Global Stabilization Control of Acrobot Based on Equivalent-Input-Disturbance Approach

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Abstract: This paper concerns the global stabilization control of an underactuated two-link acrobot in a vertical plane using a new control method based on an equivalent-input-disturbance (EID) approach. The design procedure consists of two steps: (1) A homeomorphous coordinate transformation transforms the acrobot system into a new nonlinear system. (2) The new system is divided into linear and nonlinear parts, and the nonlinear part is taken to be an artificial disturbance. Then, the EID-based approach is used to globally and asymptotically stabilize the system at the origin. This method enables the acrobot to be swung up from any initial position and balanced at the straight-up position. Unlike the most commonly used switching control method, ours features a single controller for both swing-up and balancing control. Simulation results demonstrate the validity of the method.

Keywords: Acrobot, Global stabilization, Underactuated system, Equivalent-input-disturbance (EID).

1. INTRODUCTION

An underactuated mechanical system is a system that has fewer actuators than configuration variables. It is more energy efficient, lighter, and more compact than a fully actuated one. It is also useful for describing a fully actuated system in the event that one of the actuators fails. For these reasons, many researchers are studying underactuated systems, see Jain et al. (1993), Rosas-Flores et al. (2000), Olfati-Saber (2002) and Grizzle et al. (2005). However, the complexity of the internal dynamics and the nonholonomic constraints imposed on such systems make their control a challenging problem.

An acrobot is a two-link manipulator moving in a vertical plane (Fig. 1). It is a typical example of an underactuated robotic system. It has an actuator at the second joint, but not the first. It is a very simple model of a gymnast on a high bar: The first and second joints represent the gymnast’s hands and hips, respectively. An acrobot exhibits strong nonlinearities throughout the whole motion space. Although gravity makes it small-time local controllable (STLC), see Mullhaupt et al. (2002), it still is not full-state feedback linearizable (FL). In addition, there is a second-order nonholonomic constraint imposed on the robotic system. This makes its motion control very difficult.

A common control objective is to swing the acrobot up from the straight-down equilibrium position and balance it at the straight-up equilibrium position. The most common strategy for doing that involves dividing the motion space into two subspaces (the attractive area around the straight-up position, and the swing-up area, which is the rest); and designing a controller for each subspace. Switching from one controller to the other accomplishes the control objective. Many methods have been developed for both swing-up control in the swing-up area [e.g., a partial feedback linearization method in Spong (1995), Henmi et al. (2006); an energy-based method in Xin et al. (2007), Lai et al. (2009a); and an intelligent method in Brown et al. (1997), Lai et al. (2009b), etc.] and balancing control in the attractive area [e.g., a linear quadratic regulator (LQR) in Spong (1995), approximate linearization in Hauser et al. (1990), and a Takagi-Sugeno fuzzy model in Lai et al. (1999), etc.].

Although a switching strategy may be effective, it does not guarantee the global stability of the control system. So recently, attempts have been made to use a single controller for the motion control of the acrobot. For instance, Willson et al. (2009) presented a quotient method, which is local-control method. It is applied to the region around the straight-up position, and a suitable choice of control gains enables the attractive area to be expanded to cover the whole motion space. This allows the acrobot to be swung up from any initial position. However, its validity was just illustrated by simulations and has not been proven theoretically.

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This paper presents a new control method for an acrobot. It uses a single controller to globally stabilize the acrobot. It is based on the concept of an equivalent-input-disturbance (EID), which was first presented in She et al. (2008) to deal with disturbance rejection in a linear servo system. The validity of the EID approach has been demonstrated for many linear mechatronic systems, see e.g., She et al. (2007, 2008, 2010). In this study, we extended the idea of EID to the stabilization control of a nonlinear acrobot system. The design procedure has two steps:

**Step 1:** Carry out a homeomorphous coordinate transformation on the acrobot system that transforms it into a new simplified nonlinear system.

**Step 2:** Divide the new system into linear and nonlinear parts and take the nonlinear part to be an artificial disturbance. Then, apply the EID-based approach to globally stabilize the new system at the origin.

