Investigating the Calculation Error of the Monte-Carlo Bayesian Estimator

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Abstract: A method is suggested to investigate the calculation error of the optimal Bayesian estimate for a nonlinear problem. This estimate is represented as a ratio of two integrals. It is assumed that both of them are calculated by the Monte-Carlo method. The method suggested is based on the study of the probability density function (p.d.f.) for the ratio. The interrelation between the method considered and the so-called delta method is investigated. Some examples are considered.

Keywords: Bayesian estimation, nonlinear problem, Monte-Carlo method, error evaluation.

1. INTRODUCTION

The methods of Bayesian optimal estimation and filtering are widely used in processing of navigation data (Grewal 1993, Stepanov 1998, Bergman 1999). It is well known that the optimal Bayesian estimate with a minimum mean square error (minimum variance) is a mathematical expectation which corresponds to a posteriori p.d.f. for the parameters under estimation (Stepanov 1998, Bergman 1999). Simple algorithms for calculation of the optimal estimate in the form of the Kalman filter can only be derived in linear Gaussian problems. However, it is not infrequent that in practice we have to solve nonlinear estimation problems, in particular, in processing of navigation data. In order to calculate optimal estimates for nonlinear problems on-line aboard vehicles, it is necessary to design simplified (suboptimal) algorithms. It is a question of vital importance in processing of navigation data as applied to a new class of various applications (robots, automatic vehicles, personal navigation) (Daum 2005, Li and Jilkov 2004, Van der Merwe 2000, Doucet 2001). It is also well known how to check the error in calculation of a single integral—a numerator or denominator—with the use of the Monte-Carlo method (Sobol 1973). The accuracy in calculation of the ratio of two integrals by the important sampling method in computational practice is usually estimated by the so-called delta method (Bergman 1999, Doucet 2001, Kong 1992, Geweke 1989), which is based on the fact that the p.d.f. of the ratio tends to the Gaussian density. The question is whether it is possible and makes sense to improve the procedure of accuracy estimation, relying on a more accurate description of the p.d.f. It is the answer to this question that this paper is devoted to.

2. PROBLEM STATEMENT

Assume that the joint p.d.f. f(x,y) of the vector x being estimated and the measurement vector y are known. Then the optimal estimate with the minimum variance can be represented as follows (Stepanov 1998, Bergman 1999):

\[ \hat{x} = \frac{I_1}{I_2} = \frac{\int x f_1(x) \, dx}{\int x f_2(x) \, dx}, \]

where \( f_1(x) = x f(x,y) \) and \( f_2(x) = f(x,y) \) are the mapping of \( R^n \) in \( R^d \); \( dx \) is an elementary n-dimensional volume. Let us assume that the estimate for integral (1) is calculated by the Monte-Carlo method as

\[ \tilde{x}_N = \frac{\tilde{I}_1}{\tilde{I}_2}, \]

where

\[ \tilde{I}_1 = \frac{1}{N} \sum_{i=1}^{N} f_1(x_i), \quad \tilde{I}_2 = \frac{1}{N} \sum_{i=1}^{N} f_2(x_i), \]

and \( x_1, x_2, \ldots, x_N \) is a sequence of independent n-dimensional random vectors with a p.d.f. \( p(x) \). This p.d.f. can be chosen, for example,
based on the importance sampling method (Doucet 2001, Sobol 1973, Stepanov 2000). Now, without losing generality, we assume that $x$ is a scalar. In accordance with the delta method the procedure for evaluation of the calculation accuracy of $\hat{x}_n$ is based on the fact that $\hat{x}_n$ is asymptotically normal and the asymptotic p.d.f. for $\hat{x}_n$ takes the following form (Kong 1992, Geweke 1989):

$$d(z) = N \left( z, \hat{x}_n, \frac{\sigma_1^2}{I_1^2} + \frac{\sigma_2^2}{I_2^2} - \frac{2 \text{cov}(\hat{I}_1, \hat{I}_2)}{I_1 I_2} \right),$$  \hspace{1cm} (3)

where $\sigma_1$ and $\sigma_2$ the root mean square (RMS) values of $\hat{I}_1$ and $\hat{I}_2$. Based on (3), it is possible to determine when the confidence interval of the error $e = |\hat{x}_n - \bar{x}|$ reaches the specified level.

