Finite-Time and Asymptotic Stabilization of Car-like Kinematics with Amplitude-Limited Control Input

Maciej Michalek * Krzysztof Kozłowski *

* Chair of Control and Systems Engineering, Poznan University of Technology (PUT), Piotrowo 3A, 60-965 Poznań, Poland (e-mail: {maciej.michalek/krzysztof.kozlowski}@put.poznan.pl)

Abstract: The paper proposes novel control strategies for the front-axle driven car-like kinematics dealing with a set-point control problem in the presence of amplitude-limits imposed on the control input. Two alternative domains of a front-wheel steering angle (bounded and unbounded one) are considered. Introducing fictitious inputs and application of the Vector-Field-Orientation design concept a family of finite-time and asymptotic controllers is proposed. Simple scaling procedure applied for the nominal (unlimited) control input guarantees satisfaction of the amplitude-limits imposed by the user preserving simultaneously the vehicle’s motion geometry. Theoretical considerations are verified by numerical simulations.

Keywords: car-like kinematics, finite-time/asymptotic stabilization, control limitations

1. INTRODUCTION

Stabilization of a fixed configuration is the most difficult control task defined for nonholonomic mobile robots. It is a direct consequence of the Brockett’s theorem which excludes any continuous, time-invariant state-feedback as an asymptotic solution. In practice, the set-point motion task is the most easily defined problem since it does not require virtually any motion planning stage. Many control algorithms for nonholonomic mobile robots have been proposed in the literature, mostly for the unicycle model. Stabilization problem for the car-like kinematics – more difficult in control – has attracted some attention, see for instance Luca et al. (1998), Astolfi (1995), Samson (1993), and the recent paper Morin and Samson (2009). The most popular way of treating a set-point control problem for a car-like robot is a transformation of its model into a chained system and then application of a selected general algorithm dedicated for the latter model, Luo and Tsiotras (2000), Alamir et al. (2003). This approach, despite its generality, has some limits resulting from locality of the transformation and from difficulties in prediction of transient behavior of a controlled system in the original configuration space (where it is defined in practice).

This paper proposes an alternative control design approach based on the Vector-Field-Orientation (VFO) concept presented recently for the unicycle model in Michalek and Kozłowski (2010). Extension of the VFO method for the car-like kinematics leads to a family of finite-time and asymptotic stabilizers (the finite-time control concept for the nonholonomic chained system can be found in Wu et al. (2005); for general treatment of finite-time convergence see for example Bhat and Bernstein (2000)). The VFO controllers belong to the discontinuous stabilizers, they are derived and defined in the original configuration space of the robot (they do not suffer from locality of any auxiliary state or input transformation), and are characterized by fast-converging and non-oscillatory transients in a closed-loop system and very simple parameter tuning (non-sensitive to vehicle initial conditions). In the proposed method the amplitude-limits of control inputs are directly taken into account on a design stage guaranteeing preservation of imposed bounds within the whole control time-horizon. Two domains of a vehicle steering angle (bounded and unbounded one), possible for the car-like robots, are considered with explanation of their influence on time-evolution of a controlled vehicle.

1.1 Problem formulation

A design objective considered in this paper is to determine a feedback stabilizing control law for the front-axle driven car-like mobile robot represented by the following kinematic model (compare Fig. 1.):

\[
\begin{bmatrix}
\dot{\beta} \\
\dot{\theta} \\
\dot{x} \\
\dot{y}
\end{bmatrix} = U_1 + 
\begin{bmatrix}
0 \\
1/L \sin \beta \\
\cos \beta \cos \theta \\
\cos \beta \sin \theta
\end{bmatrix} U_2
\]  

(1)

with a configuration vector \( q = [\beta \ \theta \ x \ y]^T \) where: \((x, y) \in \mathbb{R}^2\) are position coordinates of the guidance point \( P \) attached to the middle of the rear wheel/axle, \( \theta \in \mathbb{R} \) is an orientation of the vehicle body, and \( \beta \) is a steering angle of the front vehicle wheel. The two available kinematic control inputs of the vehicle, \( U_1 \) and \( U_2 \), have a meaning of an angular steering velocity of the front wheel, and a longitudinal driving velocity of the front wheel, respectively.

* This work was supported in part by the grant No. N N514 406236 and in part by the statutory fund DS-03/191/11.
Fig. 1. Definition of configuration coordinates and control inputs for the car-like mobile robot.

We are interested in two alternative domains for the steering angle:

- **D1:** \( \beta \in (-\infty, +\infty) \Rightarrow \beta \in \mathbb{R} \)
- **D2:** \( \beta \in [-\beta_m, +\beta_m] \), \( \beta_m = \frac{\pi}{2} \)

Both domains permit an unlimited curvature for the vehicle motion. For domain D1 an unconstrained pivoting of the front wheel is possible, while the case D2 yields a limited steering angle leading to simpler practical realization of a steering mechanism. Since both mechanical solutions are possible in robotics, the motion characteristics for the two domains will be examined in the sequel.

