An explicit formula for computation of the state coordinates for nonlinear i/o equation. ⋆

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Abstract: For a nonlinear input-output equation defined on homogeneous time scale, an explicit formula is given allowing to compute recursively the differentials of the state coordinates. The recursive formula is based on the Euclidean division algorithm.

Keywords: nonlinear control system, input-output models, state-space realization, polynomial methods, time scales.

1. INTRODUCTION

Time scale calculus unifies the theories of differential and difference equations, see Bohner and Peterson (2001). Both continuous- and discrete-time (in terms of the difference operator) cases are merged in time scale formalism into a general framework which provides not only unification but also an extension. The main concept of the time scale calculus is the so-called delta-derivative that is a generalization of both time-derivative and the difference operator (but accommodates more possibilities, e.g. q-difference operator). However, the time scale formalism does not allow to study the discrete-time systems described in terms of the more conventional shift operator because the shift operator is not a delta-derivative unlike the difference operator. Though less popular, the description of dynamical systems based on the difference operator has been advocated by many researchers (see, for example, Middleton and Goodwin (1990)). Recently, it was demonstrated that a difference operator based NARX model improves the numerical properties of the system structure detection, being known as a very difficult subtask in system identification, and provides a model that is closely linked to the continuous-time system both in terms of parameters and structure, see Anderson and Kadirikamanathan (2007).

Realization problem, i.e. the problem of recovering the state-space model from the input-output (i/o) delta-differential equation (DDE), relating the inputs and outputs of the system together with their delta derivatives, has been studied for nonlinear control systems defined on homogeneous time scales in Casagrande et al. (2010). In this paper the state coordinates were computed using a recursive algorithm. Note that, the application of the algorithm is time-demanding and second, it requires solving a system of equations, which may fail when implemented in CAS Mathematica for systems with medium complexity. In this paper we provide for a nonlinear DDE a straightforward formula for finding the differentials of the state coordinates by generalizing the results obtained for linear systems in Rapisarda and Willems (1997) and for nonlinear continuous-time systems in Tönso and Kotta (2009). The novelty of our result, besides unification, relies in the fact that the formula herein also covers the discrete-time nonlinear systems, represented in terms of the difference operator. The explicit formula is easier to implement and the program based on it gives the results much faster than the one based on the recursive algorithm. Finally, note that the realization problem for linear and nonlinear systems defined on time scales from the i/o map has been studied in Bartosiewicz and Pawluszewicz (2006) and Bartosiewicz and Pawluszewicz (2008), respectively.

The paper is organized as follows: Section 2 recalls the notions from time scale calculus used in paper. In the next section the problem setting is explained and a brief overview of algebraic framework on homogeneous time scale is given. Section 4 is devoted to the theory of noncommutative polynomial rings. In Section 5 the main result is presented. It is followed by examples and conclusions.

2. CALCULUS ON TIME SCALE

For a general introduction to the calculus on time scales, see Bohner and Peterson (2001). Here we give only those notions and facts that we need in our paper.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases include $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, but also $\mathbb{T} = q^\mathbb{Z} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$ is a time scale.

The following operators on $\mathbb{T}$ are often used:

- the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, defined by $\sigma(t) := \inf \{\tau \in \mathbb{T} : \tau > t\}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} \in \mathbb{T}$,
- the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$, defined by $\rho(t) := \inf \{\tau \in \mathbb{T} : \tau < t\}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} \in \mathbb{T}$,
- the graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$, defined by $\mu(t) := \sigma(t) - t$.

If $\mu \equiv \text{const}$ then a time scale $\mathbb{T}$ is called homogeneous. In this paper we assume that the time scale $\mathbb{T}$ is homogeneous.

Definition 1. The delta derivative of a function $f : \mathbb{T} \rightarrow \mathbb{R}$ at $t$ is the number $f^\Delta(t)$ such that for each $\varepsilon > 0$ there exists a neighborhood $U(\varepsilon)$ of $t$, $U(\varepsilon) \subset \mathbb{T}$ such that for all $\tau \in U(\varepsilon)$,

$$|f(\sigma(t)) - f(\tau) - f^\Delta(t)(\sigma(t) - \tau)| \leq \varepsilon|\sigma(t) - \tau|.$$

The typical special cases of the delta operator are summarized in the following remark.
Remark 1. (i) If $T = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if $f^\Delta(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$, i.e. iff $f$ is differentiable in the ordinary sense at $t$.

(ii) If $T = \mathbb{Z}$, where $T > 0$, then $f : \mathbb{Z}_+ \to \mathbb{R}$ is always delta differentiable at every $t \in T$ with $f^\Delta(t) = \left( f^{(n+1)}(t) - f^{(n)}(t) \right) / \Delta t$, meaning the usual forward difference operator.

Proposition 1. Let $f : T \to \mathbb{R}$, $g : T \to \mathbb{R}$ be two delta differentiable functions defined on $T$ and let $t \in T$. Then the delta derivative satisfies the following properties.

(i) $f^\Delta = f + \Delta f$.