The aim of the paper is to find a more accurate approximation of the p.d.f. for $\hat{x}_n$ in comparison with (3) and use it to improve the procedure for evaluation of the calculation accuracy of integral (1).

3. DISTRIBUTION OF A RATIO OF TWO INTEGRALS IN THE CASE THAT THEY ARE CALCULATED BY THE MONTE-CARLO METHOD

First, consider the following auxiliary problem. Let $\mathbf{I} = (\hat{I}_1, \hat{I}_2)$ be a random vector and its p.d.f. is bivariate Gaussian distribution, i.e., $\mathbf{I} \sim N(I, \Sigma)$, where $I = (I_1, I_2)$ and $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$. It can be proved (Pham-Gia 2006) that under the assumptions made above the p.d.f. for (2) will be defined as

$$g(z) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(I_1 - I_2)^2}{2(\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2 z^2)} \right) \cdot a(z),$$  \hspace{1cm} (4)

where

$$a(z) = \frac{I_2 \sigma_2^2 + I_1 \sigma_2^2 z - \rho \sigma_1 \sigma_2 (I_1 + I_2 z)}{(\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2 z^2)^{3/2}} \cdot \left( 1 - 2 F \left( \frac{I_2 \sigma_2^2 + I_1 \sigma_2^2 z - \rho \sigma_1 \sigma_2 (I_1 + I_2 z)}{\sigma_1 \sigma_2 \sqrt{(1 - \rho^2)}} \right) \cdot \left( \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2 z^2 \right)^{1/2} \right) + b(z);$$

$$b(z) = \frac{2 \sigma_1 \sigma_2 \sqrt{1 - \rho^2}}{(\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2 z^2)^{1/2}} \cdot \exp \left( -\frac{I_2 \sigma_2^2 - I_1 \rho \sigma_1 \sigma_2 + z (I_1 \sigma_2^2 - I_2 \rho \sigma_1 \sigma_2)}{2 (1 - \rho^2) \sigma_1^2 \sigma_2^2 \left( \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2 z^2 \right)^{1/2}} \right).$$

Note the following important properties of the p.d.f. $g(z)$:

1) If $I_2 \neq 0$, $\sigma_2 > 0$, $|\rho| < 1$ (which is valid in all the cases considered in this paper), the function $g(z)$ is continuous and at infinity $g(z) = O(z^{-2})$. Thus, $g(z)$ is a heavy-tailed density. Therefore, the random values distributed with the density $g(z)$ do not have any expectation and variance in the sense of the Stieltjes integral (Gnedenko 2006).

2) Let $\sigma_1 = q \cdot \sigma_2$, where $q > 0$ is a certain coefficient, then at $z \neq \frac{I_1}{I_2}$ and specified values of the parameters $I_1$, $I_2 \neq 0$, we have $\lim_{\sigma_2 \to 0} g(z) = 0$.

3) At large values of $\sigma_2$ (it is assumed that $\sigma_1 = q \cdot \sigma_2$) the function $g(z)$ is asymmetrical. At sufficiently low values of $\sigma_2$ it is bell-shaped, with a clear cut maximum at point $z_{\text{max}}$ located in the neighborhood of point $z = I_1 I_2$. At $\sigma_2 \to 0$, $z_{\text{max}} \to \frac{I_1}{I_2}$. The plots of $g(z)$ at different $\sigma_2$, at $\sigma_1 = \sigma_2$, $(q = 1)$ and at $I_1 = 2$, $I_2 = 1$, $\rho = 0.5$ are presented in Fig.1.

