A Very Relaxed Control Law for Bearing-Only Triangular Formation Control

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Abstract: The problem of bearing-only triangular formation control is considered. Each agent measures two inter-agent bearings in a local coordinate system and is tasked with establishing, and maintaining, a desired angular separation relative to its neighbours (and consequently an overall desired shape). A distributed control law is designed for each agent that is based only on the agent’s locally measured bearings. A strong convergence result is established which guarantees global exponential convergence of the formation to the desired shape. Despite the convergence guarantee, the controller is also relaxed in the sense that each agent can independently choose their control inputs within a large region of values. The proposed controller is robust to a single agent motion failure or a common group motion command.

1. INTRODUCTION

The formation shape control problem involves a group of agents tasked with forming, and then maintaining, a prescribed geometrical shape described in terms of relative geometrical constraints between some of the agents. This work details a relaxed distributed control system for triangular (three-agent) formation control based on bearing-only measurements and relative angular constraints. The formulation introduced is novel and the proposed relaxed control law is provably globally stabilizing.

The formulation of any formation control problem involves a number of steps. For instance, the sensing technology and the so-called sensing graph must be specified. The sensing graph describes which agents sense each other and the sensing technology describes what type of measurements are taken. The sensing technology may permit bearing-only [Bishop and Basiri, 2010; Basiri et al., 2010] or distance-only measurements [Cao and Morse, 2008; Cao et al., 2011]. In fact, many existing algorithms suppose that both measurements are taken (yielding the relative position), see e.g. the work in [Egerstedt and Hu, 2001; Saber and Murray, 2002; Eren et al., 2002; Baillieul and Suri, 2003; Smith et al. 2006; Anderson et al., 2008; Krick et al., 2009; Cortes, 2009; Dörfler and Francis, 2010].

In addition, albeit not independently, the control constraints and control graph must be defined [Anderson et al., 2008]. The control constraints specify the geometrical relationships between agent-pairs that should be established and maintained. The control constraints are typically either relative position (or state) or inter-agent distance-only constraints. Bearing-only control constraints (as with bearing-only measurements) in formation control are not commonly considered. It is typical for the topology of the control and sensing graph to be equivalent; i.e. agents control some geometrical relationship to those agents to which some measurements are taken.

The control graph, and the specific control constraints, together determine what desired formation shapes/scales are feasible and their uniqueness (ignoring additional kinematic constraints on the agents) [Tabuada et al., 2001]. Often, for example, a certain (well-chosen) set of distance constraints can (generically) define a unique formation shape; e.g. see the notion of graph rigidity as it applies to formation control in [Eren et al., 2002; Anderson et al., 2008; Krick et al., 2009]. Directed constraints can also be considered, where some agents are tasked at maintaining a given parameter relative to another agent while the converse is not true; e.g. see [Hendrickx et al., 2008]. Given a sensing and control architecture, the aim is then to design distributed algorithms that seek to control each agent’s motion to achieve, and maintain, the desired constraints.

There now exists a large body of literature on formation control and the references stated here represent only a subjective selection. However, the general problem remains of interest due to the various problem formulations, the distributed nature of the problem itself and the existence of undesired equilibria, see the discussions in [Cortes, 2009; Dörfler and Francis, 2010; Anderson et al., 2010], which prompts further investigation. For example, most existing algorithms consider range-only control constraints and suppose that each agent measures the relative position (or state) of its neighbour agents. The problem formulation introduced here differs from this previous work in a novel way; namely, by assuming agents only measure the inter-agent bearings and then seek to control certain angular constraints.

1.1 Contribution Statement

Before discussing the novelty of this work, the motivation for considering the formation control problem with only three agents is detailed. Of course, a general formulation with an arbitrary number of agents would be of interest. However, the problem is difficult and multiple technical issues arise. Thus, it is reasonable to outline and rigorously analyze the problem with a limited number of agents and then conjecture, initially, about extensions to more agents. For example, in formation control where the range between certain agents is the control constraint, and the relative position is measured, the typical algorithms examined [Krick et al., 2009; Dörfler and Francis, 2010] often exhibit multiple equilibria. For this reason, the three [Anderson et al., 2007; Cao et al., 2007, 2008b,a; Chen and Tian, 2009] and four [Anderson et al., 2010] agent cases for range-based formation control have been studied extensively
to gain an insight into the system dynamics. Moreover, the algorithm proposed in this work is relaxed (in a sense to be defined) and thus an initial study with three agents is useful.

In this work each agent measures the bearing to the other two agents in a local coordinate system. The controlled constraint is the angle subtended at the agent by the other two agents. Each agent is given a desired value for this angle and is tasked at establishing and maintaining this constraint. This differs from much of the existing work in formation control where range constraints are considered and each agent typically measures the relative position (or state) of their neighbours. A large literature on bearing-only localization, see [Nardone et al., 1984; Gavish and Weiss, 1992], motivates the notion of bearing-only sensing. The problem of vision-based distributed formation control is also related since video sensors act as bearing sensors; see [Das et al., 2002; Moshtagh et al., 2009].

