State observer design for networked control systems with unknown disturbances

Ignacio Peñarrocha and Roberto Sanchis

Departament d’Enginyeria de Sistemes Industrial i Dissenya, Universitat Jaume I de Castelló, Spain, 12007 (e-mail: ipenarro@esid.uji.es, rsanchis@esid.uji.es).

Abstract: In this work, the design of an observer for systems whose output measurements are acquired through a network is addressed. The measurements are assumed to scarcely available, and arriving randomly on time. The noises of the measurement devices are assumed to be known, but the system disturbances are not. A new parameter that can be calculated on line and that measures the relationship between noise measurement and the output estimation error is defined. This parameter is related to the unknown disturbance and, hence, an observer depending on that parameter is defined. A parametric optimization problem is stated for the observer design, leading to an stable observer that minimizes the effect of both the noise and the unknown disturbance. The proposed approach is compared with the Kalman filter approach under the assumption of known and unknown disturbances, showing the validity of the approach.

Keywords: Networked control, unknown disturbance, state observer.

1. INTRODUCTION

This paper presents an adaptive filter design method for networked control systems with packet dropout, that takes into account the scarcely available measurements received through the network and the knowledge of the sensor noise in order to minimize the state estimation error due to unknown disturbances and initial estimation error. The general idea of the approach is to define a parameter that is the ratio between the a priori error of the measurable outputs, and the noise variance. This parameter is included in an optimization procedure to obtain an observer gain that is a function of the estimated value of that parameter and the availability of received measurements, in order to assure the stability of the observer and the adequate attenuation of the effect of the unknown disturbances over the state estimation error.

In order to overcome those drawbacks, other works, like Sanchis et al. (2007), take into account the knowledge of the disturbances and measurement noises, as well as the scarce data availability, to minimize the estimation error and reach a better behaviour. The present work is an extension of those works, taking into account the realistic possibility of having precise information about the sensor measurement noises, but no information about the state disturbances, leading to an LMI procedure to design an adaptive filter for networked control systems.

The filtering problem when dealing with packet dropout has been studied in several works in the literature Liang et al. (2009); Sahebsara et al. (2007); Xu and Chen (2003); Zhang et al. (2004). The different approaches depend on the model used to quantify the packet dropout on the network. Several authors use the probability of having a successful received message to model the network behavior as the transition probabilities of Markov chain packet dropout models, leading to a constant gain that assures the stability of the observer for all the probable sampling intervals between received samples. However, the techniques used in those works do not deal with the proper attenuation of the disturbances, as the optimization procedure does not take into account their characteristics. This leads to a wrong behavior of the observer when dealing with scarce data availability, high disturbances, or unstable systems.

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In this work, a time variant observer gain, that depends on the elapsed time between consecutive received measurements through the network, and on the ratio between the a priori output quadratic error and the sensor noise variance, is defined. That gain is calculated by means of an optimization procedure that leads to an observer with guaranteed stability and that maximizes the disturbance and sensor noise rejection, even when the disturbance is unknown. The optimization procedure is run off line, and hence, the computational cost of the on line implementation of the observer is low.

The layout of the paper is as follows: in section 2 the process model and the proposed observer algorithm are described, deriving also the equation that defines the estimation error. In section 3, the main contribution of the paper is developed, that is, the LMI based observer design. In section 4, some examples demonstrate the validity of the proposed approach, comparing it with a Kalman filter with several assumptions about the disturbance knowledge. Finally, in section 5 the main conclusions are summarized.

2. PROBLEM STATEMENT

2.1 The plant

The ZOH discrete equivalent of a continuous linear time invariant process at period T is assumed to be described by the equation

\[ x[t+1] = A x[t] + B u[t] + w[t], \]

(1a)

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control input and \( w \in \mathbb{R}^n \) is the state disturbance, considered unknown. The output measured by the sensor is assumed to be

\[ y[t] = c x[t] + v[t], \]

(1b)

where \( y[t] \in \mathbb{R} \) is the measured output, and \( v[t] \in \mathbb{R} \) is the measurement noise, whose variance \( \sigma_v^2 = \mathbb{E}\{v[t]^2\} \) is assumed to be known.