Since the coordinate transformation is homeomorphous, the acrobot is stabilized globally and asymptotically.

### 2. DYNAMICS OF ACROBOT

The parameters and variables of the model of an acrobot (Fig. 1) are \((i = 1, 2)\):

- \(q_1\): angle of first link relative to the vertical;
- \(q_2\): angle of second link relative to the first link;
- \(m_i\): mass of \(i\)-th link;
- \(L_i\): length of \(i\)-th link;
- \(L_{ci}\): distance from \(i\)-th joint to the center of mass (COM) of the \(i\)-th link;
- \(J_i\): moment of inertia around the COM of the \(i\)-th link;
- \(\tau_2\): torque applied to the second joint;
- \(g\): gravitational acceleration (= 9.806 m/s²).

Let

\[
q(t) := [q_1(t), q_2(t)]^T, \\
\dot{q}(t) := dq(t)/dt, \\
\ddot{q}(t) := d\dot{q}(t)/dt,
\]

and

\[
\zeta(t) := [q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t)]^T.
\]

The kinetic energy, \(K(\zeta(t))\), and the potential energy, \(P(\zeta(t))\), of the acrobot are

\[
K(\zeta(t)) = \frac{1}{2}q^T(t)M(q(t))\dot{q}(t), \quad (1)
\]

and

\[
P(\zeta(t)) = b_1 \cos q_1(t) + b_2 \cos[q_1(t) + q_2(t)], \quad (2)
\]

where

\[
M(q(t)) = \begin{bmatrix}
m_{11}(q_2(t)) & m_{12}(q_2(t)) \\
m_{21}(q_2(t)) & m_{22}
\end{bmatrix} =
\begin{bmatrix}
a_1 + a_2 + 2a_3 \cos q_2(t) & a_2 + a_3 \cos q_2(t) \\
a_2 + a_3 \cos q_2(t) & a_2
\end{bmatrix}
\]

is the symmetric positive-definite inertia matrix, and

\[
a_1 := m_1L_{c1}^2 + m_2L_1^2 + J_1, \\
a_2 := m_2L_{c2}^2 + J_2, \\
a_3 := m_2L_1L_{c2}, \\
b_1 := (m_1L_{c1} + m_2L_1)g, \\
b_2 := m_2L_{c2}g.
\]

We assume that there is no friction at the joints of the acrobot and take \(L(\zeta(t)) = K(\zeta(t)) - P(\zeta(t))\) to be a Lagrangian of the system. Then, the dynamic equation of the robot is obtained from the Euler-Lagrange equation:

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L(\zeta(t))}{\partial \dot{q}_i} \right) - \frac{\partial L(\zeta(t))}{\partial q_i} &= 0, \\
\frac{d}{dt} \left( \frac{\partial L(\zeta(t))}{\partial \dot{q}_2} \right) - \frac{\partial L(\zeta(t))}{\partial q_2} &= \tau_2(t).
\end{align*}
\]

A simple calculation yields

\[
M(q(t))\ddot{q}(t) + H(q(t), \dot{q}(t)) + G(q(t)) = \begin{bmatrix} 0 \\ \tau_2(t) \end{bmatrix},
\]

where

\[
H(q(t), \dot{q}(t)) = \begin{bmatrix} H_1(q_1(t), \dot{q}_1(t)) \\ H_2(q_2(t), \dot{q}_2(t)) \end{bmatrix} =
\begin{bmatrix}
-a_3(2q_1(t) + \dot{q}_1(t)q_2(t)) \sin q_2(t) \\
a_3q_1(t) \sin q_2(t)
\end{bmatrix},
\]

\[
G(q(t)) = \begin{bmatrix} G_1(q(t)) \\ G_2(q(t)) \end{bmatrix} =
\begin{bmatrix}
-b_1 \sin q_1(t) - b_2 \sin(q_1(t) + q_2(t)) \\
-b_2 \sin(q_1(t) + q_2(t))
\end{bmatrix}.
\]

