Abstract: This paper is concerned with $L_2$-gain disturbance attenuation problem for linear multivariable systems. First, high gain output feedback theorem for $L_2$-gain disturbance attenuation problem for general (nonminimum phase) systems. As a control law, P-SPR-D and/or P-SPR-D+I control is applied (SPR is an abbreviation for strict positive real). We can solve the problem by changing the general system to minimum phase one with the use of P, SPR, D elements and by applying the above high gain output feedback theorem.

1. INTRODUCTION

As a method of system stabilization, high gain output feedback control (Isidori (1995); Khalil (2002); Shimizu (2004)) is well known. A system can be asymptotically stabilized by high gain output feedback, provided that the system has relative degree 1 and is minimum phase.

We first review high gain output feedback theorem for linear minimum phase system as preliminaries. Next, we study $L_2$-gain disturbance attenuation problem for MIMO minimum phase system. We derive high gain output feedback theorem for the $L_2$-gain disturbance attenuation of minimum phase system, applying Lyapunov’s method.

The $L_2$-gain disturbance attenuation was investigated (Schaft (2000); Marino et al. (1994); Isidori (1996)) and extended (Isidori (1999); Shen (2002)) as a basis of $H_\infty$ control, based on the small gain theorem and passivity property. It is also discussed as the generalized K-Y-P lemma in relation to positive boundedness of transfer function, strict positive realness, the Riccati inequality, LMI, etc. and it contributes to a basis of robust control. Meanwhile, we solve it here as high gain output feedback for $L_2$-gain disturbance attenuation, applying Lyapunov’s direct method (LaSalle and Lefschetz (1961)).

Finally, we study the $L_2$-gain disturbance attenuation problem for general (nonminimum) phase systems. As a control law, P-SPR-D control (Tamura and Shimizu (2008); Shimizu (2009)) is applied. The P-SPR-D and/or P-SPR-D+I control is a structured controller as PID control using a SPR (strict positive real) element. We can solve the $L_2$-gain disturbance attenuation problem of general systems by making the system be minimum phase with the use of the SPR element and by applying the high gain output feedback theorem obtained for the minimum phase system. It is noted that our approach differs from existing $H_\infty$ control theory and it possesses possibility of not only assuring small gain property of a transfer function but also improving control performance to some extent.

The effectiveness of the proposed method is confirmed with the simulation results of position control problem of 3-inertia system (Hara et al. (1995)).

2. HIGH GAIN OUTPUT FEEDBACK STABILIZATION

Consider a linear multivariable minimum phase system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  
\[ y(t) = Cx(t) \]

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^m$ are the state vector, the control input and the output, respectively. And consider an output feedback control

\[ u(t) = -Ky(t) \]

with a gain matrix $K \in R^{m \times m}$. Then the following high gain output feedback theorem is obtained.

[Theorem 1] (High Gain Output Feedback) Suppose that system (1),(2) has relative degree \{1,1,\cdots,1\} (i.e. $CB$ is nonsingular) and is minimum phase. Consider an output feedback control (3). Then there exist constants $\gamma_{i0} > 0, i = 1, 2, \cdots, m$ such that the closed-loop system (1) \sim (3) is asymptotically stable, provided that $K$ is taken as $K = (CB)^{-1}(Q_{11} + \Gamma)$ with $\Gamma = \text{diag}(\gamma_1, \gamma_2, \cdots, \gamma_r)$. $\gamma_i \geq \gamma_{i0} > 0$, where $Q_{11}$ is a partial matrix in the normal form (see eq.(5)). Namely,

\[ \dot{x}(t) = Ax(t) - BKy(t) \]

is asymptotically stable.

(Proof) From the assumption of relative degree \{1,1,\cdots,1\}, by a nonsingular coordinate transformation

\[ \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} C & T \end{bmatrix} x, \quad TB = O \]

system (1),(2) can be transformed into the normal form (Isidori (1995); Sepulcure (1997))

\[ \dot{\xi} = Q_{11} \xi + Q_{12} \eta + CBu \]
\[ \dot{\eta} = Q_{21}\xi + Q_{22}\eta \]  
\[ y = \xi \]

where \( \xi \in \mathbb{R}^m \), \( \eta \in \mathbb{R}^{n-m} \) and \( y \in \mathbb{R}^m \). Then, \( \dot{\eta} = Q_{22}\eta \) is called zero dynamics. If the zero dynamics be asymptotically stable, the system (1),(2) is said minimum phase.

By executing the output feedback
\[ u(t) = -Ky(t) = -K\xi(t) \]
the closed-loop system (5)–(7) becomes as follows.

\[ \dot{\xi} = Q_{11}\xi + Q_{12}\eta - CBK\xi \]
\[ \dot{\eta} = Q_{21}\xi + Q_{22}\eta \]

Since \( \dot{\eta} = Q_{22}\eta \) is asymptotically stable from the assumption of minimum phase, the Lyapunov inequality
\[ PQ_{22} + Q_{22}^T P < 0 \]
holds for some \( P > 0 \).