The main algorithmic contribution of this work is a relaxed distributed control law that solves the stated formation shape control problem. The distributed control law is relaxed in the sense that each agent is free to choose their own heading within a relatively large region of values. That is, the controller is not a typical feedback controller which maps, e.g., a control error to a distinct control input. Rather, the control law defines a sector of the plane towards which the agent must steer, i.e., an uncountable set of headings, and the agent is free to, in a relaxed way, pick a particular heading from this set. No inter-agent communication is required. A strong convergence result is established which guarantees global exponential convergence of the formation to the desired shape. The relaxed control law also ensures that collisions are avoided naturally. Moreover, the control scheme is robust to a single agent motion failure.

This work is a solid basis for further work on the lesser studied bearing-only formation control problem. Also, the proposed controller is a novel motivator for further work on relaxed multi-agent control laws. Finally, the application of optical sensor arrangement for target localization is cited as an immediate application of this work; e.g. [Bishop et al., 2010].

1.2 Organization

The problem is introduced in Section 2 along with the proposed relaxed distributed control law. Also, global stability of the desired formation shape is proved. Illustrative examples are given in Section 3. In Section 4, some novel extensions are examined; e.g., it is shown that the relaxed controller is robust to an agent motion failure. Conclusions are given in Section 5.

2. BEARING-ONLY FORMATION CONTROL

Consider $n = 3$ agents in $\mathbb{R}^2$ indexed by $i \in \mathcal{V} = \{1, 2, 3\}$ with positions $p_i = [x_i, y_i]^T \in \mathbb{R}^2$. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to model the interactions between agent-pairs. The neighbour set at $i$ is $\mathcal{N}_i = \{(i+1), (i-1)\} \subset \mathcal{V}$ (taken modulo $n$). The set of agent points $p_i, \forall i \in \mathcal{V}$ and straight-line edges connecting $p_i$ and $p_j, j \neq i$ define a triangle on $\mathbb{R}^2$. This triangle is referred to as the formation configuration, or the formation.

Assumption 1. $p_j(0) \neq p_j(0)$ for $i \neq j$ at time $t = 0$.

Also, the agents do not share a common coordinate frame. For each $j \in \mathcal{N}_i$, agent $i$ measures the bearing $\phi_{ij} \in [-\pi, \pi]$ in its local, body-fixed, coordinates. Let $\alpha_i$ denote the interior angle subtended at agent $i$ by the two agents in $\mathcal{N}_i$. The formation shape (not scale) is completely characterized by the $\alpha_i, i \in \mathcal{V}$.

Introduce the angle

$$\theta_i = |\phi_{i(i+1)} - \phi_{i(i-1)}| \in [0, 2\pi] \tag{1}$$

that is subtended at agent $i$ by agents $i + 1$ and $i - 1$ measured positive from $\min(\phi_{i(i+1)}, \phi_{i(i-1)})$ to $\max(\phi_{i(i+1)}, \phi_{i(i-1)})$. Then, the interior $\alpha_i$ can be mathematically computed using

$$\alpha_i = \begin{cases} \theta_i & \text{if } \theta_i \leq \pi \\ 2\pi - \theta_i & \text{otherwise} \end{cases}$$

such that $\alpha_i \in [0, \pi]$. Agent $i$ can distinguish between $\alpha_i = 0$ and $\alpha_i = \pi$ and thus knows whether or not it is in between agents $i + 1$ and $i - 1$ when all three are collinear.

Define the desired steady-state angles $\alpha_i^* \in [0, \pi], \forall i \in \mathcal{V}$. The $\alpha_i^*$ characterize the shape (not scale) of the desired formation.

Assumption 2. The desired interior angles obey $\sum \alpha_i^* = \pi$. The case $\alpha_i^* = 0$, $\alpha_j^* \neq 0$ and $\alpha_k^* = \pi - \alpha_j^*$ is excluded.

Assumption 2 ensures the desired steady-state triangle is well-defined and the control objectives are simultaneously feasible.

Problem (Bearing-Only Triangular Formation Shape Control).

Design a control law for each agent $i$ that takes $\alpha_i$ and $\alpha_j^*$ as input and steers the agent so that $\alpha_i \rightarrow \alpha_i^*$ given any initial $\alpha_i$.

This statement neglects any controller design requirements, e.g., the relaxation discussed later, or the desired convergence rate etc. For example, the proposed control law will actually specify a region of acceptable control inputs and any input chosen within this region will facilitate a solution to the stated problem.

2.1 The Proposed Relaxed and Distributed Control Law

The motion of agent $i$ is governed by

$$\dot{p}_i = v_i [\cos \beta_i \sin \beta_i]^T$$

where $[v_i, \beta_i]^T$ are controls to be determined. The heading $\beta_i$ is with respect to the same reference used to define $\phi_{ij}, \forall j \in \mathcal{N}_i$.