Furthermore, we assume that the inputs \( U_1, U_2 \) must be amplitude-limited satisfying the following relation:

\[
\forall \tau \geq 0 \quad |U_1(\tau)| < u_{1M}, \quad |U_2(\tau)| < u_{2M},
\]

where \( u_{1M}, u_{2M} > 0 \) are the finite bounds imposed by physical limitations of drives or artificially imposed by a user. For clarity of the subsequent analysis let us introduce the notion of unlimited, or nominal, case of the inputs taking \( U_1 := u_1 \) and \( U_2 := u_2 \) with \( u_1, u_2 \in \mathbb{R} \). The case of amplitude-limited inputs will be indicated substituting \( U_1 := u_{1s} \) and \( U_2 := u_{2s} \) where \( u_{1s}, u_{2s} \) meet inequalities in (2).

To precisely formulate the control design problem let us define a configuration error:

\[
e(\tau) = \begin{bmatrix} e_\beta(\tau) \\ e_\theta(\tau) \\ e_x(\tau) \\ e_y(\tau) \end{bmatrix} \triangleq \begin{bmatrix} f_\beta(\beta - \beta(\tau)) \\ f_\theta(\theta - \theta(\tau)) \\ x - x(\tau) \\ y - y(\tau) \end{bmatrix},
\]

where \( f_{\beta,\theta}(\cdot) : \mathbb{R} \mapsto (-\pi, \pi] \), \( e \in (-\pi, \pi]^2 \times \mathbb{R}^2 \), and

\[
q = [\beta, \theta, x, y]^T, \quad \theta \triangleq 0, \quad \theta \in (-\pi, \pi]
\]

is a fixed reference point.

**Problem 1.** For kinematics (1) determine an amplitude-limited feedback control law \( U = U(q, v, \cdot) \), \( U = [U_1, U_2]^T \) satisfying (2) and guaranteeing convergence of the configuration error (3) in the sense that \( \lim_{\tau \to T} \| e(\tau) \| \leq \varepsilon \) with \( \varepsilon \geq 0 \), being some vicinity of the origin, and \( T \leq \infty \) being the convergence time-horizon.

Note that the above formulation admits two cases of convergence precision: asymptotic for \( \varepsilon = 0 \) and practical one if \( \varepsilon > 0 \). Also two kinds of convergence rate are included: infinite-time if \( T = \infty \) and the finite-time when \( T < \infty \).

Subsequent sections explain the novelty of solutions to Problem 1 by application of the VFO design concept presented recently for the unicycle kinematics in Michalek and Kozlowski (2010) and extended here for car-like kinematics.

### 2. VFO CONTROL DESIGN

The VFO approach is a direct consequence of geometrical features connected with the mobile robot nonholonomic kinematics. The method has been originally proposed for the unicycle model decomposing its motion control task into orienting and pushing subprocesses related to azimuthal and radial distances, respectively. Since the unicycle model is an archetypical example where application of the VFO method proves to be especially effective (see Michalek and Kozlowski (2010)), the vehicle kinematics (1) will be reformulated to obtain the unicycle-like description for the vehicle body kinematics.

#### 2.1 General concept of control design

Let us reformulate the original kinematics (1) by defining the fictitious inputs:

\[
v_1 := \frac{U_2}{L} \sin \beta, \quad v_2 := U_2 \cos \beta,
\]

which allow rewriting kinematics (1) as follows:

\[
\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ \cos \beta \theta \\ \sin \beta \theta \end{bmatrix} v_2.
\]

Subsystem (7) represents the vehicle body kinematics, while (6) models the steering angle dynamics. It is clear, comparing Fig. 1, that \( v_1 \) has a meaning of angular velocity of the vehicle platform, while \( v_2 \) is a longitudinal velocity of the guidance point \( P (v_2 = v_1 \equiv U_{2s} \) in Fig. 1).