Now we define a Lyapunov function
\[ V(\xi, \eta) = \begin{bmatrix} \xi^T & \eta^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \]

V(\xi, \eta) is a positive definite function, because of \( P > 0 \).

Calculate its time derivative along (9),(10) to get
\[ \dot{V}(\xi, \eta) = \begin{bmatrix} \dot{\xi}^T & \dot{\eta}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \]

\[ = \begin{bmatrix} Q_{11}\xi + Q_{12}\eta - CBK\xi \\ Q_{21}\xi + Q_{22}\eta \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \]

\[ = \begin{bmatrix} \xi^T & \eta^T \end{bmatrix} \begin{bmatrix} Q_{11}^T & Q_{12}^T \\ Q_{21}^T & Q_{22}^T \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} - CBK\xi^T \begin{bmatrix} \xi \\ \eta \end{bmatrix} \]

Now system (9),(10) becomes asymptotically stable.

3. L₂-GAIN DISTURBANCE ATTENUATION FOR MINIMUM PHASE SYSTEM

Let us study L₂-gain disturbance attenuation problem in case of existence of disturbances. Consider the following linear multivariable system
\[ \dot{x}(t) = Ax(t) + Bu(t) + \overline{B}w(t) \]
\[ y(t) = Cx(t) \]
\[ z(t) = CPx(t) \]

where \( x(t) \in \mathbb{R}^m, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^m, z(t) \in \mathbb{R}^m, w(t) \in \mathbb{R}^d \) are the state vector, the control input, the measurable output, the controlled output and the disturbance input, respectively. And consider an output feedback control
\[ u(t) = -KY(t) \]

with a gain matrix \( K \in \mathbb{R}^{m \times m} \).

The L₂-gain disturbance attenuation problem is defined to attain a control such that the closed-loop system satisfies the following conditions under the given disturbance attenuation level \( \gamma > 0 \).

P1. When \( w = 0 \), the closed-loop system is asymptotically stable.

P2. When \( x(0) = 0 \), the following inequality holds for the arbitrarily given \( T > 0 \).

\[ \int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|w(t)\|^2 dt \]

It is noticed that P2 is equivalent to having L₂-gain below \( \gamma \) when \( x(0) = 0 \), that is, \( \|z\|_2 \leq \gamma\|w\|_2 \). It implies that for all \( w \in L_2[0,T] \) and the supply rate \( s(z,w) = \gamma^2w^Tw - z^Tz \), the following \( \gamma \)-dissipation inequality holds with a storage function \( V(x) \) (Schaft (2000); Shen (2002)).

\[ V(x(t)) \leq \gamma^2w(t)^Tw(t) - z(t)^T z(t) \]

Now if system (14),(15) has relative degree \( \{1,1,\ldots,1\} \), then by the nonsingular coordinate transformation
\[ \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} C & T \end{bmatrix} x, \quad TB = O, \quad T\overline{B} = O \]

system (14)–(16) can be transformed into the normal form

\[ \dot{\xi} = Q_{11}\xi + Q_{12}\eta + CBu + C\overline{B}w \]
\[ \dot{\eta} = Q_{21}\xi + Q_{22}\eta \]
\[ y = \xi \]
\[ z = Q_{31}\xi + Q_{32}\eta \]

where \( \xi \in \mathbb{R}^m, \eta \in \mathbb{R}^{n-m}, y \in \mathbb{R}^m, z \in \mathbb{R}^m \) and \( w \in \mathbb{R}^d \).

Note that in general there does not exist a matrix \( T \) satisfying \( TB = 0 \) and \( T\overline{B} = 0 \). But we consider the case of \( B \) and \( \overline{B} \) for which there exists a matrix such that \( TB = 0 \) and \( T\overline{B} = 0 \) hold. It will be applied in Chapter 4.
By executing the output feedback
\[ u(t) = -Ky(t) = -K\xi(t) \]  \hspace{1cm} (24)
the closed-loop system (20)\sim(24) becomes as follows.
\[ \dot{\xi} = Q_{11}\xi + Q_{12}\eta - CBK\xi + CB\eta \]  \hspace{1cm} (25)
\[ \eta = Q_{21}\xi + Q_{22}\eta \]  \hspace{1cm} (26)
\[ z = Q_{31}\xi + Q_{32}\eta \]  \hspace{1cm} (27)
Since \( \eta = Q_{22}\eta \) is asymptotically stable from the assumption of minimum phase, the Lyapunov inequality
\[ P_{Q_{22}} + Q_{T_{2}}P < 0 \]  \hspace{1cm} (28)
holds for some \( P > 0 \).

Here let us assume :
[**Assumption 1**] There exists \( P > 0 \) such that
\[ P_{Q_{22}} + Q_{T_{2}}P + Q_{T_{2}}Q_{22} < 0 \]
The following theorem solves the \( L_2 \)-gain disturbance attenuation problem.