The speed input of agent $i$ is defined by $v_i = (\alpha_i^* - \alpha_i) k_i$ where $k_i > 0$. Let $k_i = 1, \forall i$ and instead let the focus be shifted to relaxing the design of $\beta_i$. The heading $\beta_i$ is defined by

$$\beta_i = \begin{cases} \alpha_i \beta_i + \min(\phi_{i(i+1)}, \phi_{i(i-1)}) & \text{if } \theta_i \leq \pi \\ \alpha_i \beta_i + \max(\phi_{i(i+1)}, \phi_{i(i-1)}) & \text{if } \theta_i > \pi \end{cases} \tag{2}$$

where $\beta_i$ is given by (1) and $0 < \gamma_i < 1$. Allow $\gamma_i$ to be a function of time and chosen at run-time by the agent (according to a separate reasoning process carried out by the agent).

Visually, the control law, with $v_i > 0$, drives agent $i$ toward the interior of the formation at some angle specified by $\alpha_i \gamma_i$. For example, if $\gamma_i = 1/2$ then with $v_i > 0$ the agent travels toward the interior of the formation along the bisection of $\alpha_i$. If $v_i < 0$ then the agent travels toward the exterior of the triangle.

The controller is relaxed and the only requirement is that agents independently move toward the interior or the exterior of the formation dependent on the sign of their control error. Actually, the control mapping is a one-to-many function which then allows the agent freedom in choosing a single $\beta_i$ during execution through $\gamma_i$. The reasoning or decision making process that may lead to a particular $\beta_i$ is not considered here.

For example, an agent may choose a specific heading that avoids an external obstacle. This controller may also be robust to various forms of error or to weakly measured bearing data such as that obtained from a vision-system etc.
2.2 The Dynamics of the System and a Stability Analysis

Suppose Assumptions 1 and 2 hold. It is shown later that if \( r_{ij} = r_{ji} = \|p_i - p_j\| > 0 \) at some time \( t_0, \forall i, j \) then \( r_{ij} > 0, \forall t \geq t_0 \), i.e. collisions are avoided naturally by the controller and \( \alpha_i \) (and, as it happens, \( \dot{\alpha}_i \)) is well-defined for all time \(^4\).

Consider agent \( i \) with \( v_i = \alpha_i^* - \alpha_i \) and heading \( \beta_i \) defined by (2) and note that \( \mathcal{N}_i = \{(i + 1), (i - 1)\} \). If agents \( i + 1 \) and \( i - 1 \) are initially held stationary then

\[
\dot{\alpha}_{i+1} = -\frac{v_i}{r_{i(i+1)}} \sin(\alpha_i\gamma_i) - \frac{1}{r_{i(i+1)}} \sin(\alpha_i\gamma_i)(\alpha_i^* - \alpha_i)
\]

where the negative sign arises because agents \( i + 1 \) and \( i - 1 \) are held still and thus an increase in \( \alpha_i \), i.e. due to \( (\alpha_i^* - \alpha_i) > 0 \), implies a decrease in \( \alpha_{i+1} \). Similarly, it follows that

\[
\dot{\alpha}_{i-1} = -\frac{v_i}{r_{i(i-1)}} \sin(\alpha_i(1 - \gamma_i))(\alpha_i^* - \alpha_i)
\]

and it becomes apparent that some ordering of the agents on the plane has been assumed. That is,

\[
\dot{\alpha}_{i+1} = -\frac{1}{r_{i(i+1)}} \sin(\alpha_i(1 - \gamma_i))(\alpha_i^* - \alpha_i)
\]

\[
\dot{\alpha}_{i-1} = -\frac{1}{r_{i(i-1)}} \sin(\alpha_i\gamma_i)(\alpha_i^* - \alpha_i)
\]

is an alternative, and equally valid, formulation. For the subsequent analysis it makes no difference which ordering is chosen.

Note that \( \sum \dot{\alpha}_i = 0 \). Thus, when agents \( i + 1 \) and \( i - 1 \) are held stationary then

\[
\dot{\alpha}_i = \frac{(\alpha_i^* - \alpha_i)}{r_{i(i+1)}} \sin(\alpha_i\gamma_i) + \frac{(\alpha_i^* - \alpha_i)}{r_{i(i-1)}} \sin(\alpha_i(1 - \gamma_i))
\]

\[
= \frac{r_{i(i+1)} \sin(\alpha_i(1 - \gamma_i)) + r_{i(i-1)} \sin(\alpha_i\gamma_i)}{r_{i(i+1)}r_{i(i-1)}} (\alpha_i^* - \alpha_i)
\]

Now for future notational brevity let

\[
f_{i(i+1)i} = \frac{1}{r_{i(i+1)}} \sin(\alpha_i\gamma_i)
\]

\[
f_{i(i-1)i} = \frac{1}{r_{i(i-1)}} \sin(\alpha_i(1 - \gamma_i))
\]

which fixes any ordering issue and let

\[
g_i = \frac{r_{i(i+1)} \sin(\alpha_i(1 - \gamma_i)) + r_{i(i-1)} \sin(\alpha_i\gamma_i)}{r_{i(i+1)}r_{i(i-1)}}
\]

where \( g_i \geq 0 \) and \( f_{ij} \geq 0 \) when \( \alpha_i \in [0, \pi] \) and \( \gamma_i \in (0, 1) \).