The sensor measurements are assumed to be acquired through a network with packet dropout in which, the sensor takes a measurement every period \( T \), synchronized with the control input update. The sensor tries to send the measurement during an interval lower than \( T \), and, if fails, discards that measurement. The result of the network operation is that the measurements are only available for the observer at certain instants of time \( t = t_k \) (where \( k \) denotes the number of acquired measurements). The output measurement, when it is acquired, may be expressed as

\[ y_k = c x[t_k] + v_k, \]

(1c)

where \( y_k \) is the measurement of the sensor at the \( k \)-th sampling instant, and \( v_k \) is the noise measurement at instant \( t_k \). The number of input updates from \( t_{k-1} \) to \( t_k \) is denoted by \( N_k = t_k - t_{k-1} \) and, therefore, \( t_k = \sum_{i=1}^{k} N_i \) represents the instant in which the \( t \)-th input update occurs and the \( k \)-th output sample is received. The number of input updates between received measurements (\( N_k \)) is assumed to vary irregularly, but inside a given finite known set

\[ N_k \in \mathcal{N} = \{1, \ldots, N_{\text{max}}\}. \]

This hypothesis implies that the maximum number of control periods between successful measurement receptions is never larger than \( N_{\text{max}} \). For this purpose, the network manager is assumed to guarantee the successful transmission of the sensor by giving it the maximum priority in the case that \( N_{\text{max}} \) periods without transmissions have elapsed.

2.2 Kalman filter based approach

If the disturbance covariance matrix where known, \( (W = \mathbb{E}\{w[t]w[t]^T\}) \), the Kalman filter with missing measurements, described by the following equations, could be applied

\[ \dot{x}[t|t-1] = A \dot{x}[t-1] + B u[t-1], \]

(3a)

\[ P[t|t-1] = A P[t-1] A^T + W, \]

(3b)

\[ L[t] = c P[t|t-1] c^T + \sigma_v^2 \beta[t], \]

(3c)

\[ \dot{x}[t] = \dot{x}[t|t-1] + L[t]\{y[t] - \dot{c} \dot{x}[t|t-1]\}, \]

(3d)

\[ P[t] = (I - L[t]c) P[t|t-1] \]

(3e)

where \( \beta[t] \) is a binary variable taking the value 1 when a measurement is available (\( t = t_k \)) and 0 when it is not. \( P[t] \) is the estimation of the covariance matrix of the state estimation error (i.e., \( P[t] = \mathbb{E}\{(x[t] - \hat{x}[t])[x[t] - \hat{x}[t]]^T\} \)), and \( P[t|t-1] \) is a priori estimation of that covariance matrix (i.e., \( P[t|t-1] = \mathbb{E}\{(x[t] - \hat{x}[t])[x[t] - \hat{x}[t]]^T\} \)). Note that when there is no measurement available, the state estimation \( \hat{x}[t] \) and the covariance matrix \( P[t] \) are not updated, since \( L[t] = 0 \). The term \( c P[t|t-1] c^T + \sigma_v^2 \) can be interpreted as the expected value of the a priori output estimation quadratic error (i.e., \( E\{v[t]^2\} = c visited \mathbb{E}\{\hat{y}[t]\}c^T + \sigma_v^2 \). Note that the time varying gain \( L[t] \) puts a higher weight on the measured output when the noise variance \( \sigma_v^2 \) is negligible with respect to the a priori error term due to disturbances and accumulated state estimation error, and puts a lower weight on the measured output when the expected value of \( v[t]^2 \) is around the noise variance (i.e., when there is a negligible a priori state estimation error and disturbances).

However, no reference on the bibliography has been found about implementing the Kalman filter with missing measurements when dealing with unknown disturbances, in order to assure some attenuation of the disturbances and noise measurement. Therefore, one possibility is to apply the previous Kalman filter with an arbitrary covariance matrix \( W \) in order to assure stability, but the prediction error can move away from the optimal value.