It is easy to verify that

\[
G(q(t)) = \frac{\partial P(\zeta(t))}{\partial q},
\]

and

\[
\zeta(t) = [0, 0, 0, 0]^T, \quad (7)
\]

and

\[
\zeta(t) = [\pi, 0, 0, 0]^T, \quad (8)
\]

are two open-loop equilibrium points of the system (5). The point (7) is the straight-up position, and (8) is the straight-down position.
Our control objective is to globally and asymptotically stabilize the system (5) at (7). To achieve this, we first carry out a global coordinate transformation on (5):

\[
T: \begin{align*}
    z_1(t) &= q_1(t) + \alpha(q_2(t)), \\
    z_2(t) &= \frac{\partial L}{\partial q_1} = m_{11}(q_2(t))q_1(t) + m_{12}(q_2(t))q_2(t), \\
    z_3(t) &= q_2(t), \\
    z_4(t) &= u(t),
\end{align*}
\]

where

\[
    \alpha(q_2(t)) = \int_{0}^{q_2(t)} \frac{m_{12}(s)}{m_{11}(s)} ds
    = \frac{q_2(t)}{2} + \frac{a_2 - a_1}{\sqrt{(a_1 + a_2)^2 - 4a_3^2}} \arctan\left(\frac{a_1 + a_2 - 2a_3}{a_1 + a_2 + 2a_3} \frac{q_2(t)}{2}\right).
\]

Let \( z(t) = [z_1(t), z_2(t), z_3(t), z_4(t)]^T \). Combining (4), (5), (6), and (9) yields a new dynamic equation of the acrobot in the z-state:

\[
\begin{align*}
    \dot{z}_1(t) &= f_1(z_2(t), z_3(t)), \\
    \dot{z}_2(t) &= f_2(z_1(t), z_3(t)), \\
    \dot{z}_3(t) &= z_4(t), \\
    \dot{z}_4(t) &= u(t),
\end{align*}
\]

where

\[
    f_1(z_2(t), z_3(t)) = \frac{z_2(t)}{a_1 + a_2 + 2a_3 \cos z_3(t)},
\]

\[
    f_2(z_1(t), z_3(t)) = b_1 \sin(z_1(t) - \alpha(z_3(t))) + b_2 \sin(z_1(t) - \alpha(z_3(t)) + z_3(t)).
\]

\( u(t) \) is the input of the new system, and

\[
\tau(t) = \frac{\dot{Y}(t)u(t) + m_{11}(H_2 + G_2) - m_{21}(H_1 + G_1)}{m_{11}},
\]

\[
\dot{Y}(t) = \det(M(q(t))) = m_{11}m_{22} - m_{12}m_{21} > 0.
\]

Note that, for simplicity, we omit the variables and abbreviate, for example, \( H_2(q(t), \dot{q}(t)) \) to \( H_2 \) in (11).

Since the transformation Jacobian matrix

\[
\frac{\partial z}{\partial\zeta} = \begin{bmatrix}
1 & m_{12}(q_2) & 0 & 0 \\
0 & m_{11}(q_2) & 0 & 0 \\
0 & -a_3(2\dot{q}_1 + \dot{q}_2) & m_{12}(q_2) & m_{11}(q_2) \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

is nonsingular, \( T \) is a homeomorphic coordinate transformation. As a result, asymptotically stabilizing the system (5) at (7) is equivalent to asymptotically stabilizing the system (10) at \( z(t) = 0 \). In other words, global stabilization control of the acrobot can be achieved by finding a control law, \( u(t) \), that asymptotically and globally stabilizes (10) at the origin. In the next section, we use an EID-based method to solve this problem.

Remark 1. The system (10) is a standard strict-feedback-cascade nonlinear system. For this kind of system, one may tend to choose an integrator backstepping (IBS) scheme (see Khalil (2002)) to solve the stabilization control problem. However, due to the complicated nonlinearities of \( f_1(z_2(t), z_3(t)) \) and \( f_2(z_1(t), z_3(t)) \), it is not easy to find an explicit expression for \( z_3(t) \) that asymptotically stabilizes the subsystem \( (z_1(t), z_2(t)) \) at the origin. This is an obstacle to the use of the IBS method to control an acrobot.