Now, assuming all agents move with a motion governed by their individual control laws it follows that

\[
\dot{\alpha}_i = g_i(\alpha_i^* - \alpha_i - f_{i(i+1)}(\alpha_i^* + 1 - \alpha_{i+1}) - f_{i(i-1)}(\alpha_i^* + 1 - \alpha_{i-1})
\]

with \( \alpha_i \in [0, \pi] \) and \( \sum \dot{\alpha}_i = 0 \).

Define the error \( \epsilon_i = \alpha_i - \alpha_i^* \in [-\alpha_i^*, \pi - \alpha_i^*] \subset [-\pi, \pi], \forall i \).

Then the following differential system is obtained

\[
\dot{\epsilon}_i = -g_i\epsilon_i + f_{i(i+1)}\epsilon_{i+1} + f_{i(i-1)}\epsilon_{i-1}
\]

(4) from (3). Note that \( \dot{\epsilon}_i \) is a nonlinear differential equation since, for example, \( \alpha_i = \alpha_i^* + \epsilon_i \) and \( \alpha_i \) comes into \( g_i \geq 0 \) and \( f_{ij} \geq 0 \) through sine functions. Stacking the system of differential equations (4) leads to

\[
\dot{\mathbf{e}} = \mathbf{F}(\mathbf{e})\mathbf{e}
\]

(5) where \( \mathbf{e} = [\epsilon_1, \epsilon_2, \epsilon_3]^T \) and where

\[
\mathbf{F}(\mathbf{e}) = [-g_1, g_{12}, g_{13}, \frac{f_{21}}{f_{23}}, -g_2, \frac{f_{21}}{f_{23}}, g_{31}, \frac{f_{32}}{f_{33}}]
\]

where \( \mathbf{e} \) is defined on a 2-simplex in e-space, denoted by \( \mathcal{M}_e \), with vertices \( \mathbf{e} = \lfloor \pi - \alpha_i^* - \alpha_i^* - \alpha_i^* \rfloor, \mathbf{e} = \lfloor -\alpha_i^* - \alpha_i^* - \alpha_i^* \rfloor \) and \( \mathbf{e} = \lfloor -\alpha_i^* - \alpha_i^* - \alpha_i^* \rfloor \).

Figure 1 depicts the error manifold and six disjoint regions, \( \mathcal{R}_{i\pm}, \forall i \in \{1, 2, 3\} \). The region indices will be explained.

**Fig. 1.** A plot of the open error manifold showing six distinct regions and the boundaries of the manifold (for some arbitrary \( \alpha_i^* \) values).

For distinct \( i, j, k \in \{1, 2, 3\} \), the individual error regions are chosen to exhibit the following useful properties

\[
\mathcal{R}_{1+} \Rightarrow \{\epsilon_j > 0, \epsilon_k < 0, \epsilon_i = 0, \dot{\epsilon}_i > 0\}
\]

\[
\mathcal{R}_{1-} \Rightarrow \{\epsilon_j < 0, \epsilon_k < 0, \epsilon_i > 0, \dot{\epsilon}_i < 0\}
\]

(7) where \( \epsilon_i \in [-\alpha_i^*, \pi - \alpha_i^*], \forall i \) and \( \sum \epsilon_i = 0 \) must be enforced. The error signs are chosen so the sign of the corresponding error velocity can be determined throughout the region using the given error signs and (4) with the fact \( g_i \geq 0 \) and \( f_{ij} \geq 0 \) \( \forall i, j \in \{1, 2, 3\} \) and \( \alpha_i \in [0, \pi] \). The simplex, or manifold \( \mathcal{M}_e \), shifts in the error space depending on the desired angles \( \alpha_i^* \). As such, regions \( \mathcal{R}_{1\pm} \) can grow or shrink, and can disappear altogether. For example, when \( \alpha_i^* = \alpha_i^* = 0 \) such that \( \alpha_3^* = \pi \), then the only region in existence is \( \mathcal{R}_{3+} \).

**Theorem 1.** The manifold \( \mathcal{M}_e \) is a positively invariant region.

This result is not surprising as \( \sum \epsilon_i = \sum \dot{\epsilon}_i = 0 \) immediately implies \( \mathbf{e}(t) \in \mathcal{M}_e, \forall t \). The next result ensures the formation is well-defined \( \forall t \), i.e. the \( \alpha_i \) (and \( \dot{\alpha}_i \)) are well-defined \( \forall t \).

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2 The subsequent stability analysis permits any continuously time-varying \( 0 < \gamma_i < 1 \). However, the continuity assumption can also be relaxed. In the simulation section \( 0 < \gamma_i < 1 \) can change discontinuously.
**Theorem 2.** Suppose that $p_i(t_0) \neq p_j(t_0)$ for $i \neq j$ at some time $t_0$. Then, $p_i(t) \neq p_j(t)$ for $i \neq j$ for all $t \geq t_0$.