**[Theorem 2]** (High Gain Output Feedback for \( L_2 \)-gain Disturbance Attenuation)
Suppose that system (14)\sim(16) has relative degree \( (1,1,\ldots,1) \) (i.e. \( CB \) is non-singular) and is minimum phase. Suppose also that there exists a matrix \( T \) such that \( TB = 0 \) and \( T\bar{B} = 0 \) hold. Further assume Assumption 1 holds. And consider the output feedback control (17). Then, there exist constants
\[ \gamma > 0, \ i = 1, 2, \ldots, m \] such that the closed-loop system (14)\sim(17)satisfies condition P2, provided that \( K \) is taken as
\[ K = (CB)^{-1}\left(Q_{11} + \frac{1}{\gamma}(Q_{21}Q_{31} + \frac{1}{\gamma}CB\bar{B}CT) + \Gamma\right) \]
with \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m) \), \( \gamma_i \geq \gamma_0 > 0 \). Namely,
\[ \dot{x}(t) = Ax(t) - BKy(t) + \bar{B}w(t) \]
possesses \( L_2 \)-gain less than \( \gamma \).

Furthermore, the closed-loop system (14)\sim(17) satisfies condition P1.

**(Proof)** Define a storage function (which is also a Lyapunov function)
\[ V(\xi, \eta) = \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^T \left[ \begin{array}{cc} I & O \\ O & P \end{array} \right] \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] \]
\[ \dot{V}(\xi, \eta) = \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^T \left[ \begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right] \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] - \frac{1}{\gamma}w^T w \]
which is a positive definite function, because of \( V(\xi, \eta) \) being a storage function.

\[ \dot{V}(\xi, \eta) \leq \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^T \left[ \begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right] \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] - \frac{1}{\gamma}w^T w \]

Since \( \Theta \) is a symmetric matrix, it becomes negative definite by Schur complement, if \( \Theta_{11} \) is sufficiently small negative definite in consideration of Assumption 1. Here, if we take \( K \) as
\[ K = (CB)^{-1}\left(Q_{11} + \frac{1}{\gamma}(Q_{21}Q_{31} + \frac{1}{\gamma}CB\bar{B}CT) + \Gamma\right) \]
then \( \Theta_{11} \) becomes \( \Gamma - \Gamma^T \), which is negative definite, provided that \( \Gamma \) is a positive definite matrix. Hence, if \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m) \), \( \gamma_i \geq \gamma_0 > 0 \) is sufficiently large, then \( \Theta \) becomes negative definite under Assumption 1.

Accordingly, we have
\[ \dot{V}(\xi, \eta) + \left[ \begin{array}{c} \xi \\ \eta \end{array} \right]^T \left[ \begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right] \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] - \frac{1}{\gamma}w^T w \]
\[ < - \left( \frac{1}{\gamma}C^T \xi - \gamma w \right)^T \left( \frac{1}{\gamma}C^T \xi - \gamma w \right) \leq 0 \]  \hspace{1cm} (32)
Consequently, the $\gamma$-dissipation inequality holds, and so it follows that we have $L_2$-gain below $\gamma$.

When $w = 0$, condition P1 has been already proved in Theorem 1. Q.E.D.

**Corollary 1** Let us consider the case of $z = y$. Suppose that system (14),(15) has relative degree $\{1,1,\cdots,1\}$ (i.e. $CB$ is nonsingular) and is minimum phase. Suppose also that there exists a matrix $T$ such that $TB = 0$ and $T\overline{B} = 0$ hold. Further assume there exists $P > 0$ satisfying $PQ_{22} + Q_{12}^T P + I < 0$. And consider the output feedback control (17). Then, there exist constants $\gamma_0 > 0, i = 1,2,\cdots,m$ such that the closed-loop system (14),(15),(17) satisfies condition P2, provided that $K$ is taken as $K = (CB)^{-1}\left(Q_{11} + \frac{1}{2}(I + \frac{1}{\tau}CBB^T) + \Gamma\right)$ with $\Gamma = \text{diag}(\gamma_1,\gamma_2,\cdots,\gamma_m)$, $\gamma_i \geq \gamma_0 > 0$. Namely,

$$\dot{x}(t) = Ax(t) - BKy(t) + T\overline{w}(t)$$

possesses $L_2$-gain less than $\gamma$.

Furthermore, the closed-loop system (14),(15),(17) satisfies condition P1.

**4. $L_2$-GAIN DISTURBANCE ATTENUATION BY P-SPR-D AND/OR P-SPR-D+I CONTROL**

**4.1 P-SPR-D Control**

Consider the following general MIMO system:

$$\dot{x}(t) = Ax(t) + Bu(t) + T\overline{w}(t)$$

$$y(t) = Cx(t)$$

$$z(t) = C_p x(t)$$

where $x(t) \in R^m, u(t) \in R^p, y(t) \in R^q, z(t) \in R^{m}, w(t) \in R^d$ are the state vector, the control input, the measurable output, the controlled output and the disturbance input, respectively.

We investigate P-SPR-D control for $L_2$-gain disturbance attenuation, by making the system as minimum phase and applying the high gain output feedback.

