OPTIMIZATION OF NONLINEAR MANUFACTURING SYSTEMS UNDER UNCERTAINTY

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Abstract

A new algorithm for the optimization of nonlinear systems under uncertainty is presented. The algorithm, based on a parametric programming framework, gives a complete map of the optimal solution in the space of the uncertain parameters, solving a minimum number of NLP subproblems and simplified multiparametric linear master problems. Through cumulative outer-approximations obtained from the solution of deterministic NLP at fixed values of the uncertain parameters, the master problem provides valid lower bounds that converge to the optimal values as the number of approximations is increased. Several heuristics are proposed to guide the mathematical algorithm, drastically reducing the computational requirements, still ensuring convergence to the optimal solution.

Keywords

Nonlinear programming, optimization under uncertainty, parametric programming.

Introduction

The optimization of manufacturing systems under uncertainty has been largely studied for the past two decades, trying to introduce the variations to which real processes are subject. Variations considered can be inherent to the process, like internal flows or catalyst decay, or external to it, like raw material availability.

In the mathematical programming approach, this uncertainty has been captured mainly through stochastic formulations, where a probability distribution function is assigned to each uncertain parameter, or through multi-period or scenario formulations, where fixed values of the uncertain parameters are used. In the past years, a third approach has also been considered, through parametric programming formulations. This approach has the advantage of giving a complete map of the optimal solution in the space of the uncertain parameters. Furthermore, a solution algorithm has been presented for linear problems where the computational complexity is almost independent of the dimensionality of the problem.

For nonlinear problems, however, the geometrical increase of the number of deterministic NLPs that must be solved as the number of uncertain parameters increases, have restricted the size of the problems and particularly the number of uncertain parameters considered.

A new algorithm for the solution of multiparametric nonlinear programming problems is presented here. The algorithm minimizes the number of deterministic NLPs that have to be solved using simplified linear approximations of the multiparametric problem and heuristics based on the estimation of the difference between the real parametric solution and the one obtained through the linear approximation.

In the first part of the work, the algorithms of Dua and Pistikopoulos (1) and Moncada and Acevedo (2) for the solution of these problems are briefly discussed as the bases for the proposed procedure. Then, the new algorithm is explained and exemplified through a numerical problem, where the advantages of the algorithm are shown. Finally, some conclusions are given and future lines of research are suggested.

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Mathematical formulation.

The multiparametric nonlinear programming (mpNLP) problem considered in this work is of the following form

$$
\begin{aligned}
Min \quad & f(x) \\
s.t. \quad & h(x)=0 \\
& g(x) \le b + F\theta \\
& \theta_{min} \le \theta \le \theta_{max} \\
& x \in R^n ; \quad \theta \in R^s
\end{aligned}
\qquad (1)
$$

where f(x) represents the objective function, g(x) and h(x) represent the constraints of the problem, with x as the vector of variables involved in the model. Vector b and matrix F are constant terms, and θ represents the vector of uncertain parameters. It is assumed here that the model is convex.

Dua and Pistikopoulos (1999) presented an algorithm for the solution of mpNLPs as the one in Eq. (1), through the solution of a series of deterministic NLP primal subproblems and multiparametric linear master problems (mpLP). Based on the convexity properties of the model, linear combinations of the primal subproblems represent valid upper bounds while the solution of the master mpLP represents a valid lower bound to the optimal parametric solution.

The solution of the mpLP is obtained following the algorithm of Gal (1995). In this algorithm, the solution is given by a set of "critical regions" (CR), where optimal solution functions of the variables and the objective function are defined in terms of θ. The CRs are defined by linear constraints in terms of θ, depicting hyperspaces where the optimal functions are valid.

It can be proved that, for convex models, the vertices of these CR´s represent the points of maximum error between the parametric nonlinear solution and the linear approximation (Dua and Pistikopoulos, 1999). The solution of deterministic NLPs at these vertices will then give fixed values of θ to iterate between primal NLP subproblems an the master mpLP problem. Furthermore, the convergence of the parametric linear approximations to the solution of the nonlinear problem at these vertices guarantees the convergence of the whole parametric solution to a desired tolerance δ.

This methodology presents three main disadvantages. Firstly, it requires of the solution of a large number of NLP subproblems (at every vertex of the CRs), which increases not only with the number of uncertain parameters, but also with the number of critical regions obtained from the master problem. Secondly, since the number of critical regions usually increases as the complexity of the mpLP increases, the authors use only one of those NLP solutions to reformulate the new master problem. This not only could lead to a more relaxed approximation and eventually to a larger number of outside iterations, but at the same time, the information already obtained from the solution of the rest of the vertices is ignored. Finally, it generates complex mpLP master problems by including in a cumulative way the linearizations of the nonlinear functions at the selected vertex, when many of these linearizations are redundant.

These aspects are considered in the work presented by Moncada and Acevedo (2001) where a heuristic procedure is used to reduce the computational effort, mainly by reducing the number of NLPs to be solved per iteration.