The previous result really ensures collisions are avoided naturally by the formation. The proof relies on a geometrical argument concerning the $\alpha_i$, and is omitted for brevity. It follows easily from Assumption 1 and given Theorem 2 that a solution to $\alpha_i$, and $(\dot{\alpha}_i)$, $\forall i$ exists and is unique for all $t \geq 0$.

The following result characterizes the equilibria of the system.

**Theorem 3.** Suppose that $\alpha_i^* \in [0, \pi]$ and $\gamma_i \in (0, 1)$, $\forall i \in \mathcal{V}$. The system (5) is at equilibrium $e = 0$ if and only if $e = 0$.

Proof. The sufficiency of $e = 0$ is obvious. To prove necessity, suppose firstly that $e$ is in one of the six disjoint regions $R_{i+}$ or $R_{i-}$. Using (6) or (7) it is clear $\dot{\alpha}_i \neq 0$ for at least one $i$.

Now it remains to show there are no equilibrium points on the boundaries of the error regions. Denote such a boundary via

$$
\Sigma_{i+j-} = \{ \partial R_{i+} \cap \partial R_{j-} \} \setminus \{0\} = \Sigma_{i+j-}.
$$

(8)

and note it suffices to consider only boundaries with strictly positive length, i.e. a strictly positive 1-d Hausdorff measure. From the definition of $R_{i+}$, $\forall i$, and inspection of the error space, e.g. in Figure 1, it follows easily that

$$
\begin{align*}
\mathbf{e} \in \Sigma_{i+j-} \iff & \quad -\alpha_i^* < e_i < 0 \\
& \quad 0 < e_j < \pi - \alpha_j^* \\
& \quad \dot{e}_k = 0
\end{align*}
$$

which implies, using (4) and the fact $g_j \geq 0$ and $i_{ij} \geq 0$ $\forall i, j \in \{1, 2, 3\}$, that $\dot{e}_i > 0$ and $\dot{e}_j < 0$ and thus $\mathbf{e} \neq 0$. ■

The following theorem forms the basis of the stability proof.

**Theorem 4** (Poincare-Bendixson [Wiggins, 1990]). Let $M \subset \mathbb{R}^2$ be a compact, positively invariant two-manifold containing a finite number of fixed points. Let $x \in M$ and consider the $\omega$-limit set $\omega(x)$. Then one of the following possibilities holds:

1. $\omega(x)$ is an equilibrium point;
2. $\omega(x)$ is a closed orbit;
3. $\omega(x)$ consists of a finite number of fixed points $x_1, \ldots, x_m$ and orbits $\zeta$ with $\alpha(\zeta) = x$, and $\omega(\zeta) = x$, where $\alpha(\zeta)$ means the $\alpha$-limit set of every point $\zeta$.

The intuition behind the Poincare-Bendixson theorem is that all bounded trajectories in a planar region (or two-manifold) must converge to an equilibrium point, a limit cycle, or a union of fixed points and the trajectories connecting them. The next result concerns the closed orbits of (5).

**Theorem 5.** Suppose that $\alpha_i^* \in [0, \pi]$ and $\gamma_i \in (0, 1)$, $\forall i \in \mathcal{V}$. Then the system (5) has no closed orbits in $M$.

Proof. Consider the arcs between adjacent regions given by (8) with a strictly positive 1-d Hausdorff measure. There are six such ‘well-defined’ sets $\Sigma_{i+j-} = \Sigma_{i+j-}$. Now define

$$
\Sigma = \Sigma_{3+1-} \cup \Sigma_{1+2-} \cup \Sigma_{2+3-} \cup \\
\Sigma_{3-1+} \cup \Sigma_{1+2-} \cup \Sigma_{2+3+}
$$

and note $\Sigma \setminus \{0\} = \emptyset$. Any closed orbit must enclose the origin [Wiggins, 1990] and thus intersect every well-defined boundary $\Sigma_{i+j-}$. Consequently, if the origin is on a vertex of the simplex $M$, i.e. if the desired configuration is a collinear formation, then no closed orbits exist. Otherwise, the strategy is to show that any positive orbit $\psi^+(e)$ of (5) intersects $\Sigma$ in a strictly monotone sequence approaching the origin (if it intersects it in more than a point). That is, if $e_{m+1}$ is the $(m+1)$th intersection of $\Sigma$ then $|e_{m+1}| < |e_m|$.

**Proof.** Suppose that $e_{m+1}$ is the $(m+1)$th intersection of $\Sigma$ and thus intersect every well-defined boundary $\Sigma_{i+j-}$. Consider the arc between adjacent regions given by (8) and orbits $(\mathbf{e}, t)$ satisfying

$$
\begin{align*}
e & \in \Sigma_{i+j-} \Rightarrow e_i > 0, e_j < 0 \quad \text{and} \quad \dot{e}_j < 0
\end{align*}
$$

It follows that $|e_{m+1}| < |e_m|$. One can restart the argument at time $t = t_{m+1}$.