Since (7) has a form of the unicycle model, one may directly apply the VFO control approach to this subsystem defining the desired motion behavior for a vehicle body – it will be presented in subsection 2.3. Let us assume for a moment that fictitious inputs \( v_1 \) and \( v_2 \) of (7) have been designed in the VFO framework. Since the designed inputs \( v_1 \) and \( v_2 \) represent now the desired control action for a vehicle body, one is interested in realizing them by the physical inputs \( U_1 \) and \( U_2 \) available in the original car-like kinematics. Considerations in this subsection will be presented for the case of nominal (unlimited) control inputs. We indicate it replacing \( U_1 \) and \( U_2 \) by the nominal inputs \( u_1 \) and \( u_2 \) in the kinematics (1), and consequently in equations (5) and (6). Combining equations (5) allows recovering two simple relations for the nominal case:

\[
u_2 = v_2 \cos \beta + L v_1 \sin \beta,
\]

\[
\tan \beta = \frac{u_2 - v_2}{u_1 - v_1} \quad \text{for} \quad \| v \| \neq 0,
\]

where \( v = [v_1, v_2]^T \) and \( \| v \| = \sqrt{v_1^2 + v_2^2} \). They describe how to define input \( u_2 \) and how to shape time-behavior of \( \beta \) to obtain the control action of fictitious inputs \( v_1 \) and \( v_2 \). In general, (9) cannot be met instantaneously due to the steering dynamics represented by (6). Hence, we propose to introduce an auxiliary steering variable \( \beta_a \), defined in different ways for particular domains of a steering angle, as follows:

\[
\beta_a \triangleq \begin{cases} \text{Atan}2c(\kappa L v_1, \kappa v_2) & \text{for D1}, \\ \text{arctan} \left( L v_1/v_2 \right) & \text{for D2}, \end{cases}
\]

where \( \kappa \triangleq \text{sgn}(u_2), \text{sgn}(\cdot) \in \{-1, +1\} \), \( u_2 \) is computed according to (8), and \( \text{Atan}2c(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \) is a continuous function.

3498
version\(^1\) of the four-quadrant function Atan2 (·, ·) : \(\mathbb{R} \times \mathbb{R} \mapsto (-\pi, \pi]\). Defining now an auxiliary steering error
\[
e_{\beta a}(\tau) \triangleq \beta_a(\tau) - \beta(\tau),
\]
allows one to satisfy condition (9) by making \(e_{\beta a}(\tau)\) converge to zero. We propose a family of steering laws which accomplishes this objective, namely:
\[
u_1 \triangleq k_\delta \text{sign}(e_{\beta a})|e_{\beta a}|^\delta + \beta_a, \quad \delta \in (0, 1],
\]
where \(k_\delta > 0\) and \(\delta\) are design parameters, \(\text{sign}(\cdot) \in \{-1, 0, +1\}\), and
\[
\dot{\beta}_a = \frac{L(v_1v_2 - v_1v_2)}{L^2v_1^2 + v_2^2}
\]
is a time-derivative of the auxiliary steering variable given by (10). Taking \(\delta = 1\) in (12) yields a linear proportional feedback controller \(u_1 = k_\delta e_{\beta a} + \beta_a\) with a velocity feed-forward term \(\dot{\beta}_a\). Taking \(\delta\) from the range (0, 1) leads to a set of finite-time controllers (see Bhat and Bernstein (2000)) — more detailed convergence analysis will be conducted in Section 3.

Remark 1. Formulas (10) and (13) are not determined for instantaneous \(t\) when \(\|v(\tau)\| = 0\). In this case we may introduce additional definitions, for instance \(\beta_a(\tau) := \lim_{\tau \to t^-} \beta_a(\tau)\) and \(\beta_a(\tau) := 0\) activated at every \(\tau\) when \(\|v(\tau)\| = 0\). In practice, one may prefer replace the last equality condition by inequality \(\|v(\tau)\| < \xi\) with \(\xi > 0\) being a sufficiently small neighborhood of zero.

Summarizing, the general control scheme (in the nominal case) for kinematics (1) consists of three stages: 1\(^o\) determine the fictitious inputs \(v_1, v_2\) for the vehicle body (unicycle-like) subsystem (7) applying the VFO approach, 2\(^o\) using designed fictitious inputs determine the control input \(u_2\) according to (8), and then compute auxiliary signals (10), (11), and time-derivative term (13), 3\(^o\) compute the nominal steering control input according to (12).

Subsection 2.3 explains stage 1\(^o\) of the above scheme.

2.2 Scaling procedure for control input limiting

So far, we have not considered control input limitations imposed in (2) — the nominal control scheme proposed above does not guarantee satisfaction of control bounds. In order to meet the control limits we propose to apply a simple scaling procedure, which directly follows the nominal control scheme computations. Denote by \(u = [u_1 u_2]^T\) the nominal control input with components computed according to (12) and (8), respectively. Introducing now the strictly positive scaling function \(s(\tau) : \mathbb{R}_0^+ \mapsto (0, 1]\) as
\[
s(\tau) \triangleq \frac{1}{d(\tau)}, \quad d(\tau) = \max\left\{\frac{|u_1(\tau)|}{u_{1M}}, \frac{|u_2(\tau)|}{u_{2M}}\right\},
\]
the amplitude-limited control input \(u_s = [u_{1s} u_{2s}]^T\) is applicable to kinematics (1) and satisfying the bounds imposed in (2) takes the following form:
\[
\nu_s(\tau) \triangleq s(\tau) \cdot \nu(\tau).\]
Note that after scaling, the direction and sign of the nominal control input \(\nu\) is preserved: \(u_s \parallel u\). It is an important feature of the above procedure leading to preservation of the vehicle motion geometry.

