OPTIMIZATION UNDER UNCERTAINTY: STATE-OF-THE-ART AND OPPORTUNITIES

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Abstract

A large number of problems in production planning and scheduling, location, transportation, finance, and engineering design require that decisions be made in the presence of uncertainty. Uncertainty, for instance, governs the prices of fuels, the availability of electricity, and the demand for chemicals.

A key difficulty in optimization under uncertainty is in dealing with an uncertainty space that is huge and frequently leads to very large-scale optimization models. Decision-making under uncertainty is often further complicated by the presence of integer decision variables to model logical and other discrete decisions in a multi-period or multi-stage setting.

This paper reviews theory and methodology that have been developed to cope with the complexity of optimization problems under uncertainty. We discuss and contrast the classical recourse-based stochastic programming, robust stochastic programming, probabilistic (chance-constraint) programming, and fuzzy programming. The advantages and shortcomings of these models are reviewed and illustrated through examples. Applications and the state-of-the-art in computations are also reviewed.

Finally, we discuss several main areas for future development in this field. These include development of polynomial-time approximation schemes for multi-stage stochastic programs and the application of global optimization algorithms to two-stage and chance-constraint formulations.

Keywords

Stochastic programming, Global optimization, Approximation schemes, Fuzzy Programming.

Introduction

Over the second half of the twentieth century, optimization found widespread applications in the study of physical and chemical systems, production planning and scheduling systems, location and transportation problems, resource allocation in financial systems, and engineering design. From the very beginning of the application of optimization in these problems, it was recognized that analysts of natural and technological systems are almost always confronted with uncertainty.

Beginning with the seminal works of Beale (1955), Dantzig (1955), Tintner (1955), Charnes & Cooper (1959), and Bellman & Zadeh (1970), mathematical programming under uncertainty has experienced rapid development in both theory and algorithms. Today, Dantzig still considers planning under uncertainty as one of the most important open problems in optimization (Horner 1999).

It is beyond the scope of this paper to provide a detailed coverage of optimization under uncertainty. The main purpose of this paper is to provide a short overview of the field and give pointers to the literature that can be used as starting points for further study. For additional details and information, we refer the reader to the recent textbooks of Zimmermann (1991), Kall & Wallace (1994), Prékopa (1995), and Birge & Louveaux (1997), and the very informative *Stochastic Programming Web Site* (2002).

Approaches to optimization under uncertainty have followed a variety of modeling philosophies, including expectation minimization, minimization of deviations from goals, minimization of maximum costs, and optimization over soft constraints. We begin with an overview of the two main approaches to optimization under uncertainty: stochastic programming (recourse models, robust stochastic programming, and probabilistic models) and fuzzy programming (flexible and possibilistic programming). Then, we review applications and the state-of-the-art in computations, as well as important algorithmic developments by the process systems engineering community. Finally, we draw connections between models for optimization under uncertainty and global optimization. Throughout the presentation, we point out the fundamental differences of different modeling philosophies in optimization under uncertainty.

Stochastic Programming

Programming with Recourse

Under the standard two-stage stochastic programming paradigm, the decision variables of an optimization problem under uncertainty are partitioned into two sets. The *first-stage* variables are those that have to be decided before the actual realization of the uncertain parameters. Subsequently, once the random events have presented themselves, further design or operational policy improvements can be made by selecting, at a certain cost, the values of the *second-stage*, or *recourse*, variables. Traditionally, the second-stage variables are interpreted as corrective measures or recourse against any infeasibilities arising due to a particular realization of uncertainty. However, the second-stage problem may also be an operational-level decision problem following a first-stage plan and the uncertainty realization. Due to uncertainty, the second-stage cost is a random variable. The objective is to choose the first-stage variables in a way that the sum of the first-stage costs and the expected value of the random second-stage costs is minimized. The concept of recourse has been applied to linear, integer, and nonlinear programming.