2.3 Proposed Algorithm

A model based predictor is proposed to estimate the state at period \( T \) from the scarcely acquired measurements through the network. The state is initially estimated running the model in open loop, leading to

\[ \dot{x}[t|t-1] = A \dot{x}[t-1] + B u[t-1]. \]

(4a)

When a new measurement is received at \( t = t_k \), the a priori error is calculated as

\[ e_k = y_k - \dot{c} \dot{x}[t_k | t_k - 1] \]

(4b)

and its square value is filtered with some low-pass filter in order to obtain \( \sigma_{e,k}^2 \), that is an estimation of the mean
square a priori error $\|\epsilon_k\|_{RMS}^2$. A first order filter can be used as
\[
\sigma^2_{e,k} = p \cdot \sigma^2_{e,k-1} + (1-p) \cdot e^2_k, \quad 0 < p \leq 1.
\] (4c)
With this estimation of the mean square a priori error, the ratio between the a priori error and the noise variance is defined as
\[
\alpha_k = \begin{cases} 
0, & \sigma^2_{e,k} < \sigma^2_v \\
\frac{\sigma^2_v - \sigma^2_{e,k}}{\sigma^2_{e,k}}, & \sigma^2_{e,k} > \sigma^2_v
\end{cases}
\] (4d)
that takes values in the interval $\alpha_k \in [0,1]$, and it is updated every time a new measurement is received. The estimated state is then updated as
\[
\hat{x}[t_k] = \hat{x}[t_k|t_k-1] + L_k(y_k - c\hat{x}[t_k|t_k-1])
\] (4e)
where $L_k$ is the gain vector used to update the estimated state with the measurement $y_k$. The dynamics of the state estimation error depends on the vector gain $L_k$, defined at measuring instants $(t = t_k)$, that must be designed to assure: the observer stability, robustness to the irregular data availability and a proper attenuation of the disturbances and measurement noises.

The strategy proposed in this work consists of defining a variant gain $L_k$ that depends on the values $N_k$ and $\alpha_k$, that are known or are calculated, respectively, for each acquired measurement.

Taking into account the assumption expressed in equation (2), the gains $L_k$ are proposed to be calculated as
\[
L_k = L'(N_k) \cdot \alpha_k
\] (5)
where $\alpha_k$ is the ratio between the a priori error and the noise variance, that is calculated every time a new measurement is received, by using equation (4d), and $L'(N_k)$ is a vector gain whose value depends on the number of control periods between received measurements, i.e., it is a vector that belongs to the finite set
\[
L'(N_k) \in \mathcal{L} = \{ L'(1), L'(2), \ldots , L'(N_{\text{max}}) \}
\] (6)
Every time a new measurement arrives, one of the gains of this set $\mathcal{L}$ is used, depending on the value of $N_k$, to compute the equation (4e) with gain (5).

2.4 Prediction error

\textbf{Theorem 1.} (Prediction error dynamics). The prediction error dynamics of the algorithm (4) applied to system (1) when there is no modelling error and there is one measurement received every $N_k$ input periods (with $N_k$ time variant), is described by the linear time-varying system
\[
\bar{x}_k = A(N_k) \bar{x}_{k-1} + B(N_k) w_k - L'(N_k) \alpha_k v_k
\] (7)
where $\bar{x}_k \equiv \hat{x}[t_k] - \bar{x}[t_k]$, is the state estimation error when a measurement is received at $t = t_k$, and $W_k = [w[t_k-1]^T \cdots w[t_k-N_{\text{max}}]]^T$.
\[
A(N_k) = (I - L'(N_k) \alpha_k c) A^{N_k},
\] (9)
\[
B(N_k) = (I - L'(N_k) c \alpha_k c) \Lambda(N_k)
\] (10)
with
\[
\Lambda(N_k) = [I A A^2 \cdots A^{N_k-1} 0 \cdots 0]_{n \times N_{\text{max}}n}
\] (11)
\textbf{Proof.}