2.3 VFO design strategy for the unicycle-like subsystem

Now we explain the VFO design of fictitious inputs for the vehicle-body kinematics (7). We begin by introduction of the so-called convergence vector field:
\[
h = [h_1 h_2 h_3]^T = \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix} \in \mathbb{R}^3,
\]
defined in the tangent space of (7). Assume temporarily that (16) is properly designed defining at every state point \(\overline{q} = [\theta x y]^T\) the convergence velocity vector \(h(\overline{q}, \overline{q}_1, \cdot)\) for subsystem (7). By the convergence velocity we understand here an instantaneous velocity vector \(h(\overline{q}, \overline{q}_1, \cdot)\), which, if followed by system (7), guarantees convergence to the reference point \(\overline{q}_1 = [\theta_1 x_1 y_1]^T\). Hence, we are interested hereafter in fulfilling the relation \(\lim_{\tau \to \infty} (\overline{q}(\tau) - h(\tau)) = 0\), which can be obtained by appropriately designed fictitious inputs \(v_1\) and \(v_2\). Using model (7) one can rewrite the relation as follows:
\[
\lim_{\tau \to \infty} \begin{bmatrix} v_1(\tau) - h_1(\tau) \\ v_2(\tau) \cos \theta(\tau) - h_2(\tau) \\ v_2(\tau) \sin \theta(\tau) - h_3(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.\]
Combining terms in (17) one can obtain one relation and two design formulas equivalent to (17). The former one determines desired time-evolution for \(\theta\) variable, namely:
\[
\lim_{\tau \to \infty} [\theta(\tau) - \text{Atan2c}(\text{sign}(v_1)h_1(\tau), \text{sign}(v_2)h_2(\tau))] = 0,
\]
with \(\text{sign}(\cdot) \in \{+1, -1\}\), and \(\text{Atan2c}(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}\) being a continuous version\(^2\) of function Atan2 (\(\cdot, \cdot\)) : \(\mathbb{R} \times \mathbb{R} \mapsto (-\pi, \pi]\). (18) is called the orienting condition. It explains how to reorient the second vector field of (7) — by change of \(\theta\) variable — in order to match its orientation with that defined by \(h(\overline{q}, \overline{q}_1, \cdot)\). Remaining two formulas derived from (17) represent the general VFO control strategy and take the form: \(v_1 = h_1\) and \(v_2 = h_2 \cos \theta + h_3 \sin \theta\). Noting that the last formula can be equivalently written as \(v_2 = h^* g_2^* (\theta) = \|h^*\| \cos \alpha\), where \(g_2^* (\theta) = [\cos \theta \sin \theta]^T\), \(\alpha = \angle(g_2, h^*)\), and \(\cos \alpha = (h^T g_2^*)/\|h^*\|\), we propose to define the general VFO control law as follows:
\[
v_1 \triangleq h_1, \quad v_2 \triangleq \|h^*\|^\chi \cos \alpha = \|h^*\|^{\chi-1} (h^T g_2^*),
\]
where \(\chi \in [0, 1]\) is a design parameter. Introduction of this parameter allows flexible selection between asymptotic (when \(\chi = 1\)) and finite-time (when \(\chi \in (0, 1)\)) control strategy — it will be analyzed in the sequel.

Summarizing the above reasoning, satisfaction of orienting condition (18) and usage of definitions (19) and (20) simultaneously guarantees that the general VFO control objective given by (17) will be met.

Condition (18) involves more attention, since it cannot be satisfied instantaneously due to the integral relation given by the first row of (7). Therefore we propose to introduce an auxiliary orienting variable
\[
\theta_a(\tau) \triangleq \text{Atan2c}(\sigma h_3(\tau), \sigma h_2(\tau)) \in \mathbb{R},
\]
Function Atan2c (\(\cdot, \cdot\)) has been introduced to ensure continuity of signals from (21) and (22) — for more details we refer to Michalek and Kozlowski (2010).

\(^1\) Values of \(\beta_a\) obtained using Atan2c (\(\cdot, \cdot\)) function are equivalent to time-integration of \(\beta_a\) term denoted in (13). More details on computations of Atan2c (\(\cdot, \cdot\)) function can be found in Michalek and Kozlowski (2010).

