On the Perturbation of Homogeneous LTI Descriptor Differential Systems

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Abstract: In the literature of linear system and mathematical control theory, several different techniques have been developed for the solution of homogeneous LTI descriptor differential systems. In this paper, we apply the Weierstrass canonical form, and we investigate the conditions under which a system with specific structure and desirable properties is constructed. Consequently, we define the family of perturbed pencils such as the solutions of the initial and the relative perturbed system are $r$-close with respect to a Frobenius distance. A step-algorithm is also presented enhancing more the results of the paper.

Keywords: Linear Control Systems; Perturbation Theory; System Theory.

1. INTRODUCTION

In the literature of linear system and mathematical control theory, see for instance Campbell (1980, 1982), Dai (1989), Kalogeropoulos (1985), Grispos (1991) et al., linear internal models are called descriptor (or generalized or differential algebraic) systems, and they have a key role in the modelling and simulation process of constrained dynamical systems in many industrial applications. As a result, such kinds of systems have been intensively studied, theoretically as well as numerically, in the last few decades. Primarily, descriptor systems have been described in terms of first order ordinary nonlinear equations and they are the standard state-space descriptions of the implicit type

\[
F(x, \dot{x}) = 0, \tag{1.1}
\]

where $x$ is the vector of all internal model variables. Obviously, in the linear case, the (1.1) system can be reduced to the matrix pencil model, which is defined by $F\dot{x} = Gx$. When the inputs $u$, and the outputs $y$ have been defined, then the nonlinear control model is given by

\[
F(x, \dot{x}, u) = 0, \quad \dot{y} = G(x, \dot{x}, u), \tag{1.2}
\]

and in the linear case is expressed by the singular model

\[
F\dot{x} = Gx + Bu, \quad \dot{y} = Cx. \tag{1.3}
\]

Several different approaches have been developed for the solution of descriptor systems. However, only recently, the perturbation theory for those systems has been studied; see for instance Nye (1985), Karageorgos, Pantelous and Kalogeropoulos (2010) et al. As a consequence of our previous work, in the present paper, we want to investigate the conditions under which an unstable (i.e. with positive finite elementary divisors) or with some other properties, matrix pencil is close (in the sense of the Frobenius distance) to one which is stable (i.e. with negative finite elementary divisors) or with some more desirable properties, respectively; see for instance the ideas presented in the paper proposed by Kalogeropoulos, Karageorgos and Pantelous (2008). Thus, if we assume that $(F, G)$ is the matrix pencil of the initial system, we want to define the desirable family of perturbed matrix pencils $(F + \Delta F, G + \Delta G)$ such as the solutions of the initial and the perturbed systems are $r$-close (where $r$ is a specific radius, see next section).

2. PRELIMINARY RESULTS

In this section, some basic tools for what it follows and the fundamental notion of matrix pencil theory are briefly presented. Thus, here, we should define the Frobenius distance of matrices $A, B \in \mathcal{M}(n \times m; F)$ such as

\[
\rho(A, B) = \rho\left([A_1, A_2], [B_1, B_2]\right) = \rho(A_1 - A_2, B_1 - B_2) = \|A_1 - A_2\|^2 + \|B_1 - B_2\|^2.
\]

We consider the homogeneous linear time-invariant (LTI) descriptor (or generalized state space) differential system

\[
F\dot{x}(t) = Gx(t), \quad \text{with} \quad \dot{x}(t_0) = x_0 \neq 0, \tag{2.1}
\]

where the state vector $x \in \mathbb{C}^n(F, \mathcal{M}(n \times 1; F))$ (and $\mathcal{M}$ is an algebra of matrices/vectors; $F = \mathbb{R}$ or $\mathbb{C}$, i.e. $F$ is the field of real or complex numbers) is the state vector, and $F, G \in \mathcal{M}(n \times n; F)$ are square matrices, with $\det F = 0$. For the sake of simplicity we set in the sequel $\mathcal{M}_n = \mathcal{M}(n \times n; F)$ and $\mathcal{M}_{n \times m} = \mathcal{M}(n \times m; F)$. Now, let $B_1, B_2, ..., B_r$ be elements of $\mathcal{M}_n$. The direct sum of them denoted by $B_1 \oplus B_2 \oplus ... \oplus B_r$ is...
the block diag \( \{B_1, B_2, \ldots, B_r\} \).

