + \epsilon^{-1} \tilde{N}_{z}^T \tilde{N}_{z} - M_2 < 0. \]

Using (16), (19) and Lemma 3.1, we have
\[ M_1 - M_2 = (\tilde{G}_{z} + \tilde{M} F(k) N_{z})^T \tilde{P} (\tilde{G}_{z} + \tilde{M} F(k) N_{z}) \]
\[ \leq (\tilde{P} \tilde{G}_{z})^T (\tilde{P} - \epsilon \tilde{P} \tilde{M} \tilde{M}^T \tilde{P})^{-1} \tilde{P} \tilde{G}_{z} + \epsilon^{-1} \tilde{N}_{z}^T \tilde{N}_{z} - M_2 < 0. \]

By consequence
\[ \sum_{k=0}^{\infty} (y(k) - y_d(k))^T (y(k) - y_d(k)) < \gamma^2 \sum_{k=0}^{\infty} \tilde{w}(k)^T \tilde{w}(k). \]

Thus, the $H_\infty$ output tracking performance is achieved with the prescribed attenuation level $\gamma$. On the other hand, it follows from (9) and (20) that $\Delta V(\tilde{x}) < 0$ for $\tilde{w}(k) = 0$. Hence, the uncertain system (6) with $\tilde{w}(k) = 0$ is robustly asymptotically stable.

4. $H_\infty$ FUZZY TRACKING CONTROLLER SYNTHESIS

In this section, a cone complementarity formulation Ghaoui et al. (1997) is used to solve the bilinearity involved in (9). The idea is based on converting the conditions (9) to convex and nonconvex parts and then casting them into an optimization problem subject to some LMIs. For this, first recall the following lemma, which generalizes the result of Ghaoui et al. (1997).

Lemma 4.1. Nachidi et al. (2008) Let $P_i \in \mathbb{R}^{n \times n}, Q_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, N$ be any symmetric positive definite matrices, then the following statements are equivalent:

(a): $P_i Q_i = I, \quad i = 1, \ldots, N.$

(b): $\begin{bmatrix} \sum_{i=1}^{N} \text{Tr}(P_i Q_i) & N \times n \\ \begin{bmatrix} P_i \\ I Q_i \end{bmatrix} \end{bmatrix} \geq 0, \quad 1 \leq i \leq N.$
Then, taking $P_r = Q_r^{-1}$, the stability condition (9) can be rewritten as follows
\[ \Omega_{ijs} \equiv \begin{bmatrix} -Q_r & 0 & G_{ijs} \ W_i & \tilde{M} \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & H_i \\ * & * & -P_r & 0 \\ * & * & 0 & -\gamma^2 I \\ * & * & 0 & -\epsilon^{-1} I \end{bmatrix} \prec 0, \quad (21) \]

\[ 1 \leq i, j, s, r \leq N, \]

\[ P_r Q_r = I, \quad 1 \leq r \leq N. \quad (22) \]

Before giving the final formulation of the problem in hand, we suggest to relax the LMIs (21) from the point of view number of LMIs to be satisfied, for this, let us recall the following lemma.

**Lemma 4.2.** Nachidi et al. (2008) Consider the following matrix $\mathbf{A} = \sum_{i,j,s=1}^{N} \alpha_{ijs} \mathbf{A}_{ijs}$, where $\alpha_{ijs} = \alpha_i \alpha_j \alpha_s$ and $\sum_{i=1}^{N} \alpha_i = 1$. Then, $\mathbf{A}$ can be expressed as follows

\[ \mathbf{A} = \sum_{i=1}^{N} \alpha_i^3 \mathbf{A}_{iii} + \sum_{s \geq j > i}^{N} \alpha_{ijs} (\mathbf{A}_{ijs} + \mathbf{A}_{jsi} + \mathbf{A}_{sij}) + \sum_{i < j < s}^{N} \alpha_{ijs} (\mathbf{A}_{sji} + \mathbf{A}_{sij} + \mathbf{A}_{sjs}), \]