In case of regulation problem, PID control is given as

$$u(t) = -K_PY(t) + K_I \int_0^t -y(\tau)d\tau - K_D \dot{y}(t)$$

where $K_P, K_I, K_D \in R^{p \times m}$ are P, I, D gains.

In this paper, we propose the following P-SPR-D control

$$\dot{z}(t) = Dz(t) - y(t), \quad z(0) = 0, \quad D < 0$$

$$u(t) = -K_PY(t) + KS\zeta(t) - K_D\dot{y}(t)$$

where (37) represents a SPR element with negative definite $D \in R^{m \times m}$, and $K_S \in R^{n \times m}$ denotes the SPR gain.

Since it holds from (33),(34) that

$$\dot{y}(t) = C(Ax(t) + Bu(t) + T\overline{w}(t))$$

we have

$$u(t) = -K_PCx(t) + KS\zeta(t)$$

$$-K_D(C(Ax(t) + Bu(t) + T\overline{w}(t))$$

by substituting (34),(39) into (38).

Furthermore, arranging this equation, we obtain

$$u(t) = -(I_r + K_DCB)^{-1}(K_PC + K_DCA)x(t)$$

$$+ (I_r + K_DCB)^{-1}KS\zeta(t)$$

$$- (I_r + K_DCB)^{-1}K_D\overline{w}(t)$$

$$= -K_Xz(t) + K_\Xi\zeta(t) - K_Ww(t)$$

where

$$K_X \equiv (I_r + K_DCB)^{-1}(K_PC + K_DCA)$$

$$K_\Xi \equiv (I_r + K_DCB)^{-1}KS$$

$$K_W \equiv (I_r + K_DCB)^{-1}K_D\overline{B}$$

By substituting (40) into (33), we obtain the closed-loop system

$$\dot{z}(t) = (A - BK_X)z(t) + BK_\Xi\zeta(t) - BK_Ww(t)$$

$$+ T\overline{w}(t)$$

Accordingly, by combining (41) and (37), the closed-loop system by the P-SPR-D control becomes

$$\begin{bmatrix} \dot{x}(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} A - BK_X & BK_\Xi \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ \zeta(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -BK_W & T\overline{B} \end{bmatrix} w(t)$$

Now we define the SPR gain $K_S$ as

$$K_S \equiv H_S\mathcal{L}$$

where $H_S \in R^{r \times m}$ and $\mathcal{L} \in R^{m \times m}$ are called the intermediate parameter matrix and the adjustable parameter matrix, respectively. The parameterization as (43) is made in order to apply high gain output feedback to the system (49)~(51) below. Namely, $\mathcal{L}$ corresponds to the gain of output feedback and $H_S$ is a parameter used to asymptotically stabilize the zero dynamics. The high gain output feedback makes controller parameter tuning easy for improving the control performance.

Consider the followings.

$$\dot{\zeta}(t) = \mathcal{L}' \zeta'(t) - Ly(t)$$

$$u(t) = -K_PCx(t) + H_S\zeta(t) - K_D\dot{y}(t)$$

Then, by substituting (34),(39) into (45), we have

$$u(t) = -K_PCx(t) + H_S\zeta(t) - K_D\dot{y}(t)$$

$$-K_D(C(Ax(t) + Bu(t) + T\overline{w}(t))$$

Arranging this equation as before, we obtain

$$u(t) = -K_Xz(t) + K_\Xi'\zeta(t) - K_Ww(t)$$

where

$$K_\Xi' \equiv (I_r + K_PC)H_S$$

By substituting (46) into (33), we obtain the closed-loop system
\[ \dot{x}(t) = (A - BK_X)x(t) + BK_\Xi \dot{z}(t) - BK_W w(t) + \overline{B}w(t) \] (47)

Accordingly, by combining (47) and (44), we have
\[ \begin{align*}
&\begin{bmatrix}
\dot{x}(t) \\
\dot{\zeta}(t)
\end{bmatrix} = \begin{bmatrix}
A - BK_X & BK_\Xi \\
-LC & LD^T
\end{bmatrix} \begin{bmatrix}
x(t) \\
\dot{\zeta}(t)
\end{bmatrix} \\
&\quad + \begin{bmatrix}
O \\
-BK_W + \overline{B}
\end{bmatrix} v(t) + w(t)
\end{align*} \] (48)