Then, the complex Weierstrass canonical form \( sF_n - Q_n \) of the regular pencil \( sF - G \) is defined by \( sF_n - Q_n \triangleq s\mathbf{I}_p - J_n \oplus sH_n - I_q \), see Gantmacher (1959) where the first normal Jordan type element is uniquely defined by the set of finite elementary divisors (f.e.d.) \( (s-a_j)^{p_j}, \ldots, (s-a_1)^{p_1}, \sum_{j=1}^{r} p_j = p \) of \( sF - G \) and has the form

\[
\begin{bmatrix}
I_p & -J_p & \cdots & 0 \\
0 & I_p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_p
\end{bmatrix}
\oplus
\begin{bmatrix}
J_p(a_j) \\
\vdots \\
J_p(a_1)
\end{bmatrix}
\in \mathcal{M}_p.
\]

And also, the \( q \) blocks of the second uniquely defined block \( sH_n - I_q \) correspond to the infinite elementary divisors (i.e.d.). \( \mathbb{S}_1, \ldots, \mathbb{S}_m \), \( \sum_{j=1}^{p} q_j = q \) of \( sF - G \) and has the form

\[
sH_n - I_q \triangleq sH_n - I_q \oplus \cdots \oplus sH_n - I_q.
\]

Thus the \( H_q \) is a nilpotent matrix of \( \mathcal{M}_q \) with index \( q^* = \max\{q_j : j = 1, 2, \ldots, \sigma\} \), where \( H_q^q = \mathbb{O} \), and \( J_{p_j}(a_j) \), \( H_{q_j} \) are defined as

\[
J_{p_j}(a_j) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\in \mathcal{M}_{p_j},
\]

and, for instance, we can have

\[
H_{q_j} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\in \mathcal{M}_{q_j},
\]

Now, for the solution of system (2.1) with consistent initial conditions, we can use the Weierstrass canonical form, i.e.

\[
PFQ = I_p \oplus H_q \quad \text{and} \quad PGQ = J_p(a) \oplus I_q,
\]

where \( P, Q \in \mathcal{M}_p \) are non-singular matrices.

Consequently, the following well-known lemma is derived, see for instance Campbell (1980, 1982) and Dai (1979).

**Lemma 2.1** The analytic solution of system (2.1) is given by

\[
\begin{align*}
\tilde{y}_p(t) &= e^{\mathbb{J}_p(c)(t-t_1) - \mathbb{J}_p(t_1)} y_p(t_1), \\
y_p(t) &= 0,
\end{align*}
\]

where the Jordan super \( p = \sum_{j=1}^{r} p_j \)-block matrix \( J_p(a) \triangleq \bigoplus_{j=1}^{r} J_p(a_j) \in \mathcal{M}_p \) with eigenvalues (f.e.d.) \( a_j \in \mathbb{F} \) and \( y_p(t) \)

\[
= Q_p e^y(t).
\]

In order to be able to prove the above Lemma, we use the following transformation (2.4)

\[
\tilde{y}(t) = Q^{-1} y(t), \quad \text{with} \quad y(t) = \tilde{y}(t) = Q^{-1} y(t) \neq 0.
\]

Furthermore, as it has been also mentioned in section 1, in this paper, we want to investigate and give a concrete answer to the following significant question “if the matrix pencil is unstable -positive f.e.d.- or we want some other properties, which is the closest pencil (in the sense of the Frobenius metrics) that can give us the desired property?” Consequently, in order to be able to provide an answer to the above problem, it is similar to determine an “equivalent” to system (2.1), i.e. see (2.5). Furthermore, from all the families of the “equivalent” to systems (2.1), we consider the case where the solutions of both (2.1) and (2.5) systems are also \( r \)-close. Then,

\[
F^\mathbb{F}(t) = G^\mathbb{F}(t), \quad \text{with} \quad \mathbb{F}(t) = \mathbb{F}_n(t), \quad (2.5)
\]

where the coefficient matrices \( F, G \in \mathcal{M}_p \) of the system (2.5) are defined as \( F \triangleq F + \Delta F \in \mathcal{M}_p \) and \( G \triangleq G + \Delta G \in \mathcal{M}_p \), and the following conditions should be always satisfied:

(I) \( \det F = \det (F + \Delta F) = 0 \), i.e. to be also (2.5) descriptor.