![Fig.1. The plot of $g(z)$ at $I_1 = 2$, $I_2 = 1$, $\rho = 0.5$, $q = 1$ and three different $\sigma_2$.](image)

It is easy to see that the plot of $g(z)$ becomes more and more symmetrical as $\sigma_2$ decreases (and $\sigma_1$ as well). Note that its maximum is located closer to $\frac{I_1}{I_2} = 2$, and the extremes become higher and narrower.

4) When $q$ is close to $\frac{I_1}{I_2}$ (or equal to it), the maximum of $g(z)$ increases at sufficiently low $\sigma_2$ as $\rho$ approaches unity.
(We mean the case of $I_1 > 0$). However, if $q$ is significantly different from $I_1$, the graph shape of $g(z)$ shows weak dependence on $p$. This property is more clearly presented by two three-dimensional plots in Fig. 2. Here, the function $g(z)$ is represented as a surface as a function of two variables, $z$ and $p$. The plots of the surfaces were obtained at specified values of $I_1 = 2$, $I_2 = 1$, $\sigma_2 = 0.1$. In this case the coefficient $q = 2.5$ (Fig.2a) and $q = 7$ (Fig.2b). As evident from the plot, at $q = 2.5$, which is closer to $I_1$, when the correlation coefficient $\rho$ approaches unity, it produces a more profound effect on the behavior of the function $g(z)$ than in the case when $q = 7$.

![Fig.2. a) The plot of the function $g(z, \rho)$ at $I_1 = 2$, $I_2 = 1$, $\sigma_2 = 0.1$; q = 2.5; b) The plot of the function $g(z, \rho)$ at $I_1 = 2$, $I_2 = 1$, $\sigma_2 = 0.1$, q = 7.](image)

5) If $q = \frac{I_1}{I_2}$ and $p = 1$, $\bar{I}_1$ is linearly dependent on the random value $\bar{I}_2$, i.e., $\bar{I}_1 = \frac{I_1}{I_2} \bar{I}_2$.

6) In the case that $\frac{I_1}{I_2} < 0$, items 4–5 are valid for $q$ close to $\left|\frac{I_1}{I_2}\right|$ and $p = -1$.

Now, let us use formula (2) to derive an approximation of the p.d.f. for the ratio of two integrals calculated by the Monte-Carlo method. Notice that according to the multidimensional central limit theorem at sufficiently high $N$, the bivariate cumulative distribution function (c.d.f.) for $(\bar{I}_1, \bar{I}_2)$ can be uniformly approximated by Gaussian c.d.f. with parameters $I_1, I_2, \sigma_1, \sigma_2, \rho$ (Rao 1965). It should be remarked that $\sigma_1, \sigma_2$ depend on $N$. Next, using the formula for the ratio c.d.f. (Gnedenko 2006), it is possible to deduce that for any interval $[a, b]$ the probability $\Pr(\bar{I}_1 / \bar{I}_2 \in [a, b]) = \int_a^b g(z)dz$ and the approximation error can be made arbitrarily small by increasing $N$. All the above-said makes the basis for the method of error evaluation presented below.

It should be mentioned that at sufficiently low $\sigma_1$ and $\sigma_2$, density (4) can be replaced with (3) by neglecting $b(z)$ and substituting $z = \frac{I_1}{I_2}$ into (4) everywhere, except the numerator of the exponent. This replacement corresponds to the delta method (Kong 1992, Geweke 1989) so that in the limit, the suggested method and the delta method yield the same results.

4. EVALUATING THE CALCULATION ERROR OF THE RATIO OF TWO INTEGRALS BY THE MONTE-CARLO METHOD

As it was mentioned at the end of Section 3, the p.d.f. for $\bar{x}_N = \bar{I}_1 / \bar{I}_2$ is approximated with any accuracy by the distribution of the form (4) as $N$ increases. Therefore the method suggested for error evaluation is as follows:

1) $N$ random vectors $x_1, x_2, \ldots, x_N$ with the density $p(x)$ are simulated.