### 3. EID-BASED CONTROLLER DESIGN

Note that the first-order approximation of \( f_1(z_2(t), z_3(t)) \) and \( f_2(z_1(t), z_3(t)) \) at \( z(t) = 0 \) is

\[
    f_1(z_2(t), z_3(t)) \approx \mu_1 z_2(t), \quad f_2(z_1(t), z_3(t)) \approx \mu_2 z_1(t) + \mu_3 z_3(t),
\]

where

\[
    \mu_1 = \frac{1}{a_1 + a_2 + 2a_3}, \quad \mu_2 = b_1 + b_2, \quad \mu_3 = \frac{b_2(a_1 + a_3) - b_1(a_2 + a_3)}{a_1 + a_2 + 2a_3}.
\]

We divide the right side of (10) into linear and nonlinear parts, choose \([z_1(t), z_2(t), z_3(t)]^T\) as the output of the system, and rewrite it as

\[
\begin{align*}
    \dot{z}(t) &= Az(t) + Bu(t) + \varepsilon(t), \\
    y(t) &= Cz(t),
\end{align*}
\]

\[
A = \begin{bmatrix}
0 & \mu_1 & 0 & 0 \\
0 & 0 & \mu_3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

\[
\varepsilon(t) := \Phi(z(t)) = \begin{bmatrix}
    f_1(z(t)) - \mu_1 z_2(t) \\
    f_2(z(t)) - \mu_2 z_1(t) - \mu_3 z_3(t) \\
    0
\end{bmatrix}. \tag{19}
\]

It is not difficult to validate that \((A, B)\) is controllable, \((C, A)\) is observable, and that \((A, B, C)\) has no zeros on the imaginary axis. The remainder of this section solves the problem of globally stabilizing the system (16) based on the concept of an EID.

First, we take the nonlinear part \( \varepsilon(t) \) to be an artificial state-dependent disturbance of the system (16). As proven in She et al. (2008), the perturbed plant (16) can be considered to be a plant with an EID, \( d_e(t) \), on the control input channel; and we can write the plant as

\[
\begin{align*}
    \dot{z}(t) &= Az(t) + B[u(t) + d_e(t)], \\
    y(t) &= Cz(t),
\end{align*}
\]

Here, we use the same notation, \( z(t) \), for the state of the plant in both (16) and (20). This should not cause confusion.

Then, an EID-based control system is constructed (Fig. 2). It contains three parts in addition to the plant: a disturbance estimator, a state feedback controller, and a state observer. The design of each is described below.
Fig. 2. Configuration of EID-based control system.

3.1 Design of Disturbance Estimator and State Feedback

We define \( u_f(t) = u(t) + d_e(t) \) in (20) and construct a Luenberger observer:

\[
\dot{\hat{z}}(t) = A\hat{z}(t) + Bu_f(t) + L[y(t) - C\hat{z}],
\]

(21)

where \( L \in \mathbb{R}^{4 \times 3} \) is an observer gain that stabilizes \( A - LC \).

The design of the gain is explained in the next subsection. Let

\[
\Delta z(t) = \hat{z}(t) - z(t).
\]

(22)

Combining it with (20) yields

\[
\dot{\hat{z}}(t) = A\hat{z}(t) + Bu(t) + Bd_e(t) + [\Delta \hat{z}(t) - A\Delta z(t)].
\]

(23)

We assume that there exists a control input, \( \Delta d_e(t) \), that satisfies

\[
\Delta \dot{z}(t) - A\Delta z(t) = B\Delta d_e(t).
\]

(24)

Substituting (24) into (23) and denoting

\[
\hat{d}_e(t) = d_e(t) + \Delta d_e(t),
\]

(25)

yield

\[
\dot{\hat{z}}(t) = A\hat{z}(t) + B[u(t) + \hat{d}_e(t)].
\]