(ii) $(\alpha f + \beta g)^\Delta = \alpha f^\Delta + \beta g^\Delta$, for any constants $\alpha$ and $\beta$.

(iii) $(fg)^\Delta = f^\Delta g + f g^\Delta$.

(iv) if $gg^\Delta \neq 0$, then $(f/g)^\Delta = (f^\Delta - g f^\Delta) / (gg^\Delta)$.

For a function $f : T \to \mathbb{R}$, we define second delta derivative as $f^{[2]} := f^\Delta^\Delta$. Then $f^\Delta$ is delta differentiable on $T$. We can similarly define higher order derivatives $f^{[n]}$.

Denote $\sigma^n := \sigma \circ \cdots \circ \sigma$ and $f^\sigma := f \circ \sigma^n$.

Proposition 2. (Kotta et al. (2009)). Let $f$ and $g^\Delta$ be delta differentiable functions on homogeneous time scale $\mathbb{T}$. Then

(i) $f^\Delta = f + \Delta f$.

(ii) $f^n = \sum_{k=0}^{n} C_k \mu^k f^{(k)}$.

3. PROBLEM STATEMENT AND ALGEBRAIC FRAMEWORK

For $f : T \to \mathbb{R}$ and $i \leq k$, denote $f^{[i,k]} := (f^{[i]} , \ldots , f^{[k]})$. Let $y : T \to Y \subseteq \mathbb{R}$ and $u : T \to U \subseteq \mathbb{R}$ be the output and the input functions respectively, such that $y$ is delta-differentiable up to the order $n$ and there exists the delta-derivative of any order of $u$. Consider a single-input single-output dynamical system described by a higher order input-output (i/o) DDE on homogeneous time scale $T$

\[ y[n] = \phi (y, \ldots , y[n-1], u, \ldots , u[s]), \quad (1) \]

where $\phi$ is a real analytic function defined on $Y^n \times U^{s+1}$. Moreover, for causality, one assumes that $s < n$, where $n$ and $s$ are nonnegative integers.

The realization problem is defined as follows. Given a nonlinear system, described by the i/o equation of the form (1), find, if possible, the state coordinates $x \in X \subseteq \mathbb{R}^n$, $x = \psi(y, \ldots , y[n-1], u, \ldots , u[s])$ such that in these coordinates the system takes the classical state-space form

\[ x^\Delta = f(x, u) \]

\[ y = h(x) \quad (2) \]

called the realization of (1).

Below we briefly recall the algebraic formalism for nonlinear control systems defined on homogeneous time scales, described in Bartosiewicz et al. (2007), Casagrande et al. (2010), Kotta et al. (2009). Let $\mathcal{H}$ denote the field of meromorphic functions in a finite number of the (independent) variables

\[ \mathcal{C} = \{y, y^\Delta, \ldots , y[n-1], u[k], k \geq 0 \}. \]

The operators $\sigma : \mathcal{H} \to \mathcal{H}$ and $\Delta : \mathcal{H} \to \mathcal{H}$ are defined as follows

\[ \sigma(F) (y, \ldots , y[n-1], u, \ldots , u[k]) := F(y^\sigma, \ldots , y[n-1]^\sigma, u^\sigma, \ldots , u[k]^\sigma), \]

\[ \Delta(F) (y, \ldots , y[n-1], u, \ldots , u[k+1]) := \int_0^1 \{ \gamma F(y^{[n-1]}, u^{[0:s]}, \phi(y^{[0:n-1]}, u^{[0:s]})) \}, \quad (3) \]

\[ \Delta(F) y = \int_0^1 \{ \gamma F(y^{[n-1]}, u^{[0:s]}, \phi(y^{[0:n-1]}, u^{[0:s]})) \}, \quad (4) \]

The operator $\Delta$ satisfies a generalization of Leibnitz rule:

\[ (FG)^\Delta = F^\sigma G^\Delta + F^\Delta G, \]

for $F, G \in \mathcal{H}$. The derivation satisfying rule (6) is called a “$\sigma$-derivation” (see for example Cohn (1965)). Then $\mathcal{H}$ is a field equipped with $\sigma$-derivation $\Delta$. We assume furthermore that the operator $\sigma : \mathcal{H} \to \mathcal{H}$ is an automorphism $1$. Under the latter assumption, the pair $(\mathcal{H}; \Delta)$ is an inverse $\sigma$-differential field meaning that the backward jump operator $\rho$ is well-defined, i.e. for function $\phi \in \mathcal{H}$ there exists $\psi$ such that $\psi^\sigma = \phi$, or alternatively, $\phi^\sigma = \psi$.