\(^2\) Function Atan2c (\(\cdot, \cdot\)) has been introduced to ensure continuity of signals from (21) and (22) — for more details we refer to Michalek and Kozlowski (2010).
with \( \sigma \in \{ +1, -1 \} \) being a decision factor, which allows one to choose desired motion strategy for the vehicle body (forward/backward) – it will be defined further. Defining now an auxiliary orienting error
\[
e_{\theta a}(\tau) \triangleq \dot{\theta}_a(\tau) - \theta(\tau), \quad e_{\theta a}(\tau) \in \mathbb{R}
\]
(22) one may consider satisfaction of (18) as equivalent to driving (22) to zero. From (7) and (19) it is evident that one can accomplish it using the first fictitious input \( v_1 \) together with a properly defined \( h_1 \) component. Let us propose to take:
\[
h_1 \triangleq k_b \text{sgn}(e_{\theta a}) [e_{\theta a}]^{\xi} + \dot{\theta}_a
\]
(23) where \( k_b > 0 \) and \( \xi \in (0, 1] \) are the design parameters, and it is a time-derivative of (21). The remaining part of the convergence vector field we propose to define as follows:
\[
\begin{aligned}
&h^* \triangleq k_p e^* + \nu^* = \left[ k_p e_x + \nu_x \right], \\
&\nu^* = \left[ \nu_{y} \right] \triangleq -\eta \sigma \left\| e^* \right\| g_2^T(\theta) \quad (26)
\end{aligned}
\]
with an additional design factor \( \eta \in (0, k_p) \), and \( g_2^T(\theta) = [\cos \theta_1 \sin \theta_1]^T \). The term given by (26) can be treated as a feed-forward virtual velocity term, which proves to be helpful in shaping the transient states of the vehicle body.

The last term which remains to be determined is the decision factor \( \sigma \) used in (21) and (26). We propose to take it as follows
\[
\sigma \triangleq \text{sgn}(e_{\theta a}),
\]
(27) where \( e_{\theta a}^0 \equiv e_{\theta a}(\tau = 0) \) and \( e_{\theta a}^0 = e_x \cos \theta_1 + e_y \sin \theta_1 \) is an error component along \( z \)-axis expressed in a local frame attached to the reference point.

Summarizing, one can say that the VFO control strategy for the fictitious inputs results directly from definitions (19) and (20) with particular terms determined by (21) to (27).

3. CONVERGENCE ANALYSIS AND THE MAIN RESULT

Now we analyze time-evolution of stabilization error (3) in the resultant closed-loop system obtained by application to the car-like kinematics (1) the nominal inputs given by (12) and (8) together with fictitious inputs defined by (19) and (20). The subsequent analysis will be conducted simultaneously for asymptotic and finite-time controllers resulting from the choice of \( \delta, \chi, \) and \( \xi \) exponents introduced in (12), (20), and (23), respectively (not considered in Michalek and Kozlowski (2010)). We assume hereafter, that for the asymptotic case one takes \( \delta = \chi = \xi = 1 \). For the finite-time case all the exponents should be taken from the range \((0, 1)\) (see Bhat and Bernstein (2000)).

Firstly, we take into account behavior of auxiliary error \( e_{3\beta a}(\tau) \). Let us introduce the positive-definite function \( V_1 = \frac{1}{2} e_{3\beta a}^2 \). Its time-derivative can be calculated as follows:
\[
\dot{V}_1 = e_{3\beta a} \dot{e}_{3\beta a} = e_{3\beta a} (\dot{\beta}_a - \beta) = e_{3\beta a} (\dot{\beta}_a - \theta_1) = (12) - k_b \left| e_{3\beta a} \right|^\alpha + 1 = -k_b \left| e_{3\beta a} \right|^\alpha + 1 = -k_b \sqrt{2^{\alpha + 1}} V_1 (\dot{\theta}_a)^2. \]

For \( \delta = 1 \) one obtains \( \dot{V}_1 = -2k_b V_1 \) and asymptotic (exponential) convergence of \( e_{3\beta a} \) error:
\[
\lim_{\tau \to \infty} e_{3\beta a}(\tau) = 0. \quad (28)
\]

On the other hand, choosing \( \delta \in (0, 1) \) implies finite-time convergence (according to results presented in Bhat and Bernstein (2000)), namely:
\[
\lim_{\tau \to \tau_1} e_{3\beta a}(\tau) = 0, \quad \text{where} \quad \tau_1 = \frac{2V_1}{c_1 (1 - \delta)}, \quad (29)
\]
and \( c_1 = k_b \sqrt{2^{\alpha + 1}} \), \( V_1 = \frac{1}{2} e_{3\beta a}^2 \). Using now (10) one can write the following useful relation (valid for \( \| v \| \neq 0 \)):
\[
\lim_{\beta \to \beta_a} \left[ \tan \beta - \frac{L v}{v_2} \right] = 0, \quad (30)
\]
which will be utilized in the sequel.