Case (i): If $\psi^+(e_m)$ is in $\Sigma_{i+j-}$ then $t_{m+} = t_{m+1}$ and $0 < e_j(t_{m+1}) < |e_j(t_m)|$, $|e_i(t_{m+1})| < |e_i(t_m)|$, $\forall i \neq m$ in $(t_m, t_{m+1}]$. The relevant boundaries of $R_{i+}$ are $\Sigma_{i+j-}$ and $\Sigma_{i+k-}$ for distinct $i, j, k \in \{1, 2, 3\}$. Now if $e_{m+1} \in \Sigma_{i+j-}$ then

$$
\int_{t_m}^{t_{m+1}} e_j(t) dt = 0 \Rightarrow \int_{t_m}^{t_{m+1}} e_j(t) dt < 0
$$

which immediately implies $|e_j(t_{m+1})| < |e_j(t_m)|$. Using (9) it follows that $|e_{m+1}| < |e_m|$ and one can then restart the argument at time $t = t_{m+1}$. Now if instead $e_{m+1} \in \Sigma_{i+k-}$ then $-\dot{e}_j > \dot{e}_k$ implies

$$
\int_{t_m}^{t_{m+1}} e_j(t) dt = -e_j(t_m) \Rightarrow \int_{t_m}^{t_{m+1}} e_j(t) dt < e_j(t_m)
$$

and since $e_j(t_m) = 0$ it follows that $|e_{k(t_{m+1})}| < |e_j(t_m)|$. The consequence of this last fact is that $|e_{m+1}| < |e_m|$ and one can restart the argument at time $t = t_{m+1}$.

Case (ii): If $\psi^+(e_m)$ is in $\Sigma_{i+k-}$ then the argument follows similarly to that given in case (i).

The preceding proof almost implies a solution of (5) converges to the origin. The next result makes this convergence precise.

**Theorem 6** (The Main Result). Suppose that $\alpha_i^* \in [0, \pi]$ and $\gamma_i \in (0, 1)$, $\forall i \in \mathcal{V}$. The equilibrium $e = 0$ of the error system (5) is globally asymptotically stable.

Proof. Consider $M^- = \text{cl}(M_i)$ where $\text{cl}(\cdot)$ denotes set closure. Note that $M^- \subset M$ is compact with a single equilibrium point and no closed orbits, via Theorems 3 and 5. Clearly, e(0) lives in $M_\mathcal{I} \subset M^-$ and $M_\mathcal{E}$ is positively invariant, via Theorem 1. Poincare-Bendixson then claims the $\omega$-set of any e(0) in $M^- \subset M$ contains only e = 0. Global asymptotic stability follows.

It is noteworthy that a formation of agents at equilibrium, i.e. with $\alpha_i = \alpha_i^*$, is invariant to dilation, rotation and translation of the entire formation or reflection of any agent i about the triangle edge formed by agents $i + 1$ and $i - 1$.

The main result concerns the global asymptotic stability of all desired configurations. However, using a linearization argument, e.g. see the Hartman-Grobman theorem [Wiggins, 1990], it is possible to comment on the convergence rate.
**Theorem 7.** If \( \alpha_i^* \in (0, \pi) \) and \( \gamma_i \in (0, 1) \) then solutions of (5) with any initial condition in \( \mathcal{M}_e \) will converge asymptotically to the origin and there exists a neighbourhood \( \mathcal{U} \) of the origin within which solutions converge at an exponential rate.

The analysis in this section holds for any continuously \(^3\) time-varying \( \gamma_i \in (0, 1) \). Thus, if each agent simply moves with a heading chosen within the range of values defined by (2) with \( \gamma_i \in (0, 1) \) then the desired formation will be achieved.

The stability result relies on a novel characterization of the error state space where certain regions are identified and within which, for any \( \gamma_i \in (0, 1) \), certain (sign) inequalities on the differential error equations can be established. These inequalities can be exploited to infer certain dynamical properties of the error system; e.g. the number of equilibria, closed orbits etc.

### 3. EXAMPLES OF TRIANGULAR FORMATION CONTROL WITH THE RELAXED CONTROL LAW

#### 3.1 Triangle to Triangle with Fixed Arbitrary \( \gamma_i \) Values

The first example shows how the formation converges to a desired triangle given a random initial triangle configuration. The desired formation is characterized by \( \alpha_1^* = \pi/4 \), \( \alpha_2^* = \pi/6 \) and \( \alpha_3^* = 7\pi/12 \). Here, \( \gamma_i = 0.1 \) for \( i \in \{1, 2\} \) and \( \gamma_3 = 0.7 \). The formation motion is illustrated in Figure 2 along with the convergence of \( |e_i| \) to zero.