Over the $\sigma$-differential field $\mathcal{H}$ one can define the vector space $\mathcal{E} = \{ \delta \phi \}$, the elements of $\mathcal{E}$ are called one-forms. For $F \in \mathcal{H}$ we define $d : \mathcal{H} \to \mathcal{E}$ as follows $dF := \sum_{i=0}^{n} \partial \phi / \partial y[i] + \sum_{j=0}^{s} \partial \phi / \partial u[j]$, $dF$ is said to be the total differential (or simply the differential) of the function $F$ and it is a one-form. Any element in $\mathcal{E}$ is a vector of the form $\omega = \sum \alpha_i \delta \phi_i$, where all $\alpha_i \in \mathcal{H}$ and $\phi_i$ are functions on variables from the set $\mathcal{C}$. The operators $\sigma : \mathcal{H} \to \mathcal{H}$ and $\Delta : \mathcal{H} \to \mathcal{H}$ induce the operators $\sigma : \mathcal{E} \to \mathcal{E}$ and $\Delta : \mathcal{E} \to \mathcal{E}$ by

\[ \sigma(\omega) := \sum \alpha_i d[\sigma(\phi_i)], \quad (7) \]

\[ \Delta(\omega) := \sum \Delta(\alpha_i) d[\phi_i] + \alpha_i d[\Delta(\phi_i)], \quad (8) \]

Since $\sigma(\alpha_i) = \alpha_i + \mu \Delta(\alpha_i)$, (8) may be alternatively written as

\[ \Delta(\omega) = \sum \Delta(\alpha_i) d[\phi_i] + (\alpha_i + \mu \Delta(\alpha_i)) d[\Delta(\phi_i)]. \]

1 For this assumption to hold, one has to extend the field $\mathcal{H}$ up to its inversive closure which is always possible, see Cohn (1965).
It has been proved that $\Delta(dF) = d[F^\Delta]$, $\sigma(dF) = d[F^\sigma]$ and $\Delta = \sigma\Delta$, see Bartosiewicz et al. (2007).

The relative degree $r$ of a one-form $\omega \in \mathcal{E}$ is defined to be the least integer such that $\omega^{[1]} \not\in \text{span}_{\mathcal{E}} \{dy, \ldots, dy^{[n-1]}, du, \ldots, du^{[n]}\}$. If such an integer does not exist, we set $r = \infty$. A sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ is defined by

$$
\mathcal{H}_k = \text{span}_{\mathcal{E}} \{dy, \ldots, dy^{[k-1]}, du, \ldots, du^{[n]}\}, \quad k \geq 1.
$$

Note that $\mathcal{H}_k$ contains the one-forms whose relative degree is equal to $k$ or higher than $k$. It is clear that the sequence (9) is decreasing. Denote by $k^*$ the least integer such that $\mathcal{H}_{k^*} \supset \cdots \supset \mathcal{H}_k \supset \mathcal{H}_{k+1} = \cdots = \mathcal{H}_\infty$.

In what follows we assume that the i/o delta differential equation (1) is in the irreducible form, that is, $\mathcal{H}_\infty$ is trivial Kotta et al. (2009), i.e. $\mathcal{H}_\infty = \{0\}$. An $n$th-order realization of equation (1) will be accessible if and only if system (1) is irreducible.

System (2) is said to be single-experiment observable if the observability matrix has generically full rank

$$
\text{rank}_x \begin{pmatrix} \frac{\partial(h(x), [h(x)]^{\Delta}, \ldots, [h(x)]^{[n-1]})}{\partial x} \end{pmatrix} = n.
$$

We say that $\omega \in \mathcal{E}$ is an exact one-form if there exists $\xi \in \mathcal{E}$ such that $d\xi = \omega$. A one-form $\omega$ for which $d\omega = 0$ is said to be closed. A subspace is integrable or closed, if it has a basis which consists only of closed one-forms. Note that closed one-forms are locally exact. The Frobenius theorem can be used to check the integrability of the subspace of one-forms; in the theorem presented below the symbol $\wedge$ means the exterior or wedge product.

**Theorem 1.** (Choquet-Bruhat et al. (1982)). Let $\mathcal{V} = \text{span}_{\mathcal{E}} \{\omega_1, \ldots, \omega_r\}$ be a subspace of $\mathcal{E}$. $\mathcal{V}$ is closed if and only if

$$
dw_i \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0,
$$

for all $i = 1, \ldots, r$.

**Theorem 2.** (Casagrande et al. (2010)). The nonlinear system $\Sigma$, described by the irreducible input-output delta differential equation (1), has an observable and accessible state-space realization iff for $1 \leq k \leq s+2$ the subspaces $\mathcal{H}_k$ defined by (9) are completely integrable. Moreover, the state coordinates can be obtained by integrating the basis vectors of $\mathcal{H}_{k+2}$.

## 4. POLYNOMIAL FRAMEWORK

Consider the differential field $\mathbb{K}$ with the $\sigma$-derivation $\Delta$ with $\sigma$ being an automorphism of $\mathbb{K}$. A left differential polynomial is an element which can be uniquely written in the form $a(\partial) = \sum_{i=0}^{a_0} a_i \partial^{i-s}$, $a_i \in \mathbb{K}$, where $\partial$ is a formal variable and $a(\partial) \neq 0$ if and only if at least on of the functions $a_i$, $i = 0, \ldots, n$ is nonzero. If $a_0 \neq 0$, then the positive integer $n$ is called the degree of the left polynomial $a(\partial)$, denoted by $\deg a(\partial)$. In addition, we set $\deg 0 = -\infty$. For $a \in \mathbb{K}$ let us define multiplication by the commutation rule

$$
\partial \cdot a := a \sigma \partial + a^\Delta.
$$

This rule can be uniquely extended to multiplication on monomials by $(a^m \partial^n) \cdot (b^p \partial^q) = (a \sigma^m \partial^m + b \partial^q)^n$. The ring of differential polynomials will be denoted by $\mathbb{K}[\partial; \sigma, \Delta]$.