Stochastic Linear Programming

A standard formulation of the two-stage stochastic linear program is (Birge & Louveaux 1997, Kall & Wallace 1994):

$$\begin{aligned} \min \quad & c^t x + \mathrm{E}_{\omega\in\Omega}[Q(x,\omega)] \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{1}$$

with

$$\begin{aligned} Q(x,\omega) = \quad \min \quad & f(\omega)^t y \\ \text{s.t.} \quad & D(\omega) y \geq h(\omega) + T(\omega) x \\ & y \in Y, \end{aligned} \tag{2}$$

where $X \subseteq \Re^{n_1}$ and $Y \subseteq \Re^{n_2}$ are polyhedral sets. Here, $c \in \Re^{n_1}$, ω is a random variable from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $\Omega \subseteq \Re^k$, $f : \Omega \to \Re^{n_2}$, $h : \Omega \to \Re^{m_2}$, $D : \Omega \to \Re^{m_2 \times n_2}$, $T : \Omega \to \Re^{m_2 \times n_1}$. Problem (1) with variables x constitute the first stage which needs to be decided prior to the realization of the uncertain parameters $\omega \in \Omega$. Problem (2) with variables y constitute the second stage.

Under the assumption of discrete distributions of the uncertain parameters, the problem can be equivalently formulated as a large-scale linear program which can be solved using standard LP technology. Convexity properties of the recourse function $Q(\cdot)$ (Wets 1966, Wets 1974) have been effectively used in decomposition-based solution strategies (Van Slyke & Wets 1969, Birge & Louveaux 1988).

For continuous parameter distributions, these properties have been used to develop sampling-based decomposition and approximation schemes (Van Slyke & Wets 1969, Ruszczyński 1986, Birge & Louveaux 1988, Higle & Sen 1991, Infanger 1994, Shapiro & Homem-de-Mello 1998) as well as gradient-based algorithms (Ermoliev 1983, Shapiro & Wardi 1996).

The two-stage formulation is readily extended to a multi-stage setting by modeling the uncertainty as a filtration process. Under discrete distributions, this reduces to a scenario tree of parameter realizations. Decomposition schemes that partition the time stage (Birge 1985) as well as those that partition the scenario space (Rockafellar & Wets 1991) have been developed for multi-stage linear programs.

For an extensive discussion of stochastic linear programming, the reader is referred to standard textbooks on stochastic programming (Infanger 1994, Kall & Wallace 1994, Prékopa 1995, Birge & Louveaux 1997).

Stochastic Integer Programming

Stochastic integer programming addresses instances of (1)-(2) where the set Y contains integer restrictions. Much of the early work in this area has been on the design and analysis of heuristics for two-stage stochastic integer programs (Dempster, Fisher, Jansen, Lageweg, Lenstra & Rinnooy Kan 1981, Spaccamela, Rinnooy Kan & Stougie 1984, Stougie 1985). Exact algorithmic approaches are more recent and include extensions of the decomposition strategies for stochastic linear programs (Laporte & Louveaux 1993, Carøe & Tind 1998), Lagrangian relaxation schemes (Takriti, Birge & Long 1996, Carøe & Schultz 1999), algebraic enumeration (Schultz, Stougie & van der Vlerk 1998), convexification (Higle & Sen 2000, Sherali & Fraticelli 2000), and decomposition/branch-and-bound (Ahmed, Tawarmalani & Sahinidis 2000).

For problems where the second-stage recourse matrix D possesses a special structure known as *simple recourse*, Klein Haneveld, Stougie & van der Vlerk (1995) and Klein Haneveld, Stougie & van der Vlerk (1996) proposed solution schemes based upon the construction of the convex hull of the second-stage value function. For more general recourse structure, Laporte & Louveaux (1993) proposed a decomposition-based approach for stochastic integer programs when the first-stage variables are pure binary. This restriction allows for the construction of optimality cuts that approximate the non-convex second-stage value function at only the binary first-stage solutions (but not necessarily at other points). The authors proposed a branch and bound algorithm to search the space of the first-stage variables for the globally optimal solution, while using the optimality cuts to approximate the second-stage value function. Finite termination of the algorithm is obvious since the number of first-stage solutions is finite. The method has been successfully used in solving two-stage stochastic location-routing problems (Laporte, Louveaux & Mercure 1989, Laporte, Louveaux & Mercure 1992, Laporte, Louveaux & Mercure 1994, Laporte, Louveaux & van Hamme 1994). Unfortunately, the algorithm is not applicable if any of the first-stage variables are continuous. Carøe & Tind (1998) generalized this algorithm to handle cases with mixed-integer first- and second-stage variables. The method requires the use of non-linear integer programming dual functions to approximate the second-stage value function in the space of the first-stage variables. The resulting master problem then consists of non-linear (possibly discontinuous) cuts and no practical method for its solution is currently known.