Secondly, we analyze convergence of \( e_{\theta a} \) error. To do this one may combine the second equation of (1) with (8) to write, that for \( \beta \to \beta_a \) one gets:
\[
\dot{\theta} = (1/L)(\tan \beta_2 v_2) \cos^2 \beta + v_1 \sin^2 \beta = \frac{(1/L)L V_1 \cos^2 \beta_a + v_1 \sin^2 \beta_a}{\cos^2 \beta_a} = \frac{1}{\cos \theta_1} \quad (30)
\]
Substituting now (19) into the above relation with definition (23) yields the following equation:
\[
e_{\theta a} = -k_b \text{sgn}(e_{\theta a}) |e_{\theta a}|^\xi.
\]
(31)

For \( \xi = 1 \) we obtain \( e_{\theta a} = -k_b e_{\theta a} \) and asymptotic convergence of \( e_{\theta a} \) error:
\[
\lim_{\tau \to \infty} e_{\theta a}(\tau) = 0. \quad (32)
\]
Taking \( \xi \) from the range \((0, 1)\), and assuming that \( \delta \in (0, 1) \), implies finite-time convergence:
\[
\lim_{\tau \to \tau_2} e_{\theta a}(\tau) = 0, \quad \text{where} \quad \tau_2 = \tau_1 + \frac{2V_1 (1 - \xi)/2}{c_2 (1 - \xi)}, \quad (33)
\]
with \( c_2 = k_b \sqrt{2^{\alpha + 1}} \) and \( V_2 = \frac{1}{2} e_{\theta a}^2 (\tau_1) \).

Thirdly, we examine time-behavior of the position error \( e^*(\tau) \). Since (8) is met instantaneously, as a part of control strategy, and (30) holds as a result of steering control action defined by (12), the fictitious inputs \( v_1 \) and \( v_2 \) can be treated as a direct inputs influencing the vehicle body motion represented by subsystem (7). Therefore we now proceed over our analysis considering time-evolution of subsystem (7) in response to the fictitious inputs defined in (19) and (20). Rewriting the position error as\( e^* = q_1^* - q^* \), where \( q_1^* = [x_t \ y_t]^T \), gives \( e^* = -q^* = -g_2^T v_2 \) (compare (7)). Let us write definition (20) in the form \( v_2 = \rho \bar{v}_2 \), where \( \rho = \| h^* \|^{1/\chi} \) and \( \bar{v}_2 = \| h^* \| \cos \alpha \). Therefore one can write: \( e^* = -g_2^T \bar{v}_2 + \rho \bar{v}_2 - \rho h^* \). The latter equation can be reformulated using definition (25) yielding the following differential equation of the position error dynamics:
\[
\dot{e}^* + \rho \bar{v}_2 e^* = \rho \bar{v}_2 - \rho e^* \quad (34)
\]
with the perturbing term \( r = r(e^*, e_{\theta a}, \cdot) = h^* - g_2^T v_2 \). It can be easily shown that the following relation holds:
\[
\| r \| = \| h^* \| \gamma(e_{\theta a}), \quad \gamma(e_{\theta a}) = \sqrt{1 - \cos^2 \theta_1} e_{\theta a}^\xi, \quad (35)
\]
where we used the fact that \( \cos \theta_1 = \sigma \cos \alpha \). Introducing the positive definite function \( \dot{V}_3 = \frac{\sigma}{2} e^* e^T e^* \) its time-derivative may be estimated as follows:
\( \dot{V}_3 = e^{T}T e^* \) \(^{(34)}\) \( = e^{T}T (-pk_p e^* + pr - pv^*) \) \( \leq -\rho(k_p \|e^*\|^2 + e^{*T}r + e^{*T}v^*) \) \( \leq -\rho(k_p \|e^*\|^2 - \|e^*\| r - \|e^*\| v^*) \) \( \leq -\rho \|k_p - \gamma \|e^*\|^2 \) \( = -\rho \|k_p - \gamma \|\rho \|e^*\|^2. \) \(^{(36)}\)

with \( \gamma = \gamma(e_{\theta a}) \) defined in \((35)\). Since \( \rho \) is the positive definite function, the time-derivative \( \dot{V}_3 \) will be negative definite for \( \gamma(e^*) \) > 0. It leads to the following convergence condition: \( \gamma(k_p - \gamma)/(k_p + \gamma) \), where \( \gamma > 0 \) < 1 because \( \gamma \in (0, k_p) \). Using now the definition of \( \gamma \) included in \((35)\), and recalling convergence results \((32)\) and \((33)\) we can conclude that:

\[ \exists \tau > 0: \forall \tau > \tau, \gamma(e_{\theta a}^*(\tau)) < \frac{k_p - \gamma}{k_p + \gamma}. \] \(^{(37)}\)

