Quadratic Stability of Uncertain Reset Control Systems

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Abstract: In this paper, we consider quadratically robust stability of reset control systems with uncertain output matrices. We show that, when the uncertain set is a convex polyhedron, the necessary and sufficient condition for a reset control system to be quadratically robustly stable is that the vertices of the uncertain set share a common block diagonal Lyapunov function.

Keywords: Reset control system; Hybrid Systems, robust stability.

1. INTRODUCTION

A reset control system consists of a baseline system and a reset mechanism which resets the state of the controller whenever some continuous signal crosses zero. See Krishnan et al. (1974), Horowitz et al. (1975) for instance. The simplest reset element is the Clegg integrator (CI) proposed in Clegg (1958) to overcome limitations of linear time-invariant control design. The benefits of reset control have been showed by both theoretical analysis and experimental results in literature. Refer to Beker et al. (2001), Chen (2000), Guo et al. (2009), Zheng et al. (2000), Guo et al. (2009), Beker et al. (2004), Guo et al. (2009) for instance.

In this paper, we consider robust stability of reset control systems with uncertain output matrices. Different from other uncertainties, output matrix uncertainty not only affects the structure matrices of the closed-loop system but also makes the reset time instants uncertain. It is an interesting problem that whether a nominal reset control system is robust with respect to output matrix uncertainty. This motivates the research of this paper.

The system considered in this paper consists of a plant
\[
\begin{aligned}
\dot{x}_p &= A_p x_p + B_p u \\
y &= (C_p + \Delta_p)x_p
\end{aligned}
\] (1)

and a reset controller
\[
\begin{aligned}
\dot{x}_r &= A_r x_r + B_r y, \quad y \neq 0 \\
x^+_r &= R_r x_r, \quad y = 0 \\
u &= F_r x_r + G_r y
\end{aligned}
\] (2)

with \( x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{m}, y \in \mathbb{R}^{1}, x_r \in \mathbb{R}^{n_r}, \Delta_p \in \Omega \subseteq \mathbb{R}^{1 \times n_p} \) is the uncertain term in the output matrix with \( \Omega \) the uncertainty set containing the origin. \( C_p \) is assumed to be of full rank, i.e., \( \text{rank}(C_p) = 1 \). In this case, we can always find a \( (n_p - 1) \times n_p \) matrix \( T \) such that
\[
S = \begin{bmatrix} C_p \\ T \end{bmatrix}
\]
is nonsingular. Let \( z_p = S x_p \), then (1) becomes
\[
\begin{aligned}
\dot{z}_p &= S A_p z_p + S B_p u \\
y &= (C_p S^{-1} + \Delta_p S^{-1}) z_p
\end{aligned}
\] (3)

with
\[
C_p S^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.
\]

Thus, without loss of generality, we assume that \( C_p \) is of the form \( C_p = (1 0 \cdots 0) \). \( R_r \) is assumed to be of the form
\[
R_r = \begin{bmatrix} I \\ R_p \end{bmatrix}
\]
where \( R_p \) is a \( \rho \times \rho \) matrix with \( \rho \leq n_r \). In this paper, we assume that \( R_p \) is Schur stable in the sense that \( |\lambda(R_p)| < 1 \). Denote \( \Delta = [\Delta_p 0], C = [C_p 0], C_\Delta = C + \Delta \). Combining (1) and (2) gives
\[
\begin{aligned}
\dot{x} &= A_\Delta x, \quad C_\Delta x \neq 0 \\
x^+ &= R x, \quad C_\Delta x = 0
\end{aligned}
\] (4)

where \( x = (x_p, x_r)^T \), \( A_\Delta = A + H \Delta \) and
\[
A = \begin{bmatrix} A_p + B_p G_r C_p & B_p F_r \\ B_r C_p & A_r \end{bmatrix}.
\]

\[
H = \begin{bmatrix} B_r G_r \\ B_r \end{bmatrix}, \quad R = \begin{bmatrix} I \\ R_r \end{bmatrix}.
\]

Definition 1. The reset control system (4) is said to be robustly stable with respect to \( \Omega \) if it is asymptotically stable for any vector-valued function \( \Delta_p(t) \in \Omega \). It is said...
to be quadratically stable if there is a positive matrix $P$ such that

$$A^T_{\Delta} P + P A_{\Delta} < 0, \quad \forall \Delta_p \in \Omega,$$  

(5)

$$x^T (R^T P R - P) x \leq 0, \quad \forall x \in \bigcup_{\Delta_p \in \Omega} \ker(C_{\Delta}).$$  

(6)

It is clear that quadratic stability implies robust stability. But the inverse is not true.

For any vector $\gamma = [\gamma_1, \cdots, \gamma_n]$, denote $||\gamma||_{\infty} = \max(|\gamma_i|)$. We assume that

$$||\Delta_p||_{\infty} < ||C_p||_{\infty} = 1.$$  

2. QUADRATIC STABILITY

Denote by $C_{\Delta}^\perp$ any full-rank right annihilator of $C_{\Delta}$, i.e., $C_{\Delta}^\perp$ is a $(n_p + n_r) \times (n_p + n_r - 1)$ matrix satisfying $C_{\Delta} C_{\Delta}^\perp = 0$ and $\text{rank}(C_{\Delta}) = n_p + n_r - 1$. $C_{\Delta}^\perp$ and $\Delta_p^\perp$ are defined similarly. It is easy to check that $\ker(C_{\Delta}) = \text{Im}(C_{\Delta}^\perp)$ and (6) is equivalent to

$$(C_{\Delta}^\perp)^T (R^T P R - P) C_{\Delta}^\perp \leq 0, \quad \forall \Delta_p \in \Omega.$$  

(7)