2) The values $\bar{I}_1$ and $\bar{I}_2$, standard deviations $s_1$ and $s_2$ are calculated with the use of the Monte-Carlo method. Also calculated is the estimate $\bar{\rho}$ of the correlation coefficient $\rho$ between $\bar{I}_1$ and $\bar{I}_2$ (see Statement 1 and Corollary of Appendix) by the following formula

$$\bar{\rho} = \left( \sum_{k=1}^N \left( \int x_k(x_k) \frac{1}{N^2} \right) - N \bar{I}_1 \bar{I}_2 \right) \frac{1}{N^2 s_1 s_2}$$

and the value

$$\text{cnf} = \int \frac{g(z)}{x} dz$$

The last operation is realized with a certain periodicity.

In (6), $\varepsilon$ is a specified calculation accuracy. The values $I_1, I_2, \rho, \sigma_1, \sigma_2$ in $g(z)$ are replaced with their statistical estimates $\bar{I}_1, \bar{I}_2, \bar{\rho}, s_1, s_2$, in so doing, integration is performed by some numerical method.

3) When $\text{cnf} > 1 - \alpha$, where $\alpha$ is the confidence level, the calculations stop and (2) is assumed to be an approximate value of ratio (1).

It is clear that $\bar{x}_N$ is derived with an error that is not more than $\varepsilon$, with the confidence probability $1 - \alpha$.
If the delta method is used for error evaluation, the value \( cnf \) from (6) is replaced by \( cnf = \frac{I_2^{1/2}}{I_1} \int d(z) dz \), where \( d(z) \) is determined by (3).

**Note 1.** In addition, assume that there exist the fourth moments for \( I_1 \) and \( I_2 \). Then, evaluating the error in accordance with (6), it is possible to prove that the order of convergence of \( \hat{x}_N \) is not less than \( \frac{1}{2} \) and not higher than \( N^{-1/2} \sqrt{\ln N} \).

5. EXAMPLES

The evaluation results of the ratio calculation error with the aim to compare the method suggested and the delta method are presented below for two examples. In addition to it, the evaluation of the ratio error was calculated with the use of the simplest formula (Demidovich 2006)

\[
|\hat{x} - \bar{x}_N| \leq 2 \cdot \bar{x}_N \left( \frac{\sigma_1}{I_1} + \frac{\sigma_2}{I_2} \right).
\]

**Example 1.** The simplest estimation problem.

Let there be a measurement \( y = x + v \), where \( x \) and the measurement error \( v \) are independent of each other, such that \(( X \sim N(\bar{x}, \sigma)) \), \(( e \sim N(0, r)) \). The optimal estimate can be represented for this simplest example as

\[
\hat{x}(y) = M(X \mid Y = y) = \frac{I_1}{I_2} = \frac{\int x p(y \mid x) p(x) dx}{\int p(y \mid x) p(x) dx} = \frac{\int_0^\infty f_1(x) dx}{\int_0^\infty f_2(x) dx}, \tag{8}
\]

where \( p(x) = N(x, \bar{x}, \sigma) \), \( p(y \mid x) = N(y, x, r) \). It is known that

\[
\hat{x}(y) = \bar{x} + \frac{\sigma_0^2}{\alpha^2 + r^2} (y - \bar{x}). \tag{9}
\]

Let \( \bar{x} = 1 \), \( \sigma = 1 \), and \( \alpha = 0.05 \). Table 1 presents the results of the evaluation error in calculation of an optimal estimate by the method suggested, the delta method and the simplest method for one of the typical numerical experiments at \( r = 1 \), \( r = 0.5 \), \( r = 0.1 \) for a specified measurement \( y = 1.1 \). The actual values of the estimates at different \( r \) calculated by (9) are determined as 1.0500 (\( r = 1 \)), 1.0800 (\( r = 0.5 \)), and 1.0990 (\( r = 0.1 \)).