At the instants when a measurement is received, the output is defined by the equation (1c), and therefore, the output prediction (4e) can be expressed as
\[
\hat{x}[t_k] = \hat{x}[t_k|t_k-1] + L'(N_k) c(x[t_k] - \hat{x}[t_k|t_k-1]) + v_k.
\] (12)
Vectors $x[t_k]$ and $\hat{x}[t_k|t_k-1]$ can be expressed as a function of $x[t_k-1] \equiv \hat{x}[t_k-1]$ and $\hat{x}[t_k-1] \equiv \hat{x}[t_k - N_k]$ (the instant when the last measurement was available and the state estimation was updated) is expressions (1a) and (4a) are applied recursively, leading to
\[
\hat{x}[t_k|t_k-1] = A^{N_k} \hat{x}[t_k-1] + \sum_{i=1}^{N_k} A^{i-1} Bu[t_k-i].
\] (13)
\[
x[t_k] = A^{N_k} x[t_k-1] + \sum_{i=1}^{N_k} A^{i-1} Bu[t_k-i] + \sum_{i=1}^{N_k} A^{i-1} w[t_k-i].
\] (14)
Introducing equations (13) and (14) in (12) and substracting the resulting expression from (14), leads finally to (7). \textbf{Remark 2.} The a priori output estimation error at the time when a new measurement is received, can be expressed in terms of the state estimation error at the time when the previous measurement was received, as
\[
e_k = c A^{N_k} \hat{x}_{k-1} + c A(N_k) W_k + v_k
\] (15)
The goal of the present work is to find a procedure to design the vector gain $L_k$ such that the stability of the system (7) is guaranteed, while disturbances are properly attenuated.

3. $H_\infty$ DISTURBANCE ATTENUATION BASED PREDICTOR DESIGN

The predictor design strategy must take into account the disturbances $w$, and the measurement noise $v$, as well as the scarce data availability. The use of the $H_\infty$ norm allows to formulate the design problem and convert it into a linear matrix inequality (LMI).

\textbf{Theorem 3.} Consider the predictor (4) applied to system (1a) and assume that the measurements are received every $N_k \in \{1, \ldots , N_{\text{max}}\}$ periods. For some given $\gamma_v$ and $\gamma_w$, and uncertain $\alpha \in [0,1]$, assume that there exist some matrices $P = P^T \in R^{n \times n}$, $I(x) \in R^{n \times 1}$ such that for $i = 1, \ldots , N_{\text{max}}$
\[
\begin{bmatrix}
P & M_A(i) & M_B(i) & -X(i) \alpha \\
M_A(i)^T & U_1(i) & U_2(i) & 0 \\
M_B(i)^T & U_3(i)^T & U_4(i) & 0 \\
-X(i)^T & 0 & 0 & (\gamma_v + \gamma_w)I
\end{bmatrix} > 0,
\] (16)
\[
I - \gamma_w' (cA')^T (cA') > 0
\] (17)
where
\[ M_A(i) = (P - X(i) \alpha c) A, \]
\[ M_B(i) = (P - X(i) \alpha c) \Lambda(i), \]
\[ U_1(i) = P - I + \gamma' \left( w(cA)T (cA) \right), \]
\[ U_2(i) = \gamma' \left( \Lambda(i)c \right)T (\Lambda(i)c), \]
\[ \gamma' = (1 - \alpha) \gamma_w. \]

Then, defining the predictor gain as \( L_k = L(N_k) - \alpha c \), with \( L(i) = P - X(i) \), \( i = 1, \ldots, N_{\text{max}} \), the prediction error of the algorithm defined by (4) converges asymptotically to zero under null disturbances, and, under null initial conditions, the following condition is fulfilled
\[
\|\tilde{x}_k\|^2_{\text{RMS}} < \gamma_u \sigma^2_v + \gamma_w \|e_k\|^2_{\text{RMS}} \tag{19}
\]

**Proof.** Introducing \( X(N_k) = P L(N_k) \) in (16) for any \( N_k \in \mathcal{N} \) and applying Schur complements, one obtains
\[
\begin{bmatrix}
A(N_k)T P A(N_k) - A(N_k)T P L(N_k)^T \sigma^2_v & 0 \\
0 & 0
\end{bmatrix} \geq 0, \tag{20}
\]