Since $\sigma$ is an automorphism, the ring of the left differential polynomials is an Ore ring, see McConnell and Robson (1987).

Let $\sigma^n := \sigma \circ \cdots \circ \sigma$ and denote $\sigma^n(a)$ by $\sigma^n(a)$ for $a \in \mathbb{K}$.

**Lemma 1.** (Kotta et al. (2009)). Let $a \in \mathbb{K}$. Then $\partial^n \cdot a \in \mathbb{K}[\partial; \sigma, \Delta]$, for $n \geq 0$, and $\partial^n \cdot a = \sum_{i=0}^{n} C_n^i (a^{[n-i]} \sigma^i \partial^i)$. Let $\omega_k, l := pk, l(\partial)dy + qk, l(\partial)du$ (14)

A ring $D$ is called an integral domain, or a domain, if it does not contain any zero divisors. The latter means that if $a$ and $b$ are two elements of $D$ such that $ab = 0$, then either $a = 0$ or $b = 0$ or both.

**Proposition 3.** (McConnell and Robson (1987)). The ring $\mathbb{K}[\partial; \sigma, \Delta]$ is an integral domain.

Let us define $\partial^k dy := dy^k[\partial]$ and $\partial^l du := du^l[\partial], k, l \geq 0$ in the vector space $\mathcal{E}$. Since every one-form $\omega \in \mathcal{E}$ has the following form $\omega = \sum_{i=0}^{n-1} A_i dy^i + \sum_{j=0}^{s} B_j du^j$, where $A_i, B_j \in \mathbb{K}$, so $\omega$ can be expressed in terms of the left differential polynomials in the following way $\omega = \sum_{i=0}^{n-1} A_i \partial^i dy + \sum_{j=0}^{s} B_j \partial^j du$. A left differential polynomial can be considered as an operator acting on vectors $dy$ and $du$ from $\mathcal{E}$ and $(\omega \partial)(\partial^i \partial^j) := \sum_{i=0}^{n-1} A_i \partial^i \partial^j$, with $A_i \in \mathbb{K}$ and $\partial^j \in \{dy, du\}$, where by Lemma 1, $\partial^j \partial^i = \sum_{k=0}^{n-1} C_k^j (\partial^{[j-k]} \sigma^j)^k$. It is easy to note that $\partial(\omega) = \Delta(\omega)$, for $\omega \in \mathcal{E}$.

Instead of working with equation (1), describing the control system, we can work with its differential

$$
dy^n = \sum_{i=0}^{n-1} \frac{\partial}{\partial y^i} dy^i - \sum_{j=0}^{s} \frac{\partial}{\partial u^j} du^j = 0. \quad (12)
$$

By definition, $\partial^i dy := dy^i[\partial]$ and $\partial^j du := du^j[\partial]$, therefore (12) can be rewritten as

$$
p(\partial) dy + q(\partial) du = 0, \quad (13)
$$

with $p(\partial) = \partial^n - \sum_{i=0}^{n-1} A_i \partial^i$, $q(\partial) = -\sum_{j=0}^{s} B_j \partial^j$ and $p, q \in \mathbb{K}[\partial]$. Equation (13) describes the behavior of dynamical system in terms of two differential polynomials $p(\partial), q(\partial)$ in operator $\partial$ over the $\sigma$-differential field $\mathbb{K}$.

Since $\mathbb{K}[\partial; \sigma, \Delta]$ is an Ore ring, one can construct the division ring of fractions. If $p(\partial) = p_1(\partial)p_2(\partial)$ and $\deg(p_1(\partial)) > 0$, then $p_1(\partial)$ is called a left divisor of $p(\partial)$ and $p(\partial)$ is called left divisible by $p_1(\partial)$.

To find the left divisor one can use the left Euclidean division algorithm, see Bronstein and Petkovšek (1996). To perform the left Euclidean division algorithm it is sufficient that $\sigma$ be an automorphism. For given two polynomials $p_1(\partial)$ and $p_2(\partial)$ with $\deg(p_1(\partial)) > \deg(p_2(\partial))$ there exist a unique polynomial $r(\partial)$ and a unique left remainder polynomial $r(\partial)$ such that $p_1(\partial) = p_2(\partial)r(\partial) + r(\partial)$ and $\deg(r(\partial)) < \deg(p_2(\partial))$.