Carøe & Tind (1997), Carøe (1998), and Carøe & Schultz (1999) used the scenario decomposition approach of Rockafellar & Wets (1991) to develop a branch and bound algorithm for stochastic integer programs. This method solves the Lagrangian dual, obtained by dualizing the non-anticipativity constraints, as the lower bounding problem within a standard branch and bound framework. The subproblems of the Lagrangian dual correspond to the second-stage scenarios and are difficult to solve as they include integer constraints. Furthermore, although the Lagrangian dual provides very tight bounds, its solution requires the use of subgradient methods and is computationally expensive. A potential limitation of this approach is that finite termination is guaranteed only if the first-stage variables are purely discrete, or if an $\epsilon-$optimal termination criterion with $\epsilon > 0$ is used.

Recently, Schultz et al. (1998) proposed a finite scheme for two-stage stochastic programs with discrete distributions and pure-integer second-stage variables. For this problem, Schultz et al. (1998) observe that only integer values of the right-hand-side parameters of the second-stage problem are relevant. This fact is used to identify a finite set in the space of the first-stage variables containing the optimal solution. Schultz et al. (1998) propose complete enumeration of this set to search for the optimal solution. This set may be very large and evaluation of each of its elements requires the solution of second-stage integer subproblems. Thus, this approach is, in general, computationally prohibitive.

The above papers assume discrete probability distributions for the uncertain parameters. Except for simple cases that afford closed form solutions, sampling is required when dealing with continuous distributions of the problem parameters. Thus, convergence proofs for the resulting algorithms have to be probabilistic. For continuous distributions, Norkin, Ermoliev & Ruszczyński (1998) developed a branch and bound algorithm that makes use of stochastic upper and lower bounds and proved almost sure convergence.

More recently, stochastic integer programming is receiving increased attention from the point of view of convexification. Higle & Sen (2000) and Sherali & Fraticelli (2000) have proposed algorithms that invoke ideas from lift-and-project (Balas, Ceria & Cornuejols 1993) and the reformulation-linearization technique (Adams & Sherali 1990) in the context of Benders-like decomposition approaches. These approaches are in their formative stages and no implementations have yet been reported. By exploiting some of the structural properties of stochastic integer programs, Ahmed et al. (2000) develop a finite branch and bound scheme for a class of stochastic integer programs and present some encouraging computational results on small problems.

Stochastic Nonlinear Programming

Nonlinear versions of the linear and integer programs considered in this paper have many applications, especially in engineering design, as well as planning and scheduling. For example, two-stage nonlinear stochastic programming addresses the problem:

$$\begin{aligned} \min \quad & f(x) + \mathrm{E}_{\omega\in\Omega}[Q(x,\omega)] \\ \text{s.t.} \quad & g(x) \le 0, \end{aligned} \tag{3}$$

with

$$\begin{aligned} Q(x,\omega) = \quad \min \quad & F(\omega,x,y) \\ \text{s.t.} \quad & G(\omega,x,y) \le 0 \\ & y \in Y, \end{aligned} \tag{4}$$

where $X \subseteq \Re^{n_1}$, $Y \subseteq \Re^{n_2}$, ω is a random variable from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $\Omega \subseteq \Re^k$, and the real functions f, g, F, and G have conformable dimensions.

Most of the algorithms developed for stochastic linear programming carry over to the nonlinear case. However, nonlinearities may give rise to nonconvexities and local optima. We refer the reader to the thesis of Bastin (2001) for a more detailed discussion of nonlinear stochastic programming.

Robust Stochastic Programming

The recourse-based model (1) makes a decision based on present first-stage and expected second-stage costs, *i.e.*, based on the assumption that the decision-maker is risk-neutral. To capture the notion of risk in stochastic programming, Mulvey, Vanderbei & Zenios

(1995) proposed the following modification of the objective function of (1):

$$\min \quad c^t x + \mathrm{E}_{\omega\in\Omega}[Q(x,\omega)] + \lambda f(\omega, y)$$

where f is a variability measure, such as variance, of the second-stage costs and λ is a nonnegative scalar that represents the risk tolerance of the modeler. Large values of λ result into solutions that reduce variance while small values of λ reduce expected costs.