![Fig. 2. The motion of the formation with a desired constraint of \( \alpha_1^* = \pi/6, \alpha_2^* = \pi/4 \) and \( \alpha_3^* = 7\pi/12 \). Here, \( \gamma_i = 0.1 \) for \( i \in \{1, 2\} \) and \( \gamma_3 = 0.7 \).](image)

#### 3.2 Triangle to Triangle with Time-Varying Random \( \gamma_i \) Values

The desired formation is the same as in the first example with \( \alpha_1^* = \pi/4 \), \( \alpha_2^* = \pi/6 \) and \( \alpha_3^* = 7\pi/12 \) and the initial formation is randomly generated. In this example, each \( \gamma_i \) is randomly chosen to be within \((0, 1)\) with a uniform distribution every \( \epsilon \) seconds for some small \( \epsilon > 0 \). The formation motion is illustrated in Figure 3 along with the convergence of \( |e_i| \) to zero.

![Fig. 3. The motion of the formation with a desired terminal constraint of \( \alpha_1^* = \pi/6, \alpha_2^* = \pi/4 \) and \( \alpha_3^* = 7\pi/12 \). Each \( \gamma_i \) is randomly chosen every \( \epsilon > 0 \) seconds.](image)

The figure illustrates the formation trajectory as it converges to the desired shape given randomly changing \( \gamma_i \), \( \forall i \) values.

#### 3.3 Line to Triangle with Time-Varying Random \( \gamma_i \) Values

In this example, the desired formation is a random triangle and the initial configuration is a line with all agents collinear. Each \( \gamma_i \) is randomly chosen to be within \((0, 1)\) with a uniform distribution every \( \epsilon \) seconds for some small \( \epsilon > 0 \). The formation motion is illustrated in Figure 4 along with the convergence of \( |e_i| \) to zero.

This example illustrates that the control law is not affected by initial agent collinearity.

#### 3.4 Triangle to Line with Time-Varying Random \( \gamma_i \) Values

Here, the desired formation is characterized by \( \alpha_1^* = \alpha_2^* = 0 \) and \( \alpha_3^* = \pi \) and the initial configuration is a random triangle. Each \( \gamma_i \) is randomly chosen as in the previous two examples. The formation motion is illustrated in Figure 5 along with the convergence of \( |e_i| \) to zero.

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\(^3\) In fact, \( \gamma_i \in (0, 1) \) can change discontinuously but the given analysis must be modified using the theory of discontinuous differential equations.
This example illustrates that it is possible to steer an arbitrary initial triangle formation to a collinear formation.

3.5 Line to Line with Time-Varying Random $\gamma_i$ Values

Finally, consider the case of changing from an initial collinear formation with $\alpha_1 = 0$, $\alpha_2 = \pi$ and $\alpha_3 = 0$ to another (desired) collinear formation with $\alpha_1^* = 0$, $\alpha_2^* = 0$ and $\alpha_3^* = \pi$. Again, each $\gamma_i$ is randomly chosen as in the previous three examples. The order of the agents along the line changes from the initial formation to the desired formation. The formation motion is illustrated in Figure 6 along with the convergence of $|e_i|$ to zero.

Note that agents 2 and 3 do not collide but do indeed swap places in the formation configuration.

3.6 Remarks

Note that randomly time-varying $\gamma_i \in (0, 1)$ values were chosen during most simulations for simplicity and to illustrate the algorithm’s convergence invariance to this parameter. Other time-varying or fixed strategies could be employed by each agent to decide $\gamma_i \in (0, 1)$ and thus $\beta_i$ within the acceptable region of values. The control law is said to be relaxed since each agent is not restricted to a particular heading and, indeed, is only required to move either toward the interior or the exterior of the formation dependent on the sign of its control error. Suppose Assumptions 1 and 2 hold.

4. NOVEL EXTENSIONS TO THE PROBLEM

4.1 Robustness to a Single Agent Motion Failure

The proposed control law is generally robust to a single agent motion failure, i.e. $\dot{p}_i = 0$ for one $i \in V$ with $e_i \neq 0$.

With no loss of generality, assume agent 1 cannot move in space. However, assume that $\alpha^*_i$ is specified $\forall i \in V$ and that, in general, $\alpha_i(0) \neq \alpha^*_i, \forall i$. Agents 2 and 3 both implement the previously designed control law. The modified error system is

$$\dot{e} = \begin{bmatrix} 0 & f_{12} & f_{13} \\ 0 & -g_2 & f_{23} \\ 0 & f_{32} & -g_3 \end{bmatrix} e$$

where $e = [e_1 \ e_2 \ e_3]^T$ and where $\dot{e}$ evolves on $\mathcal{M}_e$.

Theorem 8. Assume agent 1 suffers a motion failure, i.e. $\dot{p}_1 = 0$. Suppose that $\alpha^*_i \in [0, \pi)$, $\gamma_i \in (0, 1)$, $\forall i \in V$ and $\alpha_i(0) \neq \pi$ and $\alpha_i(0) \neq 0$ for $i \in \{2, 3\}$. The equilibrium $e = 0$ of the system (10) is globally asymptotically stable.

Proof. The steps used to prove Theorem 6 can be applied.