(II) The pencils \( (F, G) \) and \( (F', G') \) should be \( \varepsilon \)-close (where \( \varepsilon \) is pre-defined error) such as

\[
\rho[(F, G), (F', G')] = \rho(\Delta F, \Delta G) = \|\Delta F\|^2 + \|\Delta G\|^2 < \varepsilon,
\]

(III) The distance for the solutions is given by

\[
\rho\left[\begin{bmatrix}
\tilde{y}_p(t) \\
\tilde{y}_q(t)
\end{bmatrix}, \begin{bmatrix}
y_p(t) \\
y_q(t)
\end{bmatrix}\right] < \rho = O(\varepsilon(t)),
\]

where \( \tilde{y}_p(t) \triangleq \begin{bmatrix}
y_p(t) \\
y_q(t)
\end{bmatrix} \) and \( \mathbb{F}(t) \triangleq \begin{bmatrix}
\tilde{y}_p(t) \\
\tilde{y}_q(t)
\end{bmatrix} \) are the Weierstrass solutions of the (2.1) and (2.5) systems respectively, and \( O(\varepsilon(t)) \) is an acceptable predefined upper (error) bound, which is point-wise in \( t \).

**Remark 2.1** The pre-defined upper (error) bound \( O(\varepsilon(t)) \) determines the family of the “equal” systems with the desirable properties, for instance eigenvalues (finite elementary divisors) lay on the left imaginary axis.

Actually, in this paper, we will also determine the family of required perturbations \( \Delta F \) and \( \Delta G \) such as the condition,

\[
\|\Delta F\|^2 + \|\Delta G\|^2 < \varepsilon,
\]

should also be satisfied. The solution of the new system (2.5) is given by Lemma 2.2

**Lemma 2.2** The solution of system (2.5) is given by

\[
\begin{align*}
\tilde{y}_p(t) &= e^{\mathbb{J}_p(c)(t-t_1) - \mathbb{J}_p(t_1)} y_p(t_1), \\
y_p(t) &= 0,
\end{align*}
\]

where the Jordan super \( p = \sum_{j=1}^{r} p_j \)-block matrix \( J'_p(\beta) \triangleq \bigoplus_{j=1}^{r} J'_p(\beta) \in \mathcal{M}_p \) with eigenvalues (f.e.d.) \( a_j \in \mathbb{F} \) and \( y_p(t) \)

\[
= Q_p e^y(t).
\]
Lemma 3.1 The Frobenius distance between the solutions of (2.1) and (2.5) Weierstrass systems is given by

\[ r \triangleq \rho \left[ \begin{array}{c}
\Sigma_p(t) \\
\Sigma_q(t)
\end{array} \right] = \left[ \begin{array}{c}
\Sigma_p(t) - \Sigma_q(t) \\
\Sigma_q(t)
\end{array} \right] = \left[ \begin{array}{c}
\Sigma_p(t) - \Sigma_p(t) \\
\Sigma_q(t) - \Sigma_q(t)
\end{array} \right]. \]

Lemma 3.2 The Frobenius norm

\[ \|e^{J_p(a)(t-t_o)} - e^{J_q(\beta)(t-t_o)}\| = \frac{1}{\rho} \left[ \sum_{j=1}^{v} e^{J_p(j)(t-t_o)} e^{J_q(j)(t-t_o)} \right] \left[ \sum_{i=0}^{p_q-1} (p_j - i) \right]^2. \]

where \( p = \sum_{j=1}^{v} p_j \) and \( a_j, \beta_j \in \mathbb{F}. \)

Proof Since we have

\[ J_p(a) \triangleq \bigoplus_{j=1}^{v} J_{p_j}(a_j) \in \mathcal{M}_p \text{ and } J_q(\beta) \triangleq \bigoplus_{j=1}^{v} J_{p_j}(\beta_j) \in \mathcal{M}_q, \]

then

\[ e^{J_p(j)(t-t_o)} = \bigoplus_{j=1}^{v} e^{J_{p_j}(a_j)(t-t_o)} \in \mathcal{M}_p \]

and

\[ e^{J_q(j)(t-t_o)} = \bigoplus_{j=1}^{v} e^{J_{p_j}(\beta_j)(t-t_o)} \in \mathcal{M}_p. \]