Taking the first block of the previous matrix, and taking into account that \( I - \gamma_w (cA)^T (cA) \succ 0 \), the inequality (20) implies
\[ \tilde{x}_k^T P \tilde{x}_k - \tilde{x}_{k-1}^T P \tilde{x}_{k-1} < 0. \]

If the disturbances and measurement noise are assumed to be zero, using (7), the above expression leads, for any \( \alpha_k \in [0,1] \), to
\[ \tilde{x}_k^T P \tilde{x}_k - \tilde{x}_{k-1}^T P \tilde{x}_{k-1} < 0, \]

that assures asymptotic convergence of the prediction error if the Lyapunov function \( V_k = \tilde{x}_k^T P \tilde{x}_k \) is defined.

If the inequality (20) is left multiplied by \( [\tilde{x}_{k-1}, W_k, v_k]^T \), and right multiplied by its transpose, it leads to
\[ \tilde{x}_k^T P \tilde{x}_k - \tilde{x}_{k-1}^T P \tilde{x}_{k-1} + \tilde{x}_{k-1}^T \tilde{x}_{k-1} - \gamma_w \sigma^2_v - \gamma_w v_k^2 < 0, \]

where the predictor dynamic error equation (7) and the a priori output prediction error equation (15) have been taken into account. Assuming a null initial prediction error \( \tilde{x}_0 = 0 \) and adding from \( k = 1 \) till \( k = K \), one obtains
\[ \tilde{x}_K^T P \tilde{x}_K + \sum_{k=1}^{K} \left( \tilde{x}_k^T \tilde{x}_{k-1} - \gamma_w \sigma^2_v - \gamma_w v_k^2 \right) < 0. \tag{21}
\]

As \( P > 0 \), then \( \tilde{x}_K^T P \tilde{x}_K > 0 \), leading to
\[ \sum_{k=1}^{K} \left( \tilde{x}_k^T \tilde{x}_{k-1} - \gamma_w \sigma^2_v - \gamma_w v_k^2 \right) < 0. \tag{22}
\]

Dividing by \( K \) and taking limits as \( K \) goes to \( \infty \), the \( \text{RMS} \) norm of the signals is obtained
\[ \|\tilde{x}_k\|^2_{\text{RMS}} < \gamma_u \sigma^2_v + \gamma_w \|e_k\|^2_{\text{RMS}}, \]

where the noise has been assumed to be a zero mean white noise signal.

**Remark 4.** (Design procedure). The previous theorem allows to obtain an observer that assures stability and certain disturbance and noise attenuation levels. However, in order to obtain an optimal observer, the values of \( \sigma^2_v \) and \( \|e_k\|^2_{\text{RMS}} \) should be previously known. In that case, minimizing
\[ \gamma_u \sigma^2_v + \gamma_w \|e_k\|^2_{\text{RMS}} \tag{23}
\]

subject to LMI’s (16), would lead to the observer that minimizes the effect of noise and disturbances.

However, the disturbances and the a priori state estimation error are assumed to be unknown. For that reason the function to be minimized is proposed to be expressed as a function of the uncertain parameter \( \alpha \), in such a way that its value is taken into account in the minimization and in the LMI restrictions. Taking into account the definition of the a priori error ratio (4d), the a priori quadratic error, \( \|e_k\|^2_{\text{RMS}} \approx \sigma^2_v \), can be expressed as
\[ \|e_k\|^2_{\text{RMS}} \leq \sigma^2_v, \tag{24}
\]

Using this expression, it is easy to obtain
\[ \gamma_u \|e_k\|^2_{\text{RMS}} = \gamma_u (1 - \alpha) \|e_k\|^2_{\text{RMS}} \leq \gamma_u \sigma^2_v, \]

and the minimization of (23) is equivalent to minimizing
\[ \gamma_u + \gamma_w \tag{25}
\]

subject to LMI’s (16) for uncertain parameter \( \alpha \in [0,1] \). This optimization problem, subject to a restriction that depends on an uncertain parameter, can be solved by using robust optimization techniques described in Ben-Tal and Nemirovskii (2002); Löfberg (2008), because the uncertain parameter enters the restrictions in a special way.