Applications of this, so-called robust stochastic programming, framework and its variants have been reported in power systems capacity expansion (Malcolm & Zenios 1994), power dispatch (Beraldi, Musmanno & Triki 1998), chemical process planning (Ahmed & Sahinidis 1998), telecommunications network design (Bai, Carpenter & Mulvey 1997, Laguna 1998), and financial planning (Mulvey et al. 1995, Bai et al. 1997, Kouwenberg & Zenios 2001).

Various examples demonstrate that a straight forward deterministic reformulation of robust models may result in second-stage solutions that are suboptimal for the recourse problem (King, Takriti & Ahmed 1997, Sen & Higle 1999). This is a highly undesirable property as it may lead to an underestimation of the recourse costs. Takriti & Ahmed (2002) proposed sufficient conditions on the variability measure to remedy this problem.

Probabilistic Programming

The recourse-based approach to stochastic programming requires the decision maker to assign a cost to recourse activities that are taken to ensure feasibility of the second-stage problem. In essence, the philosophy of this approach is that infeasibilities in the second stage are allowed at a certain penalty. The approach thus focuses on the minimization of expected recourse costs. In the probabilistic or chance-constraint approach, the focus is on the reliability of the system, *i.e.*, the system's ability to meet feasibility in an uncertain environment. This reliability is expressed as a minimum requirement on the probability of satisfying constraints.

Consider the classical linear programming model:

$$\begin{aligned} \max \quad & c^t x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0, \end{aligned} \tag{5}$$

where c and x are n-vectors, b is an m-vector, and A is an $m \times n$ matrix. Assume that there is uncertainty regarding the constraint matrix A and the right-hand-side vector b, and that the system is required to satisfy the corresponding constraint with a probability $p \in (0, 1)$. Then, the probabilistic linear program corresponding to the classical (deterministic) linear program can be stated as follows:

$$\begin{aligned} \max \quad & c^t x \\ \text{s.t.} \quad & P(Ax \geq b) \geq p \\ & x \geq 0. \end{aligned} \tag{6}$$

Consider the case when $m = 1$, *i.e.*, the case of a single constraint $P(a^t x \geq b) \geq p$. Further, assume that the vector a is deterministic while the right-hand-side b is a random variable with cumulative distribution F. Let β be such that $F(\beta) = p$. Then, the constraint $P(a^t x \geq b) \geq p$ can be written as $F(a^t x) \geq p$ or $a^t x \geq \beta$. In this simple case, the probabilistic program is equivalent to a standard linear program.

For the case when the matrix A is deterministic and the vector b has a log-concave multivariate probability density function, Prékopa (1971) has shown that the feasible set of (6) is convex. Other standard cases in which probabilistic constraints can be converted to standard constraints are summarized in Prékopa (1995). However, in general, the feasible set of (6) may be nonconvex. This issue is discussed later in this paper.

Fuzzy Mathematical Programming

Like stochastic programming, fuzzy programming also addresses optimization problems under uncertainty. A principal difference between the stochastic and fuzzy optimization approaches is in the way uncertainty is modeled. In the stochastic programming case, uncertainty is modeled through discrete or continuous probability functions. On the other hand, fuzzy programming considers random parameters as fuzzy numbers and constraints are treated as fuzzy sets. Some constraint violation is allowed and the degree of satisfaction of a constraint is defined as the membership function of the constraint. For example, consider a linear constraint $a^t x \leq \beta$ in terms of the decision vector x and assume that the random right-hand-side β can take values in the range from b to $b + \Delta b$, with $\Delta b \geq 0$. Then, the linear membership function, $u(x)$, of this constraint is defined as:

$$u(x) = \begin{cases} 1, & \text{if } a^t x \leq b, \\ 1 - \frac{a^t x - b}{\Delta b}, & \text{if } b < a^t x \leq b + \Delta b, \\ 0, & \text{if } b + \Delta b < a^t x. \end{cases}$$

Although other types of membership functions are also possible, the above linear membership function is typically used. Objective functions in fuzzy mathematical programming are treated as constraints with the lower and upper bounds of these constraints defining the decision maker's expectations.