(26)

From (21) and (26), we have

\[
B[u(t) + \hat{d}_e(t) - u_f(t)] = L[y(t) - C\hat{z}(t)].
\]

(27)

Solving the above equation yields a least-squares solution for \( \hat{d}_e(t) \):

\[
\hat{d}_e(t) = B^T L[y(t) - C\hat{z}(t)] + u_f(t) - u(t).
\]

(28)

We use \( \hat{d}_e(t) \) as an estimate of the actual EID \( d_e(t) \). Since the observer (21) is stable, from (24) and (25) we know that the estimated EID, \( \hat{d}_e(t) \), asymptotically converges to the actual EID, \( d_e(t) \), as \( t \to \infty \). To guarantee the estimation accuracy, we use a low-pass filter, \( F(s) \), to select the frequency band for estimation. This gives us the filtered disturbance estimate, \( \tilde{d}_e(t) \).

On the other hand, we choose the state feedback control law to be

\[
u_f(t) = K_F \hat{z}(t), \quad K_F = -R^{-1}B^TP,
\]

(29)

where \( P = P^T > 0 \) is the solution of the Riccati equation

\[
Q + PA + A^TP - PBR^{-1}B^TP = 0,
\]

(30)

where \( Q \) and \( R \) are positive-definite weighting matrices. Since \( (A, B) \) is controllable and \( A - LC \) is stable, the optimal control law (29) asymptotically stabilizes the plant (20) at \( z(t) = 0 \). According to the definition of EID, the control law

\[
u(t) = u_f(t) - \hat{d}_e(t)
\]

(31)

makes the output of the system (16) asymptotically converge to the origin. Since \( y(t) = 0 \) is equivalent to \( z(t) = 0 \) in (16), the problem of globally stabilizing the system (16) is solved.

3.2 Design of State Observer

In this study, we chose the following first-order low pass filter for \( \hat{d}_e(t) \):

\[
F(s) = \frac{\gamma}{Ts + 1},
\]

(32)

where \( T \) and \( \gamma \) are constants. The discussion below explains how to design the observer gain, \( L \), so that it does not destroy the stability of the whole control system.

First, since \((C, A)\) is observable, \((A^T, C^T)\) is controllable. So, the Riccati equation

\[
AS + SA^T - SC^T R_L^{-1}CS + \rho Q_L = 0,
\]

(33)

has a positive-definite solution, \( S \), where \( Q_L \geq 0 \) and \( R_L > 0 \) are given weighting matrices and \( \rho > 0 \) is a scalar. So, the optimal feedback gain

\[
L_\rho = R_L^{-1}CS,
\]

(34)

stabilizes \( A^T - C^TL_\rho \). As a result, the observer gain \( L = L_\rho \) ensures the stability of the state observer (21).
In addition, combining (21), (23), and (31) yields
\[\Delta\dot{z}(t) = (A - LC)\Delta z(t) + B\tilde{d}e(t) - Bde(t). \tag{35}\]
From (22), (28), ... (45)
show that the controller (11) and (31) quickly swings the
acrobot up from the straight-down position and asymptot-

tical stability is achieved.

To consider the stability of the system, we let \(d_e(t) = 0\).
Then the transfer function from \(d_e(t)\) to \(\tilde{d}e(t)\) is obtained based
on (35) and (36):
\[G_L(s) = 1 - B^TLC[sI - (A - LC)]^{-1}B = B^T(sI - A)[sI - (A - LC)]^{-1}B. \tag{37}\]
Separation theory tells us that, to guarantee the stability of the whole system, we have to make \(A + K_FB\) stable and also ensure that the condition (Zhou et al. (1996))
\[\|G_LF\|_\infty < 1 \tag{38}\]
holds. Here, \(\|G_LF\|_\infty := \sup_{\omega \leq \infty} \sigma_{\text{max}}[G_L(j\omega)F(j\omega)]\) and \(\sigma_{\text{max}}()\) is the maximum singular-value function.