## 5. PROBLEM SOLUTION: POLYNOMIAL APPROACH

We introduce certain one-forms in terms of which the main result will be formulated. Let

$$
\omega_{k, l} := pk, l(\partial)dy + qk, l(\partial)du \quad (14)
$$
for \( k = 1, \ldots, s + 2, \ l = 1, \ldots, n \), where \( p_{k,l}(\partial) \) and \( q_{k,l}(\partial) \) are Ore polynomials, which can be recursively calculated as left quotients from equalities

\[
p_{k,l-1}(\partial) = \partial \cdot p_{k,l}(\partial) + r_{k,l}, \quad \deg r_{k,l} = 0, \qquad q_{k,l-1}(\partial) = \partial \cdot q_{k,l}(\partial) + p_{k,l}, \quad \deg p_{k,l} = 0
\]

with the initial polynomials defined as \( p_{0,0}(\partial) := \partial^n \), \( q_{0,0}(\partial) := 0 \) for \( k = 1 \) and

\[
p_{k,0}(\partial) := \partial^n - \sum_{i=n-k+1}^{n-1} p_i \partial^i, \quad q_{k,0}(\partial) := - \sum_{j=s-k+2}^{s} q_j \partial^j \quad \text{for} \ k = 2, \ldots, s + 2.
\]

Two following lemmas are necessary to prove the main result.

**Lemma 3.** The one-forms \( \omega_{k,l} \) for \( k = 1, \ldots, s + 2 \) and \( l = 1, \ldots, n \), defined by (14), satisfy the relationships

\[
\omega_{k+1,l} = w_{k,l} + \alpha(\partial) dy + \beta(\partial) du,
\]

where

\[
\alpha(\partial) = \begin{cases} 
\sum_{i=0}^{n-k-l} (-1)^i C_i^{l+1} \left( \frac{[i]}{n-k} \right)^{\partial^{n-k-l-i}}, & \text{if } 1 \leq l \leq n-k \\
0, & \text{if } n-k < l \leq n
\end{cases}
\]

and

\[
\beta(\partial) = \begin{cases} 
\sum_{i=0}^{s-k-l+1} (-1)^i C_i^{l+1} \left( \frac{[i]}{s-k+1} \right)^{\partial^{s-k-l+i}}, & \text{if } 1 \leq l \leq s-k+1 \\
0, & \text{if } s-k+1 < l \leq s.
\end{cases}
\]

**Proof.** From (14),

\[
\omega_{k+1,l} = p_{k+1,l}(\partial) dy + q_{k+1,l}(\partial) du.
\]

We find the relationships between \( p_{k+1,l}(\partial) \) and \( p_{k+1,l}(\partial) \) and between \( q_{k+1}(\partial) \) and \( q_{k+1}(\partial) \), respectively. By repeated application of the recursive formula (14), equation (18) can be replaced by a formula which allows to find \( p_{k,l}(\partial) \) directly from \( p_{0,0}(\partial) \)

\[
p_{k,0}(\partial) = \partial^k p_{k,0}(\partial) + R_{k,0}(\partial), \quad \deg R_{k,0}(\partial) < l.
\]

By (16), the relationships (19) for \( k + 1 \) may be written as

\[
p_{k,0}(\partial) - p_{n-k} \partial^{n-k} = \partial^l p_{k+1,l}(\partial) + R_{k,1,l}(\partial), \quad \deg R_{k,1,l}(\partial) < l.
\]

Equating the coefficients of the same power of \( \partial \) on both sides of formula (22) yields the system of linear equations

\[
C_i^l (\alpha_{n-k-l})^{\partial^{n-k-l-i}} = p_{n-k},
\]

\[
C_i^l (\alpha_{n-k-l-1})^{\partial^{n-k-l-i+1}} + \cdots + C_i^{l+1} (\alpha_0^{[l+1]} = 0,
\]

\[
C_i^l (\alpha_{n-k-l})^{\partial^{n-k-l-i}} = 0,
\]

\[
C_i^l (\alpha_{n-k-l})^{\partial^{n-k-l-i}} + C_i^l (\alpha_{n-k-l})^{\partial^{n-k-l-i}} = 0.
\]

It is easy to see that the system of equations is in the triangular form, and one can find a solution recursively as

\[
\alpha_{n-k-l} = p_{n-k}^{\partial^{n-k}},
\]

\[
\alpha_{n-k-l-1} = -C_i^l (p_{n-k}^{\partial^{n-k+1}}),
\]

\[
\alpha_0 = (-1)^{n-k-l} {\binom{n-k-l}{k-l}} \partial^{n-k-l}.
\]

After substitution the solution into (21), we obtain

\[
\alpha(\partial) = \sum_{i=0}^{n-k-l} (-1)^i C_i^{l+1} \left( \frac{[i]}{n-k} \right)^{\partial^{n-k-l-i}}.
\]

Thus, \( p_{k+1,l}(\partial) = p_{k,l}(\partial) + \alpha(\partial) \).

In a similar manner one can prove that

\[
q_{k+1,l}(\partial) = q_{k,l}(\partial) + \beta(\partial),
\]

where \( \beta(\partial) = 0 \) if \( s-k+1 < l \leq s \) and \( \beta(\partial) = \sum_{l=0}^{s-k-l+1} (-1)^i C_i^{l+1} \left( \frac{[i]}{s-k+1} \right)^{\partial^{s-k-l+i}} \) if \( 1 < l \leq s-k+1 \).