The analysis has shown that the delta method and the method suggested in this paper give the same results.

<table>
<thead>
<tr>
<th>( r )</th>
<th>Actual</th>
<th>Suggested</th>
<th>Delta</th>
<th>Simple</th>
<th>( \tilde{\rho} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0037</td>
<td>0.01</td>
<td>0.01</td>
<td>0.0168</td>
<td>0.6</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0025</td>
<td>0.05</td>
<td>0.05</td>
<td>0.0218</td>
<td>0.9</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.0316</td>
<td>0.997</td>
</tr>
</tbody>
</table>

It is also easy to see that the error derived by the simplest formula (7) is high. The higher is the error, the lower is the RMS of the measurement \( r \). It is explained by the fact that when \( r \) decreases, the correlation between the estimates of the numerator and denominator increases, but it is not taken into account by the simplest formula.

**Example 2.** The problem of positioning by reference beacons.

Let us estimate the vector \( (x_1, x_2) \) by the range measurements up to two fixed reference beacons whose coordinates \( (x_1^1, x_2^1) \) and \( (x_1^2, x_2^2) \) are assumed to be known (Stepanov 2008). These measurements can be presented in the following form:

\[
y^{(1)}_i = \sqrt{(x_1 - x_1^1)^2 + (x_2 - x_2^1)^2} + \nu^{(1)}_i; \tag{10}
\]

\[
y^{(2)}_i = \sqrt{(x_1 - x_1^2)^2 + (x_2 - x_2^2)^2} + \nu^{(2)}_i; \tag{11}
\]

where \( m \) is the number of the measurement pairs; the measurement errors \( \nu^{(k)}_i \sim N(0, r), i = 1, m, k = 1, 2 \) are zero-mean independent-on-each-other Gaussian random values. The a priori p.d.f. for the vector \( x = (x_1, x_2) \) is Gaussian, i.e., \( p(x) = N(x, x^0, \sigma^2 E) \), where \( x^0 = (x_0^1, x_0^2) \) and \( E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). The confidence level was assumed to be 0.9; \( x_0^1 = x_0^2 = 0 \); \( r = 30 m \). The coordinates of the two beacons were the following: \( x^{(1)} = (3000m, 0)^T \), \( x^{(2)} = (0, 3000m) \). The calculations were performed for \( \sigma = 500 m \) and \( \sigma = 1400 m \). These two cases are essentially different as in the first case the a posteriori density has one extremum, whereas in the second – two extrema. The typical examples of the calculation results are given in Table 2.

The simulation has shown that both methods also yield similar results for this example. The error derived by the simplest formula (8) is also high.
Possible discrepancy between the results of the two comparison methods caused by the dissimilarity of \( g(z) \) from the Gaussian density does not show itself in the examples considered as the relative errors in the calculations of the numerator and denominator were small at the moment when the calculations came to a halt. Asymmetry makes itself evident only at high values of these errors. As computation practice shows, for \( g(z) \), different from the Gaussian density, to cause difference in the estimates, the relative errors of the numerator and denominator must be sufficiently high. Fig. 3 shows the plots of the functions \( g(z) \) and \( d(z) \) (Gaussian density) at specified values of \( I_1, I_2, \rho, q, \sigma_2 \).

Fig. 3. The plots of \( g(z) \) and \( d(z) \) at \( I_1=1, I_2=1, \rho=0.95, q=0.5 \) and three different values of \( \sigma_2 \): a) \( \sigma_2=1 \); b) \( \sigma_2=0.4 \); c) \( \sigma_2=0.1 \)

From Fig. 3 it is easy to see that the plots are essentially different at \( \sigma_2 = 1 \) (100%) and almost identical at \( \sigma_2 = 0.1 \) (the relative error is 10%). It is clear that the values of \( c_{nf} \) calculated for \( g(z) \) and \( d(z) \) are close to each other.