Next let us consider the following hypothetical system based on the closed-loop system (48):
\[ \begin{align*}
&\begin{bmatrix}
\dot{x}(t) \\
\dot{\zeta}(t)
\end{bmatrix} = \begin{bmatrix}
A - BK_X & BK_\Xi \\
-LC & LD^T
\end{bmatrix} \begin{bmatrix}
x(t) \\
\dot{\zeta}(t)
\end{bmatrix} \\
&\quad + \begin{bmatrix}
O \\
-BK_W + \overline{B}
\end{bmatrix} v(t) \\
&\quad := \tilde{A}\dot{x}(t) + \tilde{B}v(t) + \tilde{B}w(t)
\end{align*} \] (49)

\[ \begin{align*}
\tilde{y}(t) &= [C - D'] \begin{bmatrix}
x(t) \\
\dot{\zeta}(t)
\end{bmatrix} := \tilde{C}\tilde{x}(t) \\
\tilde{z}(t) &= [C_P O] \begin{bmatrix}
x(t) \\
\dot{\zeta}(t)
\end{bmatrix} := \tilde{C}_P\tilde{x}(t)
\end{align*} \] (50) (51)

where \(\tilde{x}(t) \in R^{n+m}, v(t) \in R^m, \tilde{y} \in R^m, \tilde{z} \in R^m\) denote the state vector, the input, the measurable output and the controlled output of the hypothetical system \(\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{C}_P\}\), respectively. And let the input \(v(t)\) be given by the output feedback
\[ v(t) = -\mathcal{L}\tilde{y}(t) = -\mathcal{L}[C - D'] \begin{bmatrix}
x(t) \\
\dot{\zeta}(t)
\end{bmatrix} \] (52)

where \(\mathcal{L} \in R^{m\times m}\) is the output feedback gain. Then, the closed-loop system of the hypothetical system becomes
\[ \begin{align*}
&\begin{bmatrix}
\dot{x}(t) \\
\dot{\zeta}(t)
\end{bmatrix} = \begin{bmatrix}
A - BK_X & BK_\Xi \\
-LC & LD^T
\end{bmatrix} \begin{bmatrix}
x(t) \\
\dot{\zeta}(t)
\end{bmatrix} \\
&\quad + \begin{bmatrix}
O \\
-BK_W + \overline{B}
\end{bmatrix} w(t) \\
&\quad := \tilde{A}\dot{x}(t) + \tilde{B}w(t)
\end{align*} \] (53)

which is the same as (48).

Now if we put \(D' = \mathcal{L}\mathcal{C}\) and \(\mathcal{C}' = \mathcal{L}\mathcal{C}\), then the lower column of (53) becomes equal to (37). Therefore, if we set \(K_S = H_S\mathcal{C}\), \(D = D'\mathcal{L}\), (53) coincides with (42).

Consequently, (53) becomes equal to the closed-loop error system with the P-SPR-D control (33),(34),(37),(38), provided that the output feedback gain \(\mathcal{L}\) is taken equal to the adjustable parameter of (43).

### 4.2 Design of P-SPR-D Controller for L₂ Gain Disturbance Attenuation

Let us apply Theorem 2 to design the P-SPR-D controller for \(L_2\) gain disturbance attenuation. Consider the following general MIMO system:
\[ \begin{align*}
\dot{x}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}v(t) + \overline{B}w(t) \\
\tilde{y}(t) &= \tilde{C}\tilde{x}(t) \\
\tilde{z}(t) &= \tilde{C}_P\tilde{x}(t)
\end{align*} \] (54) (55) (56)

where \(x(t) \in R^n, v(t) \in R^m, \tilde{y}(t) \in R^m\) and \(\tilde{z}(t) \in R^m\).

If system (54),(55) has relative degree \(\{1, 1, \cdots, 1\}\) (i.e. \(\tilde{C}\tilde{B}\) is nonsingular), then the system can be transformed into the normal form. That is, by the nonsingular coordinate transformation
\[ \begin{bmatrix}
\xi \\
\eta
\end{bmatrix} = \begin{bmatrix}
\tilde{C} \\
\tilde{T}
\end{bmatrix} \begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix}, \quad \tilde{T}\tilde{B} = O, \quad \tilde{T}\overline{B} = O \] (57)

(54)–(56) can be transformed into the normal form
\[ \begin{align*}
\xi &= Q_{11}\xi + Q_{12}\eta + \tilde{C}\tilde{B}v + \tilde{C}\overline{B}w \\
\eta &= Q_{21}\xi + Q_{22}\eta \\
\tilde{y} &= \xi \\
\tilde{z} &= Q_{31}\xi + Q_{32}\eta
\end{align*} \] (58) (59) (60) (61)

where \(\xi \in R^m, \eta \in R^{N-m}, \tilde{y} \in R^m\) and \(\tilde{z} \in R^{N'}\).

Here we assume:

- **Assumption 2** There exists \(P \succ 0\) such that

\[ PQ_{22} + Q_{11}^T P + Q_{12}^T Q_{32} < 0 \]

Applying Theorem 2 to system (54)–(56) we have:

- **Proposition 1** Suppose that system \(\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{C}_P\}\) given by (54)–(56) has relative degree \(\{1, 1, \cdots, 1\}\) (i.e. \(\tilde{C}\tilde{B}\) is nonsingular) and is minimum phase. Suppose also that there exists a matrix \(\tilde{T}\) such that \(\tilde{T}\tilde{B} = 0\) and \(\tilde{T}\overline{B} = 0\) hold. Further assume Assumption 2 holds. And consider an output feedback control

\[ v(t) = -\mathcal{L}\tilde{y}(t) \] (62)

with a gain matrix \(\mathcal{L} \in R^{m\times m}\). Then, there exist constants \(\gamma_i > 0, i = 1, 2, \cdots, m\) such that the closed-loop system (54)–(56),(62) satisfies condition P2, provided that \(\mathcal{L}\) is taken as \(\mathcal{L} = (\tilde{C}\tilde{B})^{-1} \{Q_{11} + \frac{1}{2} (Q_{31}^T Q_{31} + \frac{1}{2} \tilde{C}\overline{B}^T \tilde{C}) + \Gamma\} \) with \(\Gamma = \text{diag}(\gamma_1, \gamma_2, \cdots, \gamma_m), \gamma_i \geq \gamma_i > 0\). Namely,

\[ \dot{x}(t) = \tilde{A}\tilde{x}(t) - \tilde{B}\mathcal{L}\tilde{y}(t) + \overline{B}w(t) \]

possesses L₂-gain below \(\gamma\).

Furthermore, the closed-loop system (54)–(56),(62) satisfies condition P1.

- **Proof** Obvious from Theorem 2.

Q.E.D

Now let us consider to apply Proposition 1 to the hypothetical system (49)–(51) to design the P-SPR-D controller. So check first the relative degree with respect to \(v\). Since

\[ \frac{\partial \tilde{y}(t)}{\partial v(t)} = \tilde{C}\tilde{B} = [C - D'] \begin{bmatrix}
O \\
I_m
\end{bmatrix} = -D' \]

this matrix has to be nonsingular in order to apply Proposition 1. So let \(D'\) be a nonsingular matrix.

Accordingly, we can transform system (49)–(51) into the normal form to obtain its zero dynamics. Consider the following transformation from (57):

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Corresponding to Assumption 2, we assume also:

**Assumption 3** There exist parameter matrices $K_P, H_S$, $K_D$ such that the zero dynamics (69) is asymptotically stable.

Accordingly, from (66) the zero dynamics is expressed as

$$
\dot{\eta} = T_n\left(A - B(K_X + K'_D D^{-1}C)\right) T_n^{-1}\eta
$$

(69)

In order to satisfy the minimum phase requirement, the zero dynamics (69) has to be asymptotically stable. Thus we assume:

**Assumption 4** There exists $P > 0$ such that

$$
PQ_{22} + Q_{22}^T P + Q_{02}^T Q_{32} < 0
$$

where

$$
Q_{22} \equiv T_n(A - B(K_X + K'_D D^{-1}C)T_n^{-1},
Q_{32} \equiv CPT_n^{-1}
$$

Consequently, we can apply Proposition 1 under Assumptions 3 and 4 in order to obtain $\mathcal{L}$ of the output feedback (52) which solves the $L_2$-gain disturbance attenuation problem for (49)~(51). Accordingly, by setting SPR parameter matrices as $K_S = H_S\mathcal{L}$, $D = D'L$, we obtain the P-SPR-D controller (37),(38) for the system (33)~(35).

Then, the closed-loop system (33)~(35),(37),(38) satisfies the conditions P1 and P2. Also, from the property of high gain output feedback, we can improve the convergence speed by adjusting the high gain $\mathcal{L}$ to some extent.

### 4.3 Determination of Controller Parameters

An important task in our method is to satisfy Assumption 3, that is, to determine $K_P$, $K_D$ and $H_S$ such that the zero dynamics (69), i.e.

$$
\dot{\eta} = T_n\left(A - B(I_r + K_DCB)^{-1}(K_P C + K_DC) - H_S D^{-1}C\right) T_n^{-1}\eta
$$

is asymptotically stable.

Since eigenvalues of zero dynamics (70) are the same with those of the matrix

$$
A - B(I_r + K_DCB)^{-1}(K_P C + K_DC - H_S D^{-1}C)
$$

we consider a method of finding $K_P, K_D, H_S$ such that this matrix be asymptotically stable.

We first change the matrix of (71) as follows.

$$
A - B(I_r + K_DCB)^{-1}(K_P C + K_DC - H_S D^{-1}C) = A - B(I_r + K_DCB)^{-1} [K_P - H_S D^{-1} K_D] C\frac{CA}{CA}
$$

(72)

By defining the following matrices

$$
F_{01} = (I_r + K_DCB)^{-1}(K_P - H_S D^{-1}K_D)
$$

(73)

$$
F_{02} = (I_r + K_DCB)^{-1}K_D
$$

(74)

$$
C_n = \left[ C\frac{CA}{CA} \right]
$$

(75)

(72) can be expressed as

$$
A - BF_n C_n
$$

(76)

This can be regarded as a matrix of the closed-loop system due to the output feedback $-F_n C_n$ for system $A, B, C_n$.

Accordingly, in order to get matrices $K_P, K_D, H_S$ asymptotically stabilizing the zero dynamics, we can apply any existing method of output feedback stabilization (e.g. Tamura and Shimizu (2006)) to the system $A, B, C_n$.