Moreover,

\[ \sum_{i,j,s=1}^{N} \alpha_{ijs} = \sum_{i=1}^{N} \alpha_i^3 + \sum_{s > j > i}^{N} \alpha_{ijs} + \sum_{s > j > i}^{N} \alpha_{ijs} = 1. \]

Hence, using Lemma 4.2, (21) can be rewritten as follows:

\[ \mathbf{T}_{r}^{i} \prec 0, \quad 1 \leq i, r \leq N, \]

\[ \mathbf{P}_{r}^{i} \leq 0, \quad 1 \leq i \leq j < s \leq N, \quad 1 \leq r \leq N, \]

\[ \mathbf{W}_{r}^{i} \leq 0, \quad 1 \leq i < j < s \leq N, \quad 1 \leq r \leq N, \]

where,

\[ \mathbf{T}_{ijs} \equiv \begin{bmatrix} -Q_r & 0 & G_{ijs} \ W_i & \tilde{M} \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & H_i \\ G_{T}^{T} \tilde{N}_{ijs} \ H_{i}^{T} & -P_r & 0 & 0 \\ W_{T}^{T} & 0 & 0 & -\gamma^2 I \\ M^{T} & 0 & 0 & -\epsilon^{-1} I \end{bmatrix}, \]

\[ \mathbf{P}_{ijs} \equiv \begin{bmatrix} -3Q_r & 0 & G_{ijs} + G_{jsi} + G_{sij} \ W & 3M \\ 0 & -3I & 0 & 0 \\ 0 & 0 & -3I & H_i + H_j + H_s \\ + & + & -(P_r + P_s) & 0 \\ + & 0 & 0 & -3\gamma^2 I \\ + & 0 & 0 & -3\epsilon^{-1} I \end{bmatrix}, \]

\[ \mathbf{W} = W_i + W_j + W_s. \]

From Lemma 4.2, it is only sufficient to see that Nachidi et al. (2008) give a weight $\beta$, fix a tolerance $\varepsilon$ (for example $\varepsilon = 10^{-6}$) and execute the following steps:

- **Step 1:** Set $P_i^0 = I$ and $Q_i^0 = I$, for $i = 1, \ldots, N$.
- **Step 2:** Solve the following LMI optimization problem:

\[ \begin{align*}
\text{minimize} & \quad \beta \sum_{i=1}^{N} \text{Tr}(P_{i} Q_{i} + Q_{i} P_{i}) + (1 - \beta) \gamma \\
\text{subject to} & \quad \sum_{i=1}^{N} P_{i} I \geq 0, \quad 1 \leq i \leq N.
\end{align*} \]

- **Step 3:** If $\|P_{i} - Q_{i}^{-1}\| < \varepsilon$.

While $\|P_{i} - Q_{i}^{-1}\| < \varepsilon$.

Select $\beta = \beta - 0.01$ and repeat from step 1. Else

Set $P_i^r \leftarrow P_i$, $Q_i^r \leftarrow Q_i$, and repeat from step 2.

\[ \sum_{i=1}^{N} \alpha_i(k) \Omega_{ijs}^{r} = \sum_{i=1}^{N} \alpha_i(k) \mathbf{T}_{ijs}^{r} + \sum_{i<j<s}^{N} \alpha_{ijs} \Phi_{ijs}^{r} + \sum_{i<j<s}^{N} \alpha_{ijs}(k) \Psi_{ijs}^{r}. \]

It should be noted that, Lemma 4.2 is very useful in reducing the number of LMIs to be satisfied see Nachidi et al. (2008) for more details.

Now, back to our main problem. We suggest to use Lemma 4.1 to handle the nonconvexity involved in (22), as it is clearly shown by the following theorem:

**Theorem 4.1.** Given a weight $\beta > 0$ and $\varepsilon > 0$. The augmented closed-loop system in (6) achieves the $H_{\infty}$ output tracking performance $\gamma$, if there exists positive definite matrices $P_1 > 0, \ldots, P_N > 0, Q_1 > 0, \ldots, Q_N > 0$ and controller gains $K_1, \ldots, K_N$ such that the following optimization problem is solvable and equal to $n_2 \times N$:

\[ \begin{align*}
\text{minimize} & \quad \beta \sum_{i=1}^{N} \text{Tr}(P_i Q_i) + (1 - \beta) \gamma \\
\text{subject to} & \quad (23) \quad \text{and} \quad P_i I \geq 0, \quad 1 \leq i \leq N.
\end{align*} \]

The following iterative algorithm Ghaoui et al. (1997); Nachidi et al. (2008) can be used to linearize the objective function of the optimization problem (24).