This reduction is obtained by avoiding the solution of the NLP subproblems at vertices where the error of the parametric solution, i.e. the difference between the solution of the mpLP at a given θ and the real solution at that point, is less than a given tolerance. Since the real optimal value is not known prior to the solution of the NLP at that θ, several estimation methods where studied. The best results were obtained from the estimation of the change in the objective function when moving from a vertex of known solution (θ_j) to the vertex under study (θ_i). This change can be estimated as follows:

$$\Delta Z = OF_{lin}^{\theta_j}(\theta_i) - Z^*(\theta_j) \qquad (2)$$

where $Z^*(\theta_j)$ is the known optimal solution at θ_j and $OF_{lin}^{\theta_j}(\theta_i)$ is the linearization of the objective function with respect to θ_j, evaluated at θ_i. ΔZ is considered an estimation of the error of the parametric solution at θ_i, based on the information obtained by the solution of the NLP at θ_j.

A heuristic criterion (Criterion of the Expected Error) is then applied to decide whether the NLP should be solved or not. This criterion is mathematically defined as follows:

$$\left| \frac{OF_{lin}^{\theta_j}(\theta_i) - Z^*(\theta_j)}{Z^*(\theta_j)} \right| \ge \varepsilon \left[1 + \frac{\bar{Z}(\theta_i) - OF_{lin}^{\theta_j}(\theta_i)}{Z^*(\theta_j)} \right] \qquad (3)$$

where $\bar{Z}(\theta_i)$ is the parametric solution obtained form the mpLP, evaluated at θ_i, and ε is a tolerance.

The left-hand side of Eq. (3), named here the Coefficient of the Expected Error ($CEE_{i,j}$), evaluates if the objective function is changing rapidly around the vertex where a solution has been found. If so, it is likely that the new NLP will add important information to the master problem. In this way, if the relative change is greater than the right-hand side then a new NLP must be solved. The right-hand side has a term that modifies the tolerance ε trying to define if the change of the objective function has already been captured by the parametric solution, by comparing the parametric solution to the linearized objective function. This relative difference between these two terms is added to ε, to increase the tolerance with which the solution of the NLP at that vertex is avoided. The value of ε was set at ten times the final tolerance (δ) and reduced every time no vertices were selected in an iteration, until the value of the final tolerance is reached (Moncada and Acevedo, 2001).

Proposed Algorithm

In this section, a new algorithm is developed based on the definition of a representative Coefficient of the Expected Error (CEE_{rep}). CEE_{rep} represents the estimation of the error that the parametric linear profiles could have at a given vertex based on the information obtained from *all* the deterministic NLP subproblems that have been solved through out the algorithm. This coefficient is then used to determine the vertices that are expected to have the largest errors, which will in turn be the candidate points where new deterministic NLPs will be solved. The main steps of this algorithm are described now.

Initialization

An initial optimal solution is found by solving problem (1) as a deterministic NLP, taking the vector of uncertain parameters, θ, as variables. The optimal solution found, $Z^*(\theta_0)$, is used to define outer-approximations of the model to generate the first mpLP. A tolerance (δ) for the approximation to the mpNLP is set.

Solution of the Multiparametric Master Problem

The mpLP master problem is solved according to the methodology of Gal (1995). Its solution yields a number of critical regions and the corresponding optimal functions. The vertices of the critical regions are identified (Dua and Pistikopoulos, 1999) and those infeasible are discarded.

Solution of the Primal NLP Subproblems

The vertices where new deterministic NLPs must be solved are defined by the systematic application of the Criterion of the Expected Error according to the following procedure:

The criterion in Eq. (3) is evaluated for every vertex θ_i of the new parametric solution with respect to every point θ_j where a deterministic NLP has already been solved. If $CEE_{i,j}$ is less than the right-hand side of Eq. (3) for any θ_j, vertex θ_i is discarded.

A representative Coefficient of the Expected Error ($CEE_{i,rep}$) is obtained for each vertex as follows:

$$CEE_{i,rep} = \min_j \left(CEE_{i,j}\right) = \min_j \left| \frac{OF_{lin}^{q_j}(q_i) - Z^*(q_j)}{Z^*(q_j)} \right| \quad (4)$$

The vertex with the largest CEE_{rep} represents an estimation of the point in θ where the maximum error may occur, and is therefore selected for the solution of a deterministic NLP.

The error between the parametric linear approximation and this new solution is used to define the convergence of the algorithm. If the error is greater than δ and ε, then the value of ε is update with the error and the step is repeated with the new solution of the NLP. If the error is greater than δ but less ε, then the major iteration is concluded. If the error is less than δ, but another vertex was solved in this step, the major iteration is concluded. However, if no other vertex was solved then the algorithm has converged and the last parametric solution represents an δ-approximation of the solution of the mpNLP.

The rest of the vertices with smaller $CEE_{i,j}$ are discarded and the new master problem is formulated.