6. CONCLUSIONS

The paper considers a procedure for the analysis of accuracy in calculations of the Bayesian optimal estimate, which is presented in the form of a relation of two integrals. The analysis is based on the derivation of the approximate p.d.f. for a ratio of two integrals, each of them calculated by the Monte-Carlo method. It has been shown that the p.d.f. of a ratio is a density with “heavy tails”. At certain relations between means and variances for the numerator and denominator this p.d.f. is essentially asymmetric and, hence, differs greatly from normal distribution. The p.d.f. derived was used as a basis for the formula of the criteria to stop calculations. The upper and the lower estimates of the order of convergence of the calculation process have been determined. The interrelation between the delta method and the method suggested in this paper has been established. It has been shown, in particular, that if the number of samplings is large, the derived p.d.f. converges to normal distribution with the same parameters as in the delta method. The results of the comparison of the efficiency of the two methods and the simplest formula for estimating the calculation error of the ratio in the solution of the test estimation problem and nonlinear estimation problem, which are often solved during the processing of navigation data, are presented. It has been shown that the difference between the delta method and the method suggested here only manifests itself for the case when number of samples is small. However, if the requirements to the calculation accuracy of the numerator and denominator are reasonable, both methods lead to similar results, which makes the implementation of the delta method more justified.

REFERENCES


Appendix.

Statement 1. Let \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_N, \eta_N)\) be independent realizations of the random vector \((\xi, \eta)\). Let \(\rho\) be the coefficient of correlation between \(\bar{\xi} = \frac{1}{N} \sum_{k=1}^{N} \xi_k\) and \(\bar{\eta} = \frac{1}{N} \sum_{k=1}^{N} \eta_k\). Then

\[
\rho = \frac{M[\xi \eta] - M\xi M\eta}{N\sigma_{\xi}\sigma_{\eta}},
\]

where \(\sigma_{\xi}\) and \(\sigma_{\eta}\) are mean-square deviations \(\bar{\xi}\) and \(\bar{\eta}\).

Proof: Let us calculate the covariance \(\bar{\xi} = \frac{1}{N} \sum_{k=1}^{N} \xi_k\) and \(\bar{\eta} = \frac{1}{N} \sum_{k=1}^{N} \eta_k\):

\[
\text{cov}(\bar{\xi}, \bar{\eta}) = M\left\{\frac{1}{N} \sum_{k=1}^{N} \xi_k \right\} \cdot \frac{1}{N} \sum_{k=1}^{N} \eta_k
\]

\[
- M\left\{\frac{1}{N} \sum_{k=1}^{N} \xi_k \right\} M\left\{\frac{1}{N} \sum_{k=1}^{N} \eta_k \right\} =
\]

\[
= \frac{1}{N^2} \left\{ \sum_{k=1}^{N} M[\xi_k \eta_k] + \sum_{i,j=1}^{N} M[\xi_i \eta_j] \right\} - M\xi M\eta =
\]

\[
= \frac{1}{N} M[\xi \eta] + \frac{N-1}{N} M\xi M\eta - M\xi M\eta = \frac{M[\xi \eta] - M\xi M\eta}{N}.
\]

The calculation in the chain of equations (11) makes essential use of the fact that, in view of independent random vectors, we have \(M[\xi_i \eta_j] = M\xi_i M\eta_j = M\xi M\eta\) when \(i \neq j\).

Eq. (11) proves the Statement.

Corollary: Under the conditions of Statement 1 the consistent estimate for \(\rho\) by independent realizations \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_N, \eta_N)\) will be presented by the value

\[
\bar{\rho} = \left( \frac{N}{N-1} \right) \frac{\bar{\xi} \cdot \bar{\eta}}{\sigma_{\bar{\xi}} \sigma_{\bar{\eta}}} = \frac{1}{N^2 \sigma_{\xi} \sigma_{\eta}}.
\]

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