So, we obtain

\[ e^{J_p(a)(t-t_o)} e^{J_q(\beta)(t-t_o)} = e^{J_p(a)(t-t_o)} \bigoplus_{j=1}^{v} e^{J_{p_j}(\beta_j)(t-t_o)} \in \mathcal{M}_p. \]

for every \( j = 1, 2, \ldots, v, \) where we have

\[ M_{p_j} = \left[ \begin{array}{cccc}
1 - t_o & (t-t_o)^2 & \cdots & \left( (t-t_o)^{p_j-1} \right) \\
0 & 1 & \cdots & \left( (t-t_o)^{p_j-2} \right) \\
0 & 0 & 1 & \cdots & \left( (t-t_o)^{p_j-3} \right) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{array} \right], \]

for every \( j = 1, 2, \ldots, v. \) Then, we take the difference

\[ e^{J_p(a)(t-t_o)} e^{J_q(\beta)(t-t_o)} = \bigoplus_{i=1}^{v} e^{J_{p_i}(a_i)(t-t_o)} - e^{J_{p_i}(\beta_i)(t-t_o)} \]

and

\[ \|e^{J_p(a)(t-t_o)} - e^{J_q(\beta)(t-t_o)}\| = \sum_{i=1}^{v} \left[ e^{J_{p_i}(a_i)(t-t_o)} - e^{J_{p_i}(\beta_i)(t-t_o)} \right] \left[ M_{p_i} \right] \]

Now, we will calculate

\[ \left\| M_{p_i} \right\|^2 = \left[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right] \frac{(t-t_o)^2}{(p_j-1)!} + \frac{(t-t_o)^3}{(p_j-2)!} + \cdots \]

Then, the equality (3.1) is derived.

Remark 3.1 If \( t = t_o, \) then the distance \( r \) is equal to zero. In this part of the section, we will investigate two interesting cases: (a) when \( |t-t_o| < \delta, \delta > 0 \) and (b) the asymptotic behaviour of the solution, i.e. when \( t \rightarrow \infty. \)

Lemma 3.3 When \( |t-t_o| < \delta \) (or \( t \in [t_o - \delta, t_o + \delta] \)) for \( \delta \rightarrow 0, \) an upper bound for the distance (error) between the (2.1) and (2.5) systems is provided, i.e.

\[ r \leq \sum_{j=1}^{v} \left[ \left\| \Sigma_p(t_o) \right\|^2 + 2 \delta^2 \| \Sigma_p(t_o) \|^2 \right]. \]
where \( \theta = \max_{j=1,2,\ldots,v} \left\{ 1, e^{(\text{Re}a_j)\delta}, e^{(\text{Re}\beta_j)\delta} \right\} \), \( p = \sum_{j=1}^{v} p_j \) and \( a_j, \beta_j \in \mathbb{F} \).

**Proof** Assume that we have
\[
\text{Re}a_j + z\text{Im}a_j = a_j \quad \text{and} \quad \text{Re}\beta_j + z\text{Im}\beta_j = \beta_j.
\]

So, we obtain
\[
e^{-\delta(t-t_0)} - e^{\beta_j(t-t_0)} = e^{\text{Re}a_j(t-t_0)} \left( \cos(\text{Im}a_j(t-t_0)) + z^2 \sin(\text{Im}a_j(t-t_0)) \right) - e^{\text{Re}\beta_j(t-t_0)} \left( \cos(\text{Im}\beta_j(t-t_0)) + z^2 \sin(\text{Im}\beta_j(t-t_0)) \right)
\]
\[
< e^{\text{Re}a_j(t-t_0)} + e^{\text{Re}\beta_j(t-t_0)}.
\]

Then, combining eq. (3.1) and the above results, we take
\[
\rho_M \leq \sum_{j=1}^{v} \left( e^{\text{Re}a_j(t-t_0)} + e^{\text{Re}\beta_j(t-t_0)} \right)^2 \left| \sum_{i=0}^{p_j-1} (p_j-i) \left( \frac{\delta_j}{j!} \right)^2 \right| \|y_p(t_0)\|_p.
\]

If \( |t-t_0| < \delta \), then we assume that there exists
\[
\theta = \max_{j=1,2,\ldots,v} \left\{ 1, e^{(\text{Re}a_j)\delta}, e^{(\text{Re}\beta_j)\delta} \right\},
\]

since \( e^{k(t-t_0)} \) with \( |t-t_0| < \delta \), then
\[
\text{if} \quad k > 0, \quad \text{then} \quad e^{k(t-t_0)} < e^{\delta \theta}
\]
\[
\text{if} \quad k < 0, \quad \text{then} \quad e^{k(t-t_0)} < 1.
\]