### 4. EXAMPLE

Consider the system defined by the matrices
\[ A = \begin{bmatrix} 0.14 & 0.55 \\ 0.34 & 0.36 \end{bmatrix}, \quad B = \begin{bmatrix} 0.23 \\ 0.40 \end{bmatrix}, \quad c^T = \begin{bmatrix} 0.12 \end{bmatrix} \]

Assume that a state disturbance of zero mean and covariance matrix \( W = 0.01 I \) is present, and that the sensor has a zero mean white measurement noise of variance \( \sigma^2_v = 0.01 \). Assume also that the measurements are available through a network, every \( N \) Control periods, with \( N \) irregularly varying, but in the set \( \mathcal{N} = \{1,2,\ldots,10\} \). For this system, several state estimation strategies will be applied to compare their performance. The following index has been used for comparison purposes
\[ \|\tilde{x}[t]\|_{\text{RMS}}^2 = \text{tr} \left( \frac{1}{t_{\text{max}}} \sum_{t=0}^{t_{\text{max}}} (x[t] - \hat{x}[t])^T (x[t] - \hat{x}[t]) \right) \]

where \( \text{tr} \) is the trace operator, and where \( t_{\text{max}} \) refers to the simulation time.

If the disturbances were known, a Kalman filter for synchronous random sampling, defined by (3), could be applied, leading to a state estimation error norm of \( \|\tilde{x}[t]\|_{\text{RMS}}^2 = 38.44 \cdot 10^{-3} \). If the state disturbance is assumed to be unknown, the algorithm (3) can still be applied, but using an approximate matrix \( W \). The problem is that the error can be important if a wrong guess of matrix \( W \) is used. For example, if a matrix \( W = I \) is used, the resulting error variance is \( \|\tilde{x}[t]\|_{\text{RMS}}^2 = 47.7 \cdot 10^{-3} \), that is a 24\% larger than the optimum. However, using the proposed algorithm, defined by equations (4), using a discrete pole of \( p = 0.99 \) on equation (4c) in order to filter
5. CONCLUSIONS

In this paper, the design of an observer for systems with scare and irregular measurements taken through a network, has been studied, assuming that the state disturbances are unknown. The proposed observer uses the state the a priori square error, and where the observer gains are the result of the robust optimization problem described in section 3, the resulting error is $\|\tilde{x}[t]\|_{RMS}^2 = 39.15 \cdot 10^{-3}$. This is only a 2% larger than the optimum that could be obtained by using the ideal Kalman filter with perfect disturbance known.

Let as assume now that the state disturbance covariance is reduced by a factor of 0.1 at a given instant of time, and apply again the three previous estimation algorithms: a Kalman filter with known disturbances (for comparison purposes), a Kalman filter with $W = I$, and the algorithm proposed in this work. Then, the resulting state estimation error norms are $\|\tilde{x}[t]\|_{RMS}^2 = 3.85 \cdot 10^{-3}$, $\|\tilde{x}[t]\|_{RMS}^2 = 17.64 \cdot 10^{-3}$ and $\|\tilde{x}[t]\|_{RMS}^2 = 3.93 \cdot 10^{-3}$, respectively, showing that the performance of a Kalman filter with $W = I$ is 4.6 times worse than the optimal Kalman filter, while the proposed approach results in a small deterioration of only a 2% with respect the optimal Kalman filter.

Filtered values of $\text{trace} \{\tilde{x}_k \tilde{x}_k^T\}$ at each sampling instant are shown in the figure 1 for the three approaches. A first order low-pass filter with discrete pole on 0.99 has been used to smooth the signals for better visualization. At instant $t = 5000$, a sudden reduction of the covariance matrix by a factor of 0.1 is simulated. It can be observed how the proposed approach leads to a similar performance as the Kalman filter with known state disturbances, while the performance of the Kalman filter with arbitrary $W$ matrix deteriorates when the disturbance decrease. The value of $\alpha_k$ (the ratio between a priori error and measurement noise), used in the calculation of gain $L_k$, is shown in figure 2. At instant $t = 5000$, the average value of the parameter $\alpha_k$ changes to fit the reduction of the disturbance variance.

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