**Lemma 4.** For the one-forms \( \omega_{k,l} \), defined by (14), the property

\[
\omega_{k+1,l} = \omega_{k,l} + \partial \partial q_{k,l}(\partial) du
\]

holds, where \( \deg r_{k,l} = \deg p_{k,l} = 0, \) for \( l = 2, \ldots, n \) and \( k = 1, \ldots, s + 2 \).

**Proof.** Taking the delta derivative of (14)

\[
\omega_{k+1,l} = (p_{k,l}(\partial) dy + q_{k,l}(\partial) du)^\Delta = p_{k,l}^\Delta(\partial) dy + q_{k,l}^\Delta(\partial) du + q_{k,l}^\Delta(\partial) du = (p_{k+1,l}(\partial) dy + q_{k+1,l}(\partial) du)
\]

and using the commutation rule (11), equation (24) can be rewritten as follows

\[
\omega_{k+1,l} = (\partial \cdot p_{k,l}(\partial)) dy + (\partial \cdot q_{k,l}(\partial)) du.
\]

Next, we replace in (25) the terms \( \partial \cdot p_{k,l}(\partial) \) and \( \partial \cdot q_{k,l}(\partial) \) by the corresponding expressions from (15) to obtain

\[
\omega_{k+1,l} = (p_{k+1,l}(\partial) dy + q_{k+1,l}(\partial) - p_{k,l} dy)
\]

and use the definition of \( \omega_{k+1,l} \), given by (14), to get

\[
\omega_{k+1,l} = \omega_{k+1,l} = \omega_{k+1,l} - r_{k,l} dy - p_{k,l} du.
\]

**Theorem 3.** For the input-output model (1), the subspaces \( \mathcal{H}_k \) can be calculated as

\[
\mathcal{H}_k = \text{span}_x \{ \omega_{k,l}, du, \ldots, du^{[s-k+1]} \}
\]

for \( k = 1, \ldots, s + 1 \) and

\[
\mathcal{H}_{s+2} = \text{span}_x \{ \omega_{s+2,l} \},
\]

where \( \omega_{k,l} \) for \( l = 1, \ldots, n \) are defined by (14).

**Proof.** The proof is by induction. We first show that formula (26) holds for \( k = 1 \). From (14), it follows that
\[ p_{l+1}(\partial) = \partial^n - l \quad \text{and} \quad q_{l+1}(\partial) = 0 \quad \text{for} \quad l = 1, \ldots, n. \]

Consequently, \( \omega_{1,l} = \partial^n - l dy = dy[n-l] \) and \( \mathcal{H}_1 = \text{span}_\mathbb{F} \{dy, \ldots, dy[n-1], du, \ldots, du[s]\} \) which agrees with the definition of \( \mathcal{H}_1 \) in (9).

Assume now that formula (26) holds for \( k \) and prove it to be valid for \( k + 1 \). The proof is based on definition of the subspaces \( \mathcal{H}_k \). We have to show that \( \mathcal{H}_{k+1} = \text{span}_\mathbb{F} \{\omega_{k+1,l}, du, \ldots, du[s-k]\} \) in (26) satisfies the condition (9).

First, we show that the basis one-forms \( \omega_{k+1,l}, du, \ldots, du[s-k] \) are in \( \mathcal{H}_k \). It is obvious that \( du, \ldots, du[s-k] \in \mathcal{H}_k \). Lemma 3 represents the one-forms \( \omega_{k+1,l} \) as a linear combination of vectors \( \omega_{k,l}, dy, \ldots, dy[n-k], du, \ldots, du[n-k+1] \). It is easy to see that \( \omega_{k,l}, du, \ldots, du[s-k+1] \) are in \( \mathcal{H}_k \) by definition of subspaces (9). Though \( dy, \ldots, dy[n-k] \) are not listed explicitly among the basis vectors of \( \mathcal{H}_k \) in (26), they can be expressed as a linear combination of other basis vectors. From (15) and (16) follows that coefficients of the higher order terms of polynomials \( p_{k,l}(\partial) \) are always 1 as well as \( \deg p_{k,l}(\partial) = n-1 \) for \( l = 1, \ldots, n-1 \) and \( \deg q_{k,l} = s-l \) for \( l = 1, \ldots, s \). It means that \( p_{k,l}(\partial) \) and \( q_{k,l}(\partial) \) have the form

\[
p_{k,l}(\partial) = \partial^n - l - \sum_{j=0}^{n-k} p_{k,l,j} \partial^j, \quad q_{k,l}(\partial) = -\sum_{j=0}^{s-l} q_{k,l,j} \partial^j.
\]

For \( l = n \) we get \( p_{k,n}(\partial) = 1 \) and \( q_{k,n}(\partial) = 0 \). Consequently, \( \omega_{k,n} = dy \). The rest of the differentials \( dy^n, \ldots, dy[n-k] \) can be recursively computed from (15) as follows

\[
dy[l] = \omega_{k,n-l} + \sum_{j=0}^{l-1} p_{k,n-l,j} dy[j] + \sum_{j=0}^{s-n+l} q_{k,n-l,j} du[j]
\]

for \( l = 1, \ldots, n-k \).