So, the following expression is derived
\[
r \leq 2 \left( \sum_{j=1}^{p_j-1} \theta^2 \left| \sum_{i=0}^{p_j-1} (p_j-i) \left( \frac{\delta_j}{j!} \right)^2 \right| \right) \|y_p(t_0)\|_p.
\]

Here, we can assume that \( \delta \) is a sequence that tends to zero (very small), then we have
\[
\sum_{i=0}^{p_j-1} (p_j-i) \left( \frac{\delta_j}{j!} \right)^2 = \sum_{i=0}^{p_j-1} (p_j-i) \left( \frac{\delta_j}{j!} \right)^2 \to p_j.
\]

Consequently, the upper bound (3.2) is obtained. \( \square \)

**Theorem 3.1** Two homogeneous linear time-invariant (LTI) descriptor (or generalized state space) differential systems
\[
F\dot{x}(t) = Gx(t), \quad \text{and} \quad (F+\Delta F)\dot{x}(t) = (G+\Delta G)x(t)
\]
with \( x(t_0) = x_0 \), \( \det F = 0 \) and \( \det (F + \Delta F) = 0 \), are \( r \)-close
for \( |t-t_0| < \delta \), \( \delta \to 0 \) (very small), with \( \theta = \max_{j=1,2,\ldots,v} \left\{ 1, e^{(\text{Re}a_j)\delta}, e^{(\text{Re}\beta_j)\delta} \right\} \) if the following inequality is satisfied
\[
r \leq \rho \left[ \frac{y_p(t_0)}{y_p(t_0)} \right] \leq \mathcal{O}(F(t)) - \mathcal{O}(G(t)) \leq 0.
\]

where \( \mathcal{O}(F(t)) \) is an upper bound.

**Remark 3.2** It is very interesting that the upper (error) bound
\[
\mathcal{O}(F(t)) = 2\theta^2 \|y_p(t_0)\|^2
\]
depends on three significant parameters, i.e. the Frobenius norm of the consistent initial condition \( y_p(t_0) \), the parameter \( \theta \) (which depends on \( a_j, \beta_j \in \mathbb{F} \)), and the number of finite elementary divisors \( (p = \sum_{j=1}^{v} p_j) \). However, in our case, it should be mentioned that the parameter \( \theta \) is the most significant, since we have assumed the same initial condition \( y_p(t_0) \), and the number of finite elementary divisors \( (p) \) for both (2.1) and (2.5) systems.

**Remark 3.3** With other words, there is a \( n \)-ball for the family of the solution for the new system (2.5), see figure 1, such that
\[
\left[ \begin{array}{c} \tilde{y}_p(t) \\ \tilde{\dot{y}}_p(t) \end{array} \right] \in \mathcal{S} \left( \begin{array}{c} \mathcal{O}(F(t)) \\ \mathcal{O}(G(t)) \end{array} \right), \quad r \leq 2\theta^2 \|y_p(t_0)\|^2.
\]

![Fig. 1. n-ball with centre \( S_M \) and radius \( r \)](image)

Note also that \( 2\theta^2 \|y_p(t_0)\|^2 \) is a positive number, since the zero initial condition has been excluded.

Now, we will assume that \( t \to \infty \), then we have the following two cases, see Proposition 3.1.

**Proposition 3.1** (a) If one of the \( \text{Re}a_j, \text{Re}\beta_j \) is positive, then the two solutions do not converge even if or not the pencils \((F,-G)\) and \((F,-G)\) are \( e \)-close. (b) Now, if both \( \text{Re}a_j \) and \( \text{Re}\beta_j \) are negative, then the two solutions do converge, i.e. \( \rho_M \to 0 \).
Proof (a) Since, one of the \( \text{Re} a_j \), \( \text{Re} \beta_j \) is positive, and \( t \to \infty \), then
\[
\lim_{t \to \infty} \left( e^{Re a_j(t-t_0)} + e^{Re \beta_j(t-t_0)} \right) = +\infty,
\]

since \( \sum_{j=1}^{p} \left| e^{Re a_j(t-t_0)} - e^{Re \beta_j(t-t_0)} \right| > 0 \), and
\[
\lim_{t \to +\infty} \sum_{i=0}^{p-1} \left( p_j - i \right) \left( -\frac{\ln \left| \beta_j \right|}{\beta_j} \right)^2 = +\infty,
\]
so it is easy to prove that the metric \( r \to +\infty \) is positive, and
\[
A \to \infty, \quad \text{and consequently,} \quad r \to 0.
\]