Equations (29) and (30) ensure that \(A + K_FB\) is stable. So
next, for a given filter, \(F(s)\) in (32), we just have to choose a proper \(L\) such that the condition (38) is true. Consider the dual system of the plant \((A, B, C)\):
\[
\begin{align*}
\dot{z}_d(t) &= A^Tz_d(t) + C^Tu_d(t), \\
\dot{y}_d(t) &= B^Tz_d(t).
\end{align*} \tag{39}\]
Since the number of inputs of (39) is larger than the number of outputs and since the matrix \(A^T - C^TL_\rho\) is stable, the two conditions in Theorem 3 in Kimura (1981) are satisfied. According to Theorem 1 in Kimura (1981), it is true that
\[\lim_{\rho \to \infty} B^T[sI - (A^T - C^T L_\rho)]^{-1} = 0. \tag{40}\]
The left side of (40) is the transpose of \(sI - (A - LC)^{-1}B\), which is part of \(G_L(s)\). Hence, (40) means that the gain \(L\), which is obtained from (33) for a large enough \(\rho\), makes the condition (38) true.

4. SIMULATIONS

The numerical example in this section demonstrates the
validity of the method described above. The parameters of
the acrobot in Spong (1995) were used in the simulations
(Table 1); and the parameters in (30), (32), and (33) were
chosen to be
\[
Q = 5I_4, \quad R = T = 0.001, \quad \gamma = 0.8, \quad Q_L = 0.005I_4, \quad R_L = 5000I_3, \quad \rho = 10^5, \tag{41}\]
where \(I_n\) stands for an \(n \times n\) identity matrix.

We used the MATLAB function \texttt{lqr} to calculate optimal
control gains, and obtained
\[K_F = [4872.4, 461.26, -414.61, -76.349], \tag{42}\]

Table 1. Parameters of acrobot used in simulations.

<table>
<thead>
<tr>
<th>Link</th>
<th>(m_i) [kg]</th>
<th>(L_i) [m]</th>
<th>(L_{ci}) [m]</th>
<th>(J_i) [kg \cdot m^2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i = 1)</td>
<td>1.00</td>
<td>1.00</td>
<td>0.50</td>
<td>8.33 \times 10^{-2}</td>
</tr>
<tr>
<td>(i = 2)</td>
<td>1.00</td>
<td>2.00</td>
<td>1.00</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Fig. 3. Time response of \(q_1(t), q_1(t) \ (i = 1, 2)\) and \(q_2(t)\).

\[
L = \begin{bmatrix}
0.3258 & 1.1774 & 0.1120 & 0.0525 \\
1.1774 & 7.6267 & -0.1825 & -0.1328 \\
0.1120 & -0.1825 & 1.7128 & 0.9898 \\
0.1120 & 0.1328 & 1.7128 & 0.9898
\end{bmatrix}. \tag{43}\]

The function \texttt{norm} in MATLAB gives
\[\|G_LF\|_\infty = 0.95284 < 1. \tag{44}\]
So, the condition (38) is satisfied, and the control system is
stable.

The simulation results (Fig. 3) for the initial condition
\[\begin{bmatrix} q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t) \end{bmatrix}^T = [\pi, 0, 0, 0]^T \tag{45}\]
show that the controller (11) and (31) quickly swings the
acrobot up from the straight-down position and asymptot-
ically stabilizes it at the straight-up position. The settling time is less than 8 s.

5. CONCLUSION

This paper presents a new control method of globally stabilizing an acrobot. It has two steps:

(1) A coordinate transformation transforms the dynamics of the acrobot into a new nonlinear system.

(2) The new system is divided into linear and nonlinear parts; the nonlinear part is taken to be an artificial disturbance; and the EID approach is used to asymptotically stabilize the new system at the origin.

Since the coordinate transformation is homeomorphous, the problem of globally stabilizing the acrobot is solved. Unlike switching control strategies, which are the ones most commonly used, our method does not switch control laws: It uses a single controller for both swing-up and balancing control. This new method is effective and can be extended to the stabilization control of other types of nonlinear systems.

REFERENCES