Many of the developments in the area of fuzzy mathematical programming are based on the seminal paper by Bellman & Zadeh (1970). The field has been recently popularized by the work of Zimmermann (1991). Two types of fuzzy programming will be considered here: flexible programming and possibilistic programming. Flexible programming deals with right-hand-side coefficient uncertainties while possibilistic programming recognizes uncertainties in the objective function coefficients as well as in constraint coefficients. In both types of fuzzy programming, the membership function is used to represent the degree of satisfaction of constraints, the decision maker's expectations about the objective function level, and the range of uncertainty of coefficients.

Flexible Programming

Consider the classical linear programming model:

$$\begin{aligned} \max \quad & c^t x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0, \end{aligned} \tag{7}$$

where c and x are n-vectors, b is an m-vector, and A is an $m \times n$ matrix. Let us suppose that there is uncertainty regarding the exact values of the coefficients and some violation of the constraints is acceptable within a certain range. This means that some parts of (7) can be fuzzy. When the elements of A, b, or c are treated as fuzzy numbers rather than crisp numbers, constraints can be represented by fuzzy sets rather than by crisp inequalities, and objective functions can be represented by a fuzzy goal rather than a crisp objective function. We use $\tilde{\alpha}$ to indicate that the parameter α is fuzzy. Similarly, $a^t x \tilde{\leq} b$ means that a^t should be essentially smaller than or equal to b, *i.e.*, that this constraint is a soft constraint for which some violation is allowed. The tolerance or spread of the fuzzy parameter α will be denoted by $\Delta\alpha$.

A flexible programming problem can then be written as (Tanaka, Okuda & Asai 1974, Zimmermann 1991):

$$\begin{aligned} \tilde{\max} \quad & c^t x \\ \text{s.t.} \quad & Ax \tilde{\leq} b \\ & x \geq 0. \end{aligned} \tag{8}$$

Let us denote $(c_j, a_{1j}, \ldots, a_{mj})^t$ and $(v, b_1, \ldots, b_m)^t$ by $(\hat{a}_{0j}, \hat{a}_{1j}, \ldots, \hat{a}_{mj})^t$ and $(\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_m)^t$, respectively. Then, problem (8) can be rewritten as follows:

$$\text{Find} \quad x \quad \text{s.t.} \quad \hat{A}x \tilde{\leq} \hat{b}, \tag{9}$$

where $\hat{A}$ is an $(m+1) \times n$ matrix and $\hat{b}$ is an $(m+1)$ vector. It is assumed that the fuzzy constraints and fuzzy goal are subjectively defined by the decision maker. Let $u_i(x)$ denote the membership for the ith constraint of (8), $i = 1, \ldots, n$. Also, let $u_0(x)$ denote the membership function of the objective of (8). In addressing problem (9), Bellman & Zadeh (1970) define an optimal fuzzy decision to be:

$$x^* = \arg \max_{x \geq 0} \min_{i=0,\ldots,n} u_i(x).$$

According to this definition, the optimal solution of problem (9) can be obtained by solving the nonlinear programming problem

$$\max_{x \geq 0} \min_{i=0,\ldots,n} 1 - \frac{\hat{A}_i x - b_i}{\Delta b_i}.$$

By introducing one new variable λ, Zimmermann (1978) showed that, if all membership functions are linear, then (8) can be reduced to a classical linear program:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \bar{A}x + \lambda \leq \bar{b} \\ & x \geq 0 \\ & 0 \leq \lambda \leq 1, \end{aligned} \tag{10}$$

where the elements of $\bar{A}$ and $\bar{b}$ are $\bar{a}_{ij} = \frac{\hat{a}_{ij}}{\Delta b_i}$ and $\bar{b}_i = 1 + \frac{\hat{b}_i}{\Delta b_i}$, respectively.

Problem (10) includes one more variable and one more constraint than the original problem (8). Although a linear membership function is only a very rough approximation of the knowledge of the decision-maker about the membership function, Delgado, Herrera, Verdegay & Vila (1993) showed that the optimal solution obtained by using a linear membership function is often of the same quality as the solution obtained using a complicated nonlinear membership function. Therefore, the use of linear membership provides an efficient way to solve fuzzy programs and obtain solutions of good quality.