For simplicity, decompose $\Delta_p$ as $\Delta_p = [\delta \gamma]$ where $\delta$ is a scalar and $\gamma$ a $(n_p - 1)$-dimensional row vector. It is easy to check that

$$C_{\Delta}^\perp = \begin{bmatrix} J_{\Delta} & I_p \end{bmatrix}$$

with $I_p$ a $\rho \times \rho$ identity matrix and $J_{\Delta}$ a $(n_p + n_r - \rho) \times (n_p + n_r - \rho - 1)$ matrix defined as

$$J_{\Delta} = \begin{bmatrix} \gamma & -(1+\delta) I_{n_p-1} & \gamma \end{bmatrix}.$$  

Repartition $R$ and $P$ as

$$R = \begin{bmatrix} I_{n_p+n_r-\rho} & R_p \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}.$$  

(8)

It is trivial to check that (7) is equivalent to

$$\begin{bmatrix} 0 & J_{\Delta}^T P_{12} (R_p - I) \\ (R_p^T - I) P_{12}^T J_{\Delta} R_p P_{22} - P_{22} \end{bmatrix} \leq 0, \quad \forall \Delta_p \in \Omega.$$  

(9)

which is in turn equivalent to

$$R_p^T P_{22} R_p - P_{22} \leq 0,$$  

(10)

$$J_{\Delta}^T P_{12} (R_p - I) = 0, \quad \forall \Delta_p \in \Omega.$$  

(11)

Since $R_p$ is Schur stable, (10) is equivalent to

$$J_{\Delta}^T P_{12} = 0, \quad \forall \Delta_p \in \Omega.$$  

(12)

Decompose $P_{12}$ as

$$P_{12} = \begin{bmatrix} \Gamma_1 \\ \Gamma_{n_p-1} \\ \Gamma_{n_r-\rho} \end{bmatrix},$$  

where $\Gamma_i$ ($i = 1, n_p - 1$ or $n_r - \rho$) is an $i \times \rho$ matrix. Then (11) is equivalent to

$$\gamma^T \Gamma_1 - (1+\delta) \Gamma_{n_p-1} = 0,$$  

(13)

$$\Gamma_{n_r-\rho} = 0.$$  

By the assumption that $\Omega$ contains the origin, we can choose $\gamma = 0$. In addition, the assumption $||\Delta_p||_{\infty} <...
then the state-space equation of the plant becomes

\[
P: \begin{cases}
\dot{z}_p = \begin{bmatrix} 0.4 & -0.4 \\ 2 & -1 \end{bmatrix} z_p + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y_p = \begin{bmatrix} 1 & 0 \end{bmatrix} z_p
\end{cases}
\] (20)

and the relation between \( y \) and \( z_p \) becomes

\[
y = \begin{bmatrix} 1 & \gamma \end{bmatrix} z_p.
\]

Thus the closed-loop system is of the form (4) with \( x = [z_p^T, x_T^T]^T \), \( C_\Delta = \begin{bmatrix} 1 & \gamma & 0 \end{bmatrix} \) and

\[
A_\Delta = \begin{bmatrix} 0.4 & -0.4 & 1 \\ 2 & -1 & 0 \\ -1 & -\gamma & -1 \end{bmatrix} := A(\gamma), \quad R = \begin{bmatrix} 1 \\ 1 \\ r \end{bmatrix}
\]

Since \( r \in (-1, 1) \), according to Proposition 1, this reset control system is quadratically stable with respect to \( \gamma(t) \in [\delta_1, \delta_2] \) with \( \delta_1 < 0 \) and \( \delta_2 > 0 \) if and only if there is a block diagonal positive matrix \( P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \) where \( P_{11} \) is a \( 2 \times 2 \) positive matrix and \( P_{22} \) is a positive scalar number, such that

\[
A(\gamma)^T P + PA(\gamma) < 0, \quad \gamma = \delta_1 \text{ or } \delta_2,
\]

i.e., \( P \) is a common Lyapunov function of \( A(-\delta_1) \) and \( A(\delta_2) \). Note that we do not need to check (16) since it is equivalent to

\[
r^2 - 1 \leq 0
\]

which is naturally satisfied. By Matlab LMI toolbox, it is easy to check that (21) holds with \( \delta_1 = -0.3, \delta_2 = 0.4 \) and

\[
P = \begin{bmatrix}
2.1702 & -0.5721 \\
-0.5721 & 0.4683 \\
1.3525 & 0
\end{bmatrix}
\]

Thus this reset control is quadratically stable with respect to \( \gamma \in [-0.3, 0.4] \).

In the above analysis, it is clear that, provided that \( |r| \leq 1 \), the reset control system is quadratically stable. Thus we can adjust \( r \) within the range \( r \in [-1, 1] \) without destroying the robustness. Fig. 2, gives the output response of the nominal system \( (\gamma = 0) \) with initial condition \( x_0 = [-10, 5]^T \) and with \( r = 1 \) (i.e., the baseline system), \( r = 0 \) and \( r = -0.6 \) respectively. It shows that \( r = -0.6 \) gives good performance. The state plots are given in Fig. 3. In order to show the robustness, we choose

\[
\gamma(t) = 0.3 \sin 10t
\]

and the corresponding output responses and state plots are given in Fig. 4 and Fig.5.

4. CONCLUSION

In this paper, we have investigated robust stability of reset control systems with uncertain output matrices. A necessary and sufficient condition for quadratical stability has been obtained in terms of LMIs. An example has also been given to demonstrate the effectiveness of the result.

It is worth pointing out that a robustly stable reset control system need not to be quadratically stable. Further studies on this topic are needed in the future research.
Fig. 5. The states of the system with output matrix uncertainty $\gamma = 0.3 \sin 10t$ and with $r = 1$ (without reset), $r = 0$ and $r = -0.6$ respectively.

REFERENCES