Formulation of a New Master Problem

With the solutions at the chosen vertices, new linearizations of the nonlinear functions are determined with respect to x. Including all the approximations, however, increases the complexity of the mpLP and, in some cases, do not improve greatly the solution. The linearizations are then analyzed as follows:

Linearizations of a constraint are selected so as to ensure that the value of the constraint at each solution point obtained from the NLPs is approximated to a tolerance of ε with the minimum number of approximations.

Linearizations of the objective function at points where the calculated error of the parametric approximations is less than the tolerance ε, are analyzed in a similar way as the constraints.

Linearizations of the objective function at points where the calculated error of the parametric approximations is larger than the tolerance ε, are always included. The error between this linearization evaluated at any other point θ_k where a NLP has been solved can then be evaluated. If this error is less than the tolerance ε, then the linearization obtained from the solution at θ_k can be discarded.

The algorithm for the solution of mpNLP continues with the solution of the reformulated master problem.

This algorithm has three main features: first, it reduces considerably the number of NLPs to solve at each iteration; second, it exploits the information available from the solution of deterministic NLPs through out the algorithm, and three, it reduces the complexity of the master mpLP problem while maintaining a tight approximation of the original problem. In the next section, a numerical example is presented in detail to evaluate these features.

A Numerical Example

The example presented here is based on the structure of Example 2 solved by Dua and Pistikopoulos (1999). The original problem is a mixed-integer nonlinear problem, which is solved here for fixed values of the integer variables, considering all the processes.

The new mpNLP formulation includes 26 continuous variables, 12 equality constraints and 24 inequality constraints. In addition to the three uncertain variables considered in the original problem, another three were included in this formulation. In three of the continuous parameters, a margin of variation for the uncertain parameters of 50% or more was considered. A discrete parameter (equipment availability) is also considered. The final tolerance was set to 3%, i.e. δ=0.03.

The solution procedure runs as follows. The first optimal solution identified vertex θ_0=(1,1,1,1,1,1) as the best-case scenario. At this point linear outer-approximations were defined to formulate the first mpLP master problem. Its solution yields three critical regions delimited by 140 feasible vertices, including θ_0. The Criterion of the Expected Error is applied to these vertices finding 28 vertices to discard.

From the evaluation of the representative coefficient CEE_{rep} for the remaining 112 vertices, the vertex with the largest expected error is θ_1=(0,0,0,0,0,0). After solving the NLP at θ_1 a real error of 3.75% is determined and ε is set to 0.375.

The criterion is applied again to the 111 remaining vertices, this time with respect to θ_0 *and* θ_1, selecting 87 vertices to be discarded. The representative coefficient CEE_{rep} is re-evaluated for the 24 vertices left, identifying θ_2=(0.58, 0, 1, 1, 0, 0.16) to solve a new deterministic NLP. Vertex θ_2 is found infeasible, and a feasibility problem (Dua and Pistikopoulos, 1999) is solved to find the nearest feasible point at θ_2=(0.55, 0, 1, 1, 0, 0.203). The solution of the NLP shows that the error of the parametric approximation at this point is 2.615%, smaller than the problem tolerance, defining the end of this major iteration.

The linearizations obtained at θ_1 and θ_2 are now compared to those obtained at θ_0. The linearization of the objective function with respect to θ_1 presents a good approximation to the parametric solution in every vertex, and only this linearization is included to the next master problem.

For the nonlinear constraints, however, the new linearizations with respect to θ_1 are redundant (based on the current value of ε) with respect to those obtained at θ_0, so the first linearizations remain in the master problem.

The new mpLP obtained was solved identifying a parametric solution defined by 5 critical regions with 238 feasible vertices. After using the heuristic criterion, the largest estimated error is found at θ_3=(0.0585, 1, 0, 0, 1). Solving the NLP, a real error of 1,545% was calculated indicating that the error is within tolerance and therefore we have reached the mpNLP solution.

The computational efficiency of the algorithm has been demonstrated solving only 3 NLPs out of a total of 140 vertices in the first iteration, and only one in the second iteration, out of a total of 238 vertices. The NLPs at these final vertices were solved to confirm that they were all within the defined tolerance.

Conclusions

In this work, a parametric programming approach was proposed for the consideration of uncertainty in the optimization of manufacturing systems. A new algorithm was developed that allows the incorporation of larger numbers of uncertain parameters without increasing geometrically the computational efforts.

The estimation of an expected error allows, on one hand, the selection of suitable points for the solution of deterministic NLPs, and, on the other, the incorporation of the information obtained through out the algorithm to reduced the number of subproblems that are solved. It also allows to define good approximations to formulate the master problem. The approach uses several heuristics to guide the mathematical algorithm, which have proved to be very efficient and consistent in the examples that have been solved. Furthermore, this heuristics do not change the convergence properties of the basic algorithm.

New lines of research that this algorithm permits to visualize now are the extension to mixed-integer models and non-convex formulations.

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