Second, we have to show that the delta derivatives of the one-forms, computed according to (26) and defined by (14), also belong to \( \mathcal{H}_k \). Again, it is obvious that \( du, \ldots, du[s-k+1] \in \mathcal{H}_k \). We have to prove that \( \omega_{k+1,l}^\Delta \in \mathcal{H}_k \). Note that according to Lemma 4,

\[
\omega_{k+1,l}^\Delta = \omega_{k+1,l-1}^\Delta - r_{k+1,l} dy - \rho_{k+1,l} du,
\]

where \( \deg r_{k+1,l} = \deg q_{k+1,l} = 0 \) for \( l = 2, \ldots, n \). It was proved in the previous step that \( \omega_{k+1,l}^\Delta \) for \( l = 2, \ldots, n \) and \( dy \) are in \( \mathcal{H}_k \). Therefore, for \( l = 1 \) we have to show separately that \( \omega_{k+1,1}^\Delta \in \mathcal{H}_k \). From (14) we have

\[
\omega_{k+1,1}^\Delta = (\partial \cdot q_{k+1,1}(\partial)) dy + (\partial \cdot q_{k+1,1}(\partial)) du.
\]

Increasing \( k \) by 1 and taking \( l = 1 \) in (15) allows us to express \( \partial \cdot p_{k+1,1}(\partial), \partial \cdot q_{k+1,1}(\partial) \) and substitute them into the previous equality

\[
\omega_{k+1,1}^\Delta = (p_{k+1,0}(\partial) - r_{k+1,1}) dy + (q_{k+1,0}(\partial) - \rho_{k+1,1}) du.
\]

Replacing in the above equality initial polynomials \( \partial \cdot p_{k+1,0}(\partial) \) and \( \partial \cdot q_{k+1,0}(\partial) \) by their expressions (16) and using relations \( \partial dy = dy^{[i]} \) for \( i = n-k, \ldots, n \) and \( \partial du = du^{[j]} \) for \( j = s-k+1, \ldots, s \), we obtain

\[
\omega_{k+1,1}^\Delta = dy[n] - \sum_{i=n-k}^{n-1} p_{i} dy^{[i]} - \sum_{j=s-k+1}^{s} q_{j} du^{[j]} - r_{k+1,1} dy - \rho_{k+1,1} du.
\]

Finally, replacing \( dy[n] \) in the above equality by the right-hand side of (12), we get

\[
\omega_{k+1,1}^\Delta = \sum_{i=0}^{n-k-1} p_{i} dy^{[i]} + \sum_{j=0}^{s-k} q_{j} du^{[j]} - r_{k+1,1} dy - \rho_{k+1,1} du.
\]

The latter means that the one-forms \( \omega_{k+1,1}^\Delta \) can be expressed as a linear combination of the basis vectors of \( \mathcal{H}_k \). This completes the proof.

The differentials of the state coordinates can be found from the subspace \( \mathcal{H}_{s+2} \), see Theorem 2. Though in case of the realizable i/o equation, \( \mathcal{H}_{s+2} \), defined by (27), is completely integrable, the one-forms \( \omega_{s+2,l} \) for \( l = 1, \ldots, n \), are not necessarily always exact. Therefore, one has to find for \( \mathcal{H}_{s+2} \) a new integrable bases, using the linear transformations. From Theorem 3 the next corollary can be concluded.2

**Corollary 1.** For realizable i/o equation (1), the differentials of the state coordinates can be calculated as the integrable linear combinations of the one-forms

\[
\omega_l = p_l(dy) + q_l(\partial) du, \quad l = 1, \ldots, n
\]

where \( p_l(\partial) \) and \( q_l(\partial) \) can be computed iteratively as follows

\[
p_{l-1}(\partial) = \partial 
 p_l(\partial) + r_l, \quad q_{l-1}(\partial) = \partial \cdot q_l(\partial) + \rho_l,
\]

with the initial polynomials defined by

\[
p_0(\partial) := p(\partial), \quad q_0(\partial) := q(\partial).
\]

In the case \( T = \mathbb{R} \) (the continuous-time case) the i/o delta-differential equation (1) turns to an ordinary differential equation

\[
y^{(s)} = \phi(y, \ldots, y^{(n-1)}, u, \ldots, u^{(s)})
\]

and Theorem 3 yields the following corollary.

**Corollary 2.** For the i/o model (29) the subspaces \( \mathcal{H}_k \) can be calculated as

\[
\mathcal{H}_k = \text{span}_\mathbb{F} \{\omega_{k,l}, du, \ldots, du[s-k+1]\}
\]

for \( k = 1, \ldots, s + 1 \) and

\[
\mathcal{H}_{s+2} = \text{span}_\mathbb{F} \{\omega_{s+2,l}\},\n\]

where \( \omega_{k,l} \) for \( l = 1, \ldots, n \) are defined by (14) in which \( \partial \) has to be interpreted as \( \frac{\partial}{\partial t} \), and the commutation rule (11) is \( \partial \cdot a = a \partial + \dot{a} \).