Moreover, the finite-time convergence result \((33)\) allows concluding a stronger result, namely:

\[ \forall \tau > \tau, \gamma(e_{\theta a}^*(\tau)) = 0 \Rightarrow \forall \tau > \tau, \gamma(\gamma(\tau)) = k_p - \gamma. \] \(^{(38)}\)

To formulate conclusions about position error behavior we must refer to the exponent \( \chi \) selection in the definition of \( \rho = \|h^*\|^{-1} \). Taking \( \chi = 1 \) one obtains \( \dot{V}_3 \leq -\chi e_{\theta a}^* \|e^*\|^\chi \) \((36)\). Using \((37)\) allows one to conclude about asymptotic convergence of position error to zero:

\[ \lim_{\tau \to \infty} \|e^*(\tau)\| = 0 \] \(^{(39)}\)

with an exponential rate beginning from the time instant \( \tau_a \).

In the case of \( \chi \in (0, 1) \), we can proceed the reasoning as follows. Recalling \((25)\) we can write \( \rho = \|\vartheta\|^{-1} \|e^*\|^\chi \), where \( \vartheta = k_p \vartheta_0 - \eta \|g_3^p(\vartheta_0)\|, \vartheta_0 = e^*/\|e^*\| \). Hence, we can continue estimation from \((36)\) as follows:

\[ \dot{V}_3 \leq -\chi e_{\theta a}^* \|e^*\|^{\chi+1} \|\vartheta\|^{-1} \leq -\chi e_{\theta a}^* \|e^*\|^\chi \dot{V}_3^{(\chi+1)/2}, \] \(^{(40)}\)

where the upper bound of \( \|\vartheta\| \) obtained for \( \sigma = -1 \) was used. Now, recalling \((38)\) and the general results presented in Bhat and Bernstein \((2000)\), one concludes about finite-time convergence of position error to zero:

\[ \lim_{\tau \to \tau} \|e^*(\tau)\| = 0, \text{ where } \tau \leq \tau + \frac{2(1-\chi)}{\chi(1-\chi)} \] \(^{(41)}\)

with \( c_3 = \sqrt{2^{\chi+1}}, \frac{k_p - \gamma}{k_p + \gamma}, \text{ and } V_3 = \frac{k_p - \gamma}{k_p + \gamma} \|e^*(\tau)\|^2. \)

Let us now consider terminal time-behavior of \( e^* \) error. It can be easily shown, using definition of the term \( r(e_{\theta a}^*) \) together with \((35)\) and \((25)\), that for \( \infty \to 0 \) one obtains:

\[ \tan \theta \to \sin \theta \] \(^{(42)}\)

Thus, for \( e^* \to 0 \) one gets \( \theta \to \sin \theta \). Additionally, at the limit for \( e^* \to 0 \) holds:

\[ g_3^p(\theta) \frac{\dot{g}_3^p(\theta)}{g_3^p(\theta)} > 0. \]

The last analysis stage is devoted to the terminal convergence of the steering angle error \( e_{\theta a}^*(\tau) \). Its time-behavior depends on the domain of definition \( D1 \) or \( D2 \) \((\text{see Subsection } 1.1)\) chosen by the user. Note that despite the domain definition, the terminal value of \( \beta_1 \) variable given by \((10)\) depends on the convergence rate of \( v_1 \) and \( v_2 \) arguments - both arguments tend to zero as \( e_{\theta a} \) and \( e^* \) tend to zero. As a result of the direct effect introduced by the definition of the virtual velocity in \((26)\), the first fictitious input \( v_1 \) tends to zero terminally faster than \( v_2 \). From the definition of domain \( D2 \) one gets \( \beta_1(\tau) \to 0 \) as \( e^* \to 0 \). Now, since \( \beta_1 = 0 \) from assumption, and since \( e_{\beta a} \to 0 \) one gets \( \beta_1 \to 0 \) and \( e_{\beta a}^*(\tau) \to 0 \) as \( e^* \to 0 \). For domain \( D1 \) we cannot guarantee such a strong result, since the terminal value of \((10)\) is now strictly related to the sign of \( v_2 \) input. We can surely claim that \( \beta_1 \to 0 \) as \( e^* \to 0 \). More strictly - we observe two possibilities: \( \beta_1 \to \pm(2n + 1)\pi \) if the vehicle approaches the reference position with \( u < 0 \) or \( v_1 \to 2\pi \). Specific value of \( n \) generally depends on initial conditions of the vehicle configuration. If \( 39 \) is met, \( \beta_1 \) converges asymptotically. If \( 41 \) holds, the finite-time convergence for \( \beta_1 \) is obtained.