Initial line formations cannot be allowed if the agent experiencing motion failure is in between the remaining two agents (because the remaining two agents will drive directly toward the agent experiencing motion failure until collision). The idea in this subsection is applicable to a leader-follower scenario where the agent that fails to apply its relevant control law can be interpreted as the leader while the other two agents are forced to form the required triangular shape alone.
The next example in Figure 7 illustrates the algorithm’s robustness to a single agent motion failure. Again, $\gamma_i$, $\forall i$ is randomly (uniformly) chosen in (0, 1) every $\epsilon$ seconds for some small $\epsilon > 0$. The desired angles are $\alpha_1^* = \alpha_2^* = \alpha_3^* = \pi/3$ and the initial triangle is randomly generated. Agent 1 does not move for the duration of the simulation. Figure 7 illustrates that all errors converge to zero even when a single agent does not move.

Other $\gamma_i$ values can be considered. In this section it is shown that not only is the controller relaxed but it is also robust.

4.2 Stabilizing an Arbitrary Moving Triangular Shape

The proposed control law can be applied to a group of agents which are otherwise undergoing motion. Let $\overline{p}_i = \overline{p}_i [\cos \beta_i, \sin \beta_i]^T$ where $\overline{p}_i$ and $\beta_i$ are control inputs to be defined. Consider a group speed input $v_g$ and heading $\beta_g$ known to all agents. Then, define the control inputs for agent $i$ as

$$\overline{p}_i = v_i^2 + v_g^2 + 2v_i v_g \cos (\beta_g - \beta_i)$$

$$\beta_i = \arctan \left( \frac{v_i \sin \beta_i + v_g \sin \beta_g}{v_i \cos \beta_i + v_g \cos \beta_g} \right)$$

where $v_i = |(\alpha_i^* - \alpha_i)|k$ with constant $k > 0$ and

$$\beta_i = \begin{cases} 
\alpha_i \gamma_i + \max(\phi_i(i+1), \phi_i(i-1)), & \gamma_i \leq \pi, \ e_i < 0 \\
\alpha_i \gamma_i + \max(\phi_i(i+1), \phi_i(i-1)), & \gamma_i > \pi, \ e_i < 0 \\
\alpha_i \gamma_i + \max(\phi_i(i+1), \phi_i(i-1)), & \gamma_i \leq \pi, \ e_i > 0 \\
\alpha_i \gamma_i + \max(\phi_i(i+1), \phi_i(i-1)), & \gamma_i > \pi, \ e_i > 0 
\end{cases}$$

where $\phi_i = |\phi_i(i+1) - \phi_i(i-1)| \in [0, 2\pi]$ and $0 < \gamma_i < 1$ as before. Similarly, $\gamma_i$ can be time-varying and as long as $0 < \gamma_i < 1$ then it can be chosen at run-time by the agent.

Theorem 9. Suppose that $\alpha_i^* \in [0, \pi], \gamma_i \in (0, 1), \forall i \in V$ and suppose each agent applies the $\overline{p}_i$ and $\beta_i$ detailed in this subsection. Then, as $t \rightarrow \infty$ the angles $\alpha_i \rightarrow \alpha_i^*$ given any initial $\alpha_i$, $\forall i$, and any group input $v_g$ and heading $\beta_g$.

Figure 8 depicts three examples of formation convergence in the presence of a common group motion command. In the first and third example, for simplicity, $\gamma_i$, $\forall i$ is randomly chosen every $\epsilon > 0$ seconds within (0, 1) with a uniform distribution. The desired formation is an equilateral triangle in each example and, in the second and third example, the initial formation is a random triangle. (The axis is not square in any of the figures).

In the [Left] figure, the initial triangle is a line configuration. The common group control inputs are $v_g = 1$ and $\beta_g = \pi/4$.

In the [Middle] figure, $v_g = 1$ and $\beta_g = \pi/4$. Also, $\gamma_1 = 0.9|\sin(t/2)|, \gamma_2 = 0.9|\cos(t)|$ and $\gamma_3$ is randomly chosen every $\epsilon$ seconds within (0, 1) with a uniform distribution.

Finally, in the [Right] figure, the common group control inputs are $v_g = 1$ and $\beta_g = \sin(t/2)$.

5. CONCLUDING REMARKS

The problem of bearing-only triangular formation control was considered in this article. A distributed control law was designed for each agent that is based only on the agent’s locally measured bearings. The control law is relaxed in the sense that each agent is free to choose their own heading within a relatively large region of values. Given this relaxed control strategy, a strong convergence result was established which guarantees global convergence of the formation to the desired shape. The relaxed control law also ensures that collisions are avoided naturally. The control scheme was shown to be robust to agent motion failures and additional group motion inputs.
In [Bishop, 2011] a similar formulation to the one proposed here was introduced with four agents (except the controller was a typical feedback controller and the notion of a relaxed control law was not considered). Thus, an extension of the relaxed control notion to the four-agent case may be straightforward. In general, further investigation of the bearing-only formation control problem is warranted. The work presented here provides a novel basis on which to build any future study. Moreover, the proposed controller motivates further work on relaxed multi-agent control laws that permit individual agents some freedom of movement while still ensuring the system achieves its desired control objectives.

REFERENCES