Note that the results of Corollary 2 were proved in Tönsö and Kotta (2009).

Moreover, note that the results for the discrete-time case \( T = \mathbb{Z} \) for \( T > 0 \) with \( \partial \) being interpreted as a difference operator, see Remark 1 item (ii), are now, since Kotta and Tönsö (2008) cover the discrete-time shift-operator based case.

**Example 1:** Consider the model of the controlled van der Pol oscillator in the form of i/o DDE

\[
y^{[2]} = (\xi_1 - \xi_2 y^2)y^{[1]} - \xi_3 y + \xi_4 u,
\]

where \( \xi_i \in \mathbb{R} \). Usually, system (30) is considered separately for continuous-time \( (T = \mathbb{R}, \text{Remark 1 item (i)}) \) and discrete-time

2 The first index \( s+2 \) is omitted in Corollary 1.
\( T = Z \), Remark 1 item (ii)) cases, see for example Anderson and Kadirkamanathan (2007). Here, however, we consider ... - EUROCAST 2009, Lecture Notes in Control and Information Sciences, pages 633–640. Springer Berlin / Heidelberg, 2009.

Finally, the one-forms that define the differentials of the state coordinates can be computed, according to (28), as follows

\[
\begin{align*}
\omega_1 &= p_1(q)dy + q_1(q)du = \left(\partial - \xi_1 - \xi_2(y^2)\right)dy, \\
\omega_2 &= p_2(q)dy = dy.
\end{align*}
\]

Thus \( \mathcal{H}_3 = \text{span}_\mathbb{R} \{\omega_1, \omega_2\} = \text{span}_\mathbb{R} \{dy, dy^\Lambda\} \) and integrating the basis elements, the state coordinates are \( x_1 = y \) and \( x_2 = y^\Lambda \) yielding the state equations

\[
\begin{align*}
x_1^\Lambda &= x_2, \\
x_2^\Lambda &= (\xi_1 - \xi_2x_1^2)x_2 - \xi_3x_1 + \xi_4u, \\
y &= x_1.
\end{align*}
\]

Example 2: Consider the control system in the form of i/o DDE

\[
y^{(2)} = uy^\Lambda - \mu uy + u^2y^2
\]

that can be described by two polynomials \( p(\partial) = \partial^2 - u\partial - 2uy \) and \( q(\partial) = \mu\partial - y^2 - y^3 \).

Note that \( n = 2 \) and \( s = 1 \). To find the state coordinates, one has to, according to Corollary 1, compute the one-forms \( \omega_1 \) and \( \omega_2 \), defined by (28). From equalities \( p_0(\partial) := p(\partial) = \partial \cdot p_1(\partial) + r_1 \) and \( q_0(\partial) := q(\partial) = \partial \cdot q_1(\partial) + r_1 \) one can find that \( p_1(\partial) = \partial - \xi_1 - \xi_2(y^2) \) and \( q_1(\partial) = 0 \), respectively. Furthermore, from equality \( p_2(\partial) = \partial \cdot p_2(\partial) + r_2 \) one can find that \( p_2(\partial) = 1 \) and \( q_2(\partial) = 0 \).

Finally, the one-forms that define the differentials of the state coordinates can be computed, according to (28), as follows

\[
\begin{align*}
\omega_1 &= p_1(q)dy + q_1(q)du = (\partial - u\partial - 2uy)dy + \mu du, \\
\omega_2 &= p_2(q)dy + q_2(q)du = dy.
\end{align*}
\]

Thus \( \mathcal{H}_3 = \text{span}_\mathbb{R} \{\omega_1, \omega_2\} = \text{span}_\mathbb{R} \{dy, dy^\Lambda + \mu du, dy\} \) and integrating the basis elements, the state coordinates are \( x_1 = y \) and \( x_2 = y^\Lambda - \mu u \) yielding the state equations

\[
\begin{align*}
x_1^\Lambda &= x_2 + \mu u, \\
x_2^\Lambda &= (x_1 + x_2 - \mu u)u, \\
y &= x_1
\end{align*}
\]

6. CONCLUSION

Theorem 3 provides an alternative, polynomial method for computing the bases vectors for the subspaces \( \mathcal{H}_k \), where \( k = 1, \ldots, s+2 \). Note that the polynomial method has advantages in computer implementation. First, it is direct, meaning that there is no need to compute step-by-step all \( \mathcal{H}_k \)'s in order to find \( \mathcal{H}_{s+2} \). Second, its program code is shorter and more compact. Algebraic methods require to solve a pseudolinear system of equations, which is linear with respect to unknowns, but those coefficients are nonlinear functions, and not real numbers. If the expressions, found on previous steps, have been not enough simplified, there is a chance that Mathematica may be unable to solve the pseudolinear system of equations and the computation fails. Meanwhile, polynomial method does not require solving any system of equations. The algebraic method requires to insert the additional simplification commands into the code, while the polynomial method is able to produce the result without any intermediate simplification.

REFERENCES