The above convergence analysis was conducted for the nominal (unlimited) control inputs. However, multiplying both sides of equation \( 1 \) by the scaling function defined in \((14)\) allows us to write:

\[ s = g_1 u_{1s} + g_2(q) u_{2s}, \]

with \( g_1 \) and \( g_2(q) \) being the particular vector fields of \((1)\), and \( u_{1s}, u_{2s} \) taken from \((15)\). As a consequence, one can treat influence of \( s \) function as a time-scaling operation on the integral curve of \((1)\). The time-scaling preserves the vehicle motion geometry, but it delays the overall finite time instant assessed in \((41)\).

**Remark 2.** Note that at the reference position, \( \|e^*\| = 0 \) (and equivalently \( \|h^*\| = 0 \)) one has to cope with a discontinuity, since \((21)\) and \((24)\) together with definitions \((10)\) and \((13)\) are not determined there. Thus, to obtain the VFO controller well defined also in the neighborhood of the point \( \|e^*\| = 0 \) we proposed to introduce additional definitions:

\[ \theta_0 := \theta_a(\tau), \quad \beta_0 := \beta_a(\tau), \quad \theta_0 = \beta_0 = 0, \quad u_2 := 0 \] \(^{(43)}\)

activated for \( \tau > \tau, \) where the time instant \( \tau \) is determined by the condition \( \|e^*\| = 0 \) for some assumed vicinity \( \epsilon > 0 \).

The main result is formulated below as the final corollary.

**Corollary 3.** The VFO control law determined by the nominal inputs \((8, 12)\), fictitious inputs defined by \((19, 20)\) with additional definitions \((34)\), and scaling procedure described by \((15)\) solves the Problem 1:

\[ S1. \quad \text{for } \epsilon = 0, \quad T = \infty, \quad \{e(0) : \|e^*(0)\| \neq 0\} \quad \text{if} \quad \delta = \chi = \xi = 1, \quad \epsilon > 0, \quad \beta \neq 0, \quad D2, \]

\[ S2. \quad \text{for } \epsilon = 0, \quad T \geq \tau < \infty, \quad \{e(0) : \|e^*(0)\| \neq 0\} \quad \text{if} \quad \delta = \chi = \xi = 0, \quad \epsilon > 0, \quad \beta \neq 0, \quad D2, \]

\[ S3. \quad \text{for } \epsilon = \{e(\tau) \neq 0\} \quad \text{if} \quad \delta = \chi = \xi = 0, \quad \epsilon > 0, \quad \beta = 0, \quad D2. \]

If \( \beta \neq 0 \), convergence results related to \( S1-S3 \) for the \( \tau_0 = [\tau_0 e \tau_0] \) are preserved. For \( e_\beta \) component convergence of \( \text{tan} e_\beta (\tau) \) can be guaranteed at most, where \( e_\beta(\tau > 0) > 0 \) and \( e_\beta(\tau = 0) = 0 \).

3501
4. NUMERICAL SIMULATIONS

Two numerical tests – SimD1 and SimD2 – have been conducted for kinematics (1) with $L = 0.2$ m and for the two alternative domains D1 and D2 of the steering angle $\beta$. Amplitude-limited control inputs (15) have been applied to the vehicle with $u_{1M} = 3 \text{rad/s}$, $u_{2M} = 1 \text{m/s}$. The following parameter values have been selected: $k_\beta = 10$, $k_\theta = 5$, $k_p = 2$, $\eta = 1.8$, $\delta = \xi = \chi = 3/4$ (finite-time controller), and $\varepsilon = 0.005$ m. The reference and initial configurations have been chosen as $q_0 = 0$, $q_0 = [0 0 0 1]^T$, respectively (parallel parking maneuvers). Simulation results are illustrated in Figs. 2 and 3. Terminal error values obtained at the end of simulation tests are: $e(10) \approx [-3.14 \times 10^{-7} 5.10^{-3} -7.10^{-14}]^T$ for SimD1, $e(10) \approx [4.10^{-4} -2.10^{-10} 5.10^{-3} -7.10^{-14}]^T$ for SimD2 (values in units of $[\text{rad}] \times [\text{rad}] \times [\text{m}] \times [\text{m}]$). It is worth to emphasize non-oscillatory behavior of the vehicle with control inputs respecting the imposed bounds. The steering angle in the D1 domain converges toward $\pi \text{rad}$ due to the negative value of $u_2(t)$ obtained during the overall transient stage (the front side of the steering wheel has been denoted by the star mark (*) on the figures).

5. SUMMARY

In the paper the VFO control approach was applied to the front-axle driven car-like robot for the set-point control task. A family of finite-time and asymptotic stabilizers was proposed for two alternative domains of a steering-wheel angle. Simple scaling procedure allowed one to preserve the control input limits guaranteeing preservation of the vehicle motion geometry. Future work will be focused on robustness issues of the proposed algorithms to the vehicle model uncertainty and to feedback measurement noises.

REFERENCES