After determining the output feedback gain $F_n$ stabilizing (76), we can obtain $K_P, K_D, H_S$ stabilizing (71) from the relations (73) and (74). Besides, we can make it by trial and error, checking eigenvalues of matrix (71) directly.

From Proposition 1 the high gain $\mathcal{L}$ can be attained:

$$
\mathcal{L} = (\tilde{C}\tilde{B})^{-1}\left\{ Q_{11} + \frac{1}{2}\left[ Q_{11}^T Q_{31} + \frac{1}{\gamma^2} \tilde{C}\tilde{B}\tilde{C}\tilde{B}^T \tilde{C}\tilde{C}^T \right] + \Gamma \right\}
$$

$$
= -D^{-1}\left\{ - CKB_k D^{-1} + \frac{1}{2}\left( \frac{1}{\gamma^2} C(-BK_W + \overline{B}) \right) \times (BK_W + \overline{B})^T C^T \right\} + \Gamma
$$

$$
= -D^{-1}\left\{ - C(I_r + K_DCB)^{-1} H_S D^{-1} C \right\}
$$

$$
+ \frac{1}{2}\left( \frac{1}{\gamma^2} C(-B(I_r + K_DCB)^{-1}K_D\overline{C} + \overline{B}) \right)
$$

$$
= \left( C\frac{CA}{CA} \right)
$$

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\( x = (B(I_r + K_DCB)^{-1}K_DCB + B)^TCT + \Gamma \) \hspace{1cm} (77)

A design procedure is arranged as below.

[Algorithm 1]

[Step 1] Give a nonsingular matrix \( D' \in \mathbb{R}^{m \times m} \).

[Step 2] Determine gain matrices \( K_P, H_S, K_D \in \mathbb{R}^{p \times m} \) which make the zero dynamics matrix (71) be asymptotically stable and satisfy Assumption 4.

[Step 3] Choose \( L \) as (77), based on Proposition 1, and determine \( K_S = H_SL \) and \( D = D'L \).

[Step 4] The P-SPR-D controller is given by (37),(38).

As a practical design policy, it may be effective to decide \( \Gamma \) i.e. \( L \) subjectively, observing the output responses concretely, since \( D, K_P, K_S, K_D \) can be computed immediately given \( L \) in the proposed method.

4.4 P-SPR-D+I Coontrol

The P-SPR-D control certainly satisfies condition P1 (i.e. the closed-loop system converges to the origin \( x = 0 \) when \( w = 0 \)). But an off-set arises when \( w \neq 0 \). Hence we necessitate to add I action to P-SPR-D control to get rid of it. Moreover, in order to cope with servo problem, the I action is also needed. Our method can handle the set-point servo problem easily by adding the I action.

For that purpose let us assume that among controlled output variables \( z \), variables \( z_1 \) to track the desired value \( z_1^* \) (i.e. \( z_1 \subseteq z \)) and can be measured to use for integral control. In such a case consider the following P-SPR-D+I control adding the integral action.

\[
\begin{equation}
\dot{\zeta}(t) = D(\zeta(t) - y(t)), \quad \zeta(0) = 0, \quad D < 0
\end{equation}
\]

\[
\begin{equation}
u(t) = -K_P y(t) + K_S \zeta(t) - K_D \dot{y}(t) + K_I \int_0^t (z_1^* - z_1(\tau))d\tau
\end{equation}
\]

Since stability and \( L_2 \)-gain disturbance attenuation are guaranteed sufficiently by the high gain feedback, we devise here only a countermove for a steady state error.

Incidentally, in a special case where the controlled output \( z \) equals to the measurable output \( y \), i.e., \( y \equiv z \), the prososed methods can be simplified to a great extent.

Meanwhile, when \( w = 0 \), at an equilibrium state \( x_e \) holding the controlled output \( z \) at \( z^* \) must satisfy the following relation:

\[
0 = Ax_e + B\overline{\pi}, \quad z^* = C_P x_e
\]

Since this relation consists of \((n + m')\) equations and \((n + r)\) variables, when \( r \geq m' \), \((r - m')\) state variables \( x_{eN} \) can be set arbitrary value \( x_{eN}^* \), but the remained state variables \( x_{EB} \) and \( \overline{\pi} \) are determined dependently.

Putting such an equilibrium as \( x^* = \left[ \begin{array}{c} x_{eN}^* \\ x_{EB}(x_{eN}^*, z^*) \end{array} \right] \) and \( u^* = \overline{\pi}(x_{eN}^*, z^*) \), we have

\[
0 = Ax^* + Bu^*, \quad z^* = C_P x^*
\]

Then, the measurable output \( y \) should be \( y^* = Cx^* \). Thus we can consider the following P-SPR-D+I control:

5. SIMULATION RESULTS

Let us consider a position control problem of 3-inertia system (Hara et al. (1995)). Letting \( \theta_i, i = 1, 2, 3 \), be the rotating angle of each rotating mass and \( \tau \) the torque, \( \tau_{di}, i = 1, 2, 3 \), the disturbance torque, an equation of motion of the system is represented as

\[
J_i \ddot{\theta}_i = d_i \dot{\theta}_i - k_a(\theta_1 - \theta_2) - d_a(\dot{\theta}_1 - \dot{\theta}_2) + \tau + \tau_{di}
\]

\[
J_2 \ddot{\theta}_2 = k_a(\theta_1 - \theta_2) + d_a(\dot{\theta}_1 - \dot{\theta}_2) - d_2 \dot{\theta}_2 - k_b(\theta_2 - \theta_3) - d_b(\dot{\theta}_2 - \dot{\theta}_3) + \tau_{d2}
\]
\[ J_2 \ddot{\theta}_3 = k_b (\theta_2 - \theta_3) + d_b (\dot{\theta}_2 - \dot{\theta}_3) - d_3 \dot{\theta}_3 + \tau_d \]

where \( J_i, i = 1,2,3 \), denotes the inertia moment, \( d_i, i = a,b \), the coefficient of viscosity friction and \( k_i, i = a,b \), the spring constant. Let \( x = (\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T \), \( u = \tau \) and \( w = (\tau_d1, \tau_d2, \tau_d3)^T \), then a state space representation of the system is given as

\[
\dot{x} = Ax + Bu + Bw
\]

\[
y = Cx
\]

\[
z = C_P x
\]

where

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-ka & ka & 1 & 0 & -d_1 - d_a & da \\
ka & -ka - kb & kb & J_1 & -da - d_1 - d_b & db \\
J_2 & J_2 & J_2 & J_2 & J_2 & J_2 \\
0 & J_3 & J_3 & J_3 & J_3 & J_3
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0/1J_1 \\
0 \\
0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1/J_1 & 0 & 0 \\
0 & 1/J_2 & 0 \\
0 & 0 & 1/J_3
\end{bmatrix}
\]

\[
C = [1 \ 0 \ 0 \ 0 \ 0]
\]

\[
C_P = [0 \ 0 \ 1 \ 0 \ 0 \ 0]
\]

where plant parameters are given as \( J_1 = J_2 = 0.001, J_3 = 0.002, d_1 = 0.05, d_2 = 0.001, d_3 = 0.007, d_a = d_b = 0.001, k_a = 920, k_b = 80. \)

Using Algorithm 1, we can solve \( L_2 \)-gain disturbance attenuation problem. We set \( \gamma = 1 \). We first determine \( K_f, K_D, H_S \) asymptotically stabilizing the zero dynamics (71) and then set control parameters of P-SPR-D+I control (78),(79) as follows.

\[
D' = -1, \quad H_S = 1, \quad K_P = 4, \quad K_D = 1, \quad K_I = 10
\]

\[
\Gamma = 5, \quad \mathcal{L} = 5.5, \quad K_S = 5.5
\]

The simulation results of set-point servo problem for \( z^* = 1 \) and \( x(0) = 0, w = 0 \) is shown in Fig.1. The simulation results against a step disturbance \( w = (\tau_d1, \tau_d2, \tau_d3)^T = (1 \ 0 \ 0)^T \) is shown in Fig.2, when \( z^* = 0 \) and \( x(0) = 0 \). Next, we consider P-SPR-D+I control (80),(81), setting controller parameters as follows.

\[
D' = -1, \quad H_S = 1, \quad K_P = 10, \quad K_D = 0.5, \quad K_I = 10
\]

\[
\Gamma = 5, \quad \mathcal{L} = 5.5, \quad K_S = 5.5
\]

The simulation results of set-point servo problem for \( z^* = 1 \) and \( x(0) = 0, w = 0 \) is shown in Fig.3. In this case we set \( y^* = 1 \) corresponding to \( z^* = 1 \). The simulation results against a step disturbance \( w = (1 \ 0 \ 0)^T \) is shown in Fig.4, when we set \( z^* = 0, y^* = 0 \) and \( x(0) = 0 \). We see that convergence speed for the set-point servo problem got improved very much compared to (78),(79), though step disturbance response became a little slower.

It can be said that satisfactory control performances were obtained for both cases. Note also that even if gains \( \mathcal{K}_P, K_S, K_D, K_I \) change around the above values to some degree, the closed-loop system maintains to be asymptotically stable.

### 6. CONCLUDING REMARKS

We investigated the P-SPR-D and/or P-SPR-D+I control for the \( L_2 \)-gain disturbance attenuation problem, based on the idea that one makes a system have relative degree \( \{1, \ldots, 1\} \) and be minimum phase by use of the SPR element such that high gain output feedback can be applied. The use of SPR element contributes powerfully to stabilizing the closed-loop system. The P-SPR-D or P-SPR-D+I control is very practical because of its simple structure. Compared to \( H_S \) control, our method can handle the set-point servo problem easily by using I action. Besides, we do not have to solve the Riccati inequality or LMI for designing the controller.

Although we studied only to stabilize the system and to attain \( L_2 \)-gain below \( \gamma \) against disturbances, how to determine \( K_f, K_S, K_D, D \) giving better control performance is considered a future topic.

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