**Remark 4.1.** In the optimization problem (24), the attenuation level $\gamma$ is also included in the optimization function. Thus, a multi-objective optimization problem is solved by Algorithm 4.

5. **ILLUSTRATIVE EXAMPLE**

In this section, the proposed tracking control scheme is applied to regulate the output voltage of DC-DC converter. The model of a buck converter is described in Fig. 5. Using the Kirchoff laws, the converter of Fig. 5 can be represented by the following discrete-time nonlinear model Lian et al. (2006):
\[ x(k+1) = \begin{bmatrix} -T_s (R_L + R_c) & -T_s R(k) & 1 \end{bmatrix} \frac{L}{R(k) + R_c} x(k) + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \frac{V_{id}}{L}, \]

where \( x(k) = [i_L(k), v_{in}(k)]^T \) is the state vector, \( u(k) \) is the control vector i.e. the duty cycle of the switched \( M \), \( y(k) \) is the output vector i.e. the output voltage and \( T_s \) is the sampling period. \( T_s = 0.01 \times 1/f_0 \), where \( f_0 \) is the resonant frequency of the buck converter (25). \( R(k) \) and \( V_{in}(k) \) are uncertain parameters satisfying \( R(k) \in [R, \overline{R}] \), \( V_{in} \in [V_{in}, \overline{V}_{in}] \).

Table (1) gives the parameter values of the buck converter (Fig. 5). Similar to Lian et al. (2006), we assume that the inductor current belongs in a compact set: \( i_L(k) \in [\bar{i}_L, \tilde{i}_L] \), and select the membership functions as follows

\[ \alpha_1(k) = \frac{-i_L(k) + \tilde{i}_L}{\bar{i}_L - \tilde{i}_L}, \quad \alpha_2(k) = 1 - \alpha_1(k). \]

Then, the nonlinear system (25) can be represented by the following uncertain T-S model:

\[ x(k+1) = (A_1 + \Delta A_1)x(k) + (B_1 + \Delta B_1)u(k) + E_1w(k), \]

\[ y(k) = C_1 x(k), \quad i = 1, 2, \]

where \( A_1 = A_2 = \bar{A} + \frac{A}{2}, B_1 = \bar{B} + \frac{B}{2}, B_2 = \bar{B} - \frac{B}{2}, \bar{A}, \bar{B} \) are calculated using the min and max of the compact region of \( R(k) \) and \( \overline{V_{in}}(k) \) respectively.

\[ C_1 = C_2 = \begin{bmatrix} RR_c & R & 0 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}. \]

\[ \Delta A_1, \Delta A_2, \Delta B_1, \Delta B_2 \] can be represented in the form of (2) with \( M_1 = 0.1, M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, N_{11} = 10\eta_A, N_{12} = \eta_{B_1}, N_{21} = \eta_{B_2}, N_{22} = \eta_B \), \( \eta_A = \frac{\bar{A} - A}{2}, \eta_B = \frac{\bar{B} - B}{2} \).

In this example, the objective is to make the output voltage of the buck converter, i.e. \( v_{in} \), follow a desired signal to meet the \( H_\infty \) tracking performance of the uncertain system (25).