In the next lines, we will calculate the matrix perturbations \( \Delta F \) and \( \Delta G \), where (I)-(II) conditions are satisfied. Analyti-
ically, for the new system (2.5), we have to use the new Weierstrass canonical form, i.e.
\[
\tilde{P} F Q = I_p \oplus H_q, \quad \text{and} \quad \tilde{P} G Q = J_p(\beta) \oplus I_q.
\]

since we have also assumed that \( X(t) = Q^{-1} X(t) \).

Equivalently,
\[
FQ = \tilde{P}^{-1} I_p \oplus H_q, \quad \text{and} \quad GQ = \tilde{P}^{-1} J_p(\beta) \oplus I_q.
\]

Inevitably, combining the above two linear matrix equations, the expression (3.4) is derived,
\[
[F + G]Q = \tilde{P}^{-1} I_p \oplus H_q + J_p(\beta) \oplus I_q.
\]

Finally, considering that \( F \approx F + \Delta F \) and \( G \approx G + \Delta G \), we obtain that
\[
\tilde{P}^{-1} I_p \oplus H_q + J_p(\beta) \oplus I_q - [F + G]Q = [F + G]Q,
\]
and by denoting \( \Delta X \approx \Delta F + \Delta G \), we have the following linear matrix equation
\[
\tilde{P}^{-1} \Delta X = \left[ I_p + J_p(\beta) \oplus H_q + I_q \right]^{-1} [F + G]Q,
\]
or equivalently,
\[
\mathcal{Z} A = B,
\]
where only \( \mathcal{Z} \triangleq \left[ \tilde{P}^{-1} \Delta X \right] \in \mathcal{M}_{2n} \) is the unknown parameter.

Here, we will modify a very standard result of the gener-
ized inverses’ theory, see for instance Ben-Israel and Greville (2003), Campbell and Meyer (1979).

Theorem 3.2 Let \( A \in \mathcal{M}_{2n} \) and \( B \in \mathcal{M}_n \), then the matrix equation (3.6), \( \mathcal{Z} A = B \), is consistent if and only if, for some \( A^{(0)} \in \mathcal{M}_{2n} \), we have
\[
BA^{(0)} A = B,
\]
in which case the general solution is given by (3.8)
\[
\mathcal{Z} = BA^{(0)} + \mathcal{P} \left[ I_{2n} - AA^{(0)} \right],
\]
for arbitrary \( \mathcal{P} \in \mathcal{M}_{2n} \). \( \square \)

Remarks 3.4 Indeed, Theorem 3.2 is the principal application of \{1\}-inverses (i.e. for every finite matrix \( A \) of real or complex elements, there is a unique matrix \( X \) satisfying the equation \( AXA = A \) ) to the solution of linear systems, where they are used in much the same way as ordinary inverses in the non-singular case, see Penrose (1955).

Remark 3.5 The consistency of the solution for the linear system (3.6) is very significant, see expression (3.7). Thus, among all the possible \{1\}-inverses \( A^{(0)} \in \mathcal{M}_{2n} \), only those which satisfy the condition (3.7) should be chosen.

Consequently, if we also want to characterize the set \( \mathcal{A} \{1\} \), in terms of an arbitrary element \( A^{(0)} \) of the set, see for more details Bjorhammar (1958), we can state the following result.

Corollary 3.1 Let \( A \in \mathcal{M}_{2n} \) and \( A^{(0)} \in \mathcal{A} \{1\} \), then
\[
\mathcal{A} \{1\} = \left\{ A^{(0)} + W - A^{(0)} AA^{(0)} W \in \mathcal{M}_{2n} \right\}. \quad \square \quad (3.9)
\]

So, we have
\[
\mathcal{Z} = \left[ \tilde{P}^{-1} \Delta X \right] = \left[ \left[ B A^{(0)} \right]_1 \left[ B A^{(0)} \right]_2 \right] + \mathcal{P} \left[ I_{2n} - AA^{(0)} \right]_3 \in \mathcal{M}_{2n},
\]
and we can conclude the whole discussion, presenting Theo-
rem 3.3. Since the proof is based on the previous results of this section, it is omitted.