The reference system matrices of (5) is selected as follows

\[ A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

Let \( \beta = 0.99 \) and \( \epsilon = 1 \), using Algorithm 4, the following feasible solution is obtained after only 41 iterations:

\[ P_1 = \begin{bmatrix} 0.12 & 0.73 & 0 & -0.07 \\ 0.73 & 7.38 & 0 & -0.55 \\ 0 & 0 & 1 & 0 \\ -0.07 & -0.55 & 2.85 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.12 & 0.74 & 0 & -0.09 \\ 0.74 & 7.45 & 0 & -0.66 \\ 0 & 0 & 1.12 & 0 \\ -1.11 & -0.66 & 0 & 3.00 \end{bmatrix} \]

\[ Q_1 = \begin{bmatrix} 19.85 & -1.95 & 0 & 0.11 \\ -1.95 & 3.33 & 0 & 0.05 \\ 0 & 0 & 1 & 0 \\ 0.11 & 0.015 & 0 & 0.36 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 19.87 & -1.97 & 0 & 0.17 \\ -1.97 & 3.33 & 0 & 0.03 \\ 0 & 0 & 0.91 & 0.17 \\ 0.17 & 0.013 & 0 & 0.34 \end{bmatrix} \]

and the \( H_\infty \) output tracking performance index: \( \gamma = 2.52 \). Hence, according to (4), the SOT control law that ensures the desired trajectory tracking for (25) is given as follows:

\[ u(k) = (\alpha_1(k)K_1 + \alpha_2(k)K_2)(y(k) - y_0(k)). \]

Fig. 2 shows the evolution of the output signal of the nonlinear system (25), using the fuzzy controller, with an external disturbance input \( w(k) \) defined as \( w(k) = \frac{1}{t+1}(t+1) - T_s \frac{V_{id}}{L} \), where, \( r_0 \) is a random number taken from a uniform distribution over \([0, 2]\), the uncertain parameters are as follow

\[ R(k) = \frac{R + \bar{R}}{2} + \frac{R - \bar{R}}{2} \cos(k\pi/T_s), \]

\[ V_{in}(k) = \frac{V_{in} + \overline{V_{in}}}{2} + \frac{V_{in} - \overline{V_{in}}}{2} \cos(k\pi/T_s), \]

and the reference signal \( r(k) \), are supposed to be

\[ \begin{cases} r(k) = 12V & \text{for } 0 \leq k \leq 0.005s \\ r(k) = 6V & \text{for } 0.005s < k \leq 0.01s \\ r(k) = 24V & \text{for } k > 0.01s \end{cases} \]

Fig. 6 and Fig. 4 depict a zoom of Fig. 2 at 0 s and between 5 ms and 10 ms respectively. It can be seen that the designed fuzzy SOT controller ensures the robust stability of the nonlinear system (25) and guarantees an acceptable \( H_\infty \) trajectory tracking performance level.

6. CONCLUSION

In this article, the problem of model reference tracking control with a guaranteed \( H_\infty \) performance is solved for uncertain discrete-time fuzzy systems. Based on the fuzzy Lyapunov function and cone complementary formulation, a fuzzy SOT controller is calculated to make small as possible as the tracking error output and reject disturbances.

Table 1. Parameter values of the converter Er (2001).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input voltage, ( V_{in}(k) )</td>
<td>([10, 30] )</td>
<td>V</td>
</tr>
<tr>
<td>Current in the inductance, ( i_L )</td>
<td>(8 \times 8)</td>
<td>A</td>
</tr>
<tr>
<td>Inductance, ( L )</td>
<td>98.38</td>
<td>(\mu H)</td>
</tr>
<tr>
<td>Parasitic resistance of ( L, R_L )</td>
<td>48.5</td>
<td>(\Omega)</td>
</tr>
<tr>
<td>Capacitor, ( C )</td>
<td>202.5</td>
<td>(\mu F)</td>
</tr>
<tr>
<td>Resistance of Switch, ( R_{Sw} )</td>
<td>0.27</td>
<td>(\Omega)</td>
</tr>
<tr>
<td>Diode voltage, ( V_D )</td>
<td>0.82</td>
<td>V</td>
</tr>
<tr>
<td>Load resistance, ( R(k) )</td>
<td>([2, 10])</td>
<td>(\Omega)</td>
</tr>
</tbody>
</table>

REFERENCES


Fig. 2. Response of $y(k)$ and $y_d(k)$.

Fig. 3. Zoom on Fig. 2 at 0 s.

Fig. 4. Zoom on Fig. 2 between 5 ms and 10 ms.