Theorem 3.3 The solution of the linear matrix equation (3.4) is given by the following expressions: For an arbitrary choice of \( \Delta F \in \mathcal{M}_n \), such as \( \text{det} (F + \Delta F) = 0 \) (I) condition then
\[ \Delta G = [\mathcal{L} A^{(i)}]_1 + \mathcal{P} \left[ I_{2n} - A A^{(i)} \right]_2 - \Delta F, \]  
\quad (3.10)

such that \( \rho(\mathcal{L}, G), (\mathcal{L}, -G) = \|\Delta F\|^2 + \|\Delta G\|^2 < \varepsilon. \) (II) condition

Moreover, we have
\[
\tilde{P} = \left[ (\mathcal{L} A^{(i)})_1 + \mathcal{P} \left[ I_{2n} - A A^{(i)} \right]_2 \right]^{-1}.
\quad (3.11)

for \( A^{(i)} \in \mathcal{A}\{1\} \) and arbitrary \( \mathcal{P} \in \mathcal{M}_{k,2n}. \)

\[ \square \]

**Remark 3.6** It should be pointed out that \( \Delta G \in \mathcal{M}_n \) can be also arbitrarily chosen, and then the perturbation \( \Delta F \in \mathcal{M}_k \) is given by \( \Delta F = [\mathcal{L} A^{(i)}]_1 + \mathcal{P} \left[ I_{2n} - A A^{(i)} \right]_2 - \Delta G \), such as \( \det(F + \Delta F) = 0 \) and the expression (2.6) is always satisfied. \[ \square \]

Consequently, in order to summarize and emphasize the results of this section, the following step-algorithm is provided.

**Step-Algorithm**

1\textsuperscript{st} Step: Define the pencil \((F,G)\), and the non-singular matrices \(P, Q\) since the initial (2.1) system is known. Moreover, state what are the desirable eigenvalues, i.e. \( \beta_j \), of the new system (2.5).

2\textsuperscript{nd} Step: Define \( \mathcal{A} = \left[ [I + J_\varepsilon(\beta)] @ [H_\varepsilon + I_\beta] \right] \) and \( \mathcal{B} = [F + G]Q \). Then, calculate the \( \{1\}\)-generalized inverse of the matrix \( \mathcal{A} \in \mathcal{M}_{2n,n}, \mathcal{A}^{(i)} \in \mathcal{M}_{2n,2n} \), such as (3.7) is satisfied.

3\textsuperscript{rd} Step: Determine what is the acceptable \( \varepsilon \). Then, choose \( \Delta F \in \mathcal{M}_k \) such as \( \|\Delta F\|^2 < \varepsilon / 2 \) and \( \det(F + \Delta F) = 0 \).

4\textsuperscript{th} Step: Calculate \( \Delta G = [\mathcal{L} A^{(i)}]_1 + \mathcal{P} \left[ I_{2n} - A A^{(i)} \right]_2 - \Delta F \). In this step, we can also determine a more "specific" family of \( \mathcal{P} \) since \( \|\Delta G\|^2 < \varepsilon / 2 \).

5\textsuperscript{th} Step: Then, considering the specific family of \( \mathcal{P} \) (see 4\textsuperscript{th} step), calculate \( \tilde{P} = \left[ (\mathcal{L} A^{(i)})_1 + \mathcal{P} \left[ I_{2n} - A A^{(i)} \right]_2 \right]^{-1}. \)

6\textsuperscript{th} Step: Finally, all the necessary steps have been calculated in order to be able to find the new system (2.5), when \( |t - t_o| < \delta, \delta \to 0 \), such that

\[
\begin{align*}
\frac{\hat{x}_p(t)}{\hat{y}_p(t)} \in S_M \triangleq \left[ \frac{y_p(t)}{x_p(t)} \right]_{r \pm 2 \sigma}^{r \pm 2 \sigma},
\end{align*}
\]

Applying the Step-Algorithm, many numerical examples can be presented and discussed.

4. CONCLUSIONS

In this brief paper, we investigate the conditions under which a LTI descriptor system with a specific structure and desirable properties is being constructed. Thus, if we assume that \((F, G)\) is the matrix pencil of the initial system, we define the relative family of the perturbed matrix pencils \((F + \Delta F, G + \Delta G)\) such as the solutions of the initial and the perturbed systems are \( r \)-closed in respect to the Frobenius distance.

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