Note that the spread of the objective function, Δb_0, must be provided by the decision maker. It expresses the decision maker's aspiration in regard to the objective function value. Δb_0 can be estimated as the difference of the potential upper and lower bounds for the objective function (Zimmermann 1991).

Possibilistic Programming

When (7) involves uncertainty in constraint coefficients, the fuzzy program is called possibilistic (Tanaka & Asai 1984). A possibilistic linear programming problem can be written as follows:

$$\begin{aligned} \tilde{\max} \quad & \tilde{c}^t x \\ \text{s.t.} \quad & \tilde{A}x \leq \tilde{b} \\ & x \geq 0. \end{aligned} \tag{11}$$

Let a_{ij} and Δa_{ij}, respectively, represent the center and spread of the fuzzy number $\tilde{a}_{ij}$. Similarly, let c_j and Δc_j denote the center and spread of the fuzzy number $\tilde{c}_j$. Now, consider the following membership functions:

$$u_i(x) = \begin{cases} 1, & \text{if } A_i x \leq b_i, \\ 1 - \frac{A_i x - b_i}{\Delta A_i x + \Delta b_i}, & \text{if } b_i < A_i x < \\ & b_i + \Delta A_i x + \Delta b_i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u_0(x) = \begin{cases} 1, & \text{if } b_0 \leq c^t x, \\ 1 - \frac{b_0 - c^t x}{\Delta b_0 + \Delta c^t x}, & \text{if } b_0 - \Delta b_0 - \Delta c^t x \\ & < cx < b_0, \\ 0, & \text{otherwise,} \end{cases}$$

where $[b_0 - \Delta b_0, b_0]$ denotes the aspiration range for the objective. Then, the Bellman-Zadeh decision making criterion leads to the following equivalent of the possibilistic program after the introduction of a new variable λ:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & c^t x + \Delta c^t x(1-\lambda) \geq b_0 - \Delta b_0(1-\lambda) \\ & Ax - \Delta Ax(1-\lambda) \leq b + \Delta b(1-\lambda) \\ & x \geq 0 \\ & 0 \leq \lambda \leq 1. \end{aligned} \tag{12}$$

The possibilistic programming problem (11) has been reduced into the nonlinear programming problem (12). Here, b_0 and Δb_0 can be calculated by interval linear programming (Tong 1994). In general, (12) has a nonconvex feasible space.

Applications and Computations

Applications

The original applications of stochastic programming included agricultural economics in Iowa under land and labor constraints (Tintner 1955), the allocation of aircraft to routes with penalties for lost passengers (Ferguson & Dantzig 1956), and the production of heating oil with constraints on meeting sales and not exceeding capacity (Charnes, Cooper & Symonds 1958). More recent applications have included:

- production planning (Dempster et al. 1981, Lenstra, Rinnooy Kan & Stougie 1983, Bitran, Haas & Matsuo 1986, Escudero, Kamesam, King & Wets 1993),
- scheduling (Dempster 1982, Dempster, Fisher, Jansen, Lageweg, Lenstra & Rinnooy Kan 1983, Tayur, Thomas & Natraj 1995, Birge & Dempster 1996),
- routing (Spaccamela et al. 1984, Laporte et al. 1989, Laporte et al. 1992),
- location (Laporte, Louveaux & van Hamme 1994),
- capacity expansion (Sherali, Soyster, Murphy & Sen 1984, Davis, Dempster, Sethi & Vermes 1987, Bienstock & Shapiro 1988, Eppen, Martin & Schrage 1989, Berman, Ganz & Wagner 1994, Malcolm & Zenios 1994),
- energy investment and electricity production (Louveaux 1980, Pereira & Pinto 1991, Morton 1996, Takriti et al. 1996, Carøe, Ruszczyński & Schultz 1997),
- environmental management and control (King, Rockafellar, Somlyody & Wets 1988, Somlyody & Wets 1988, Pinter 1991, Watanabe & Ellis 1993, Wagner, Shamir & Marks 1994, Birge & Rosa 1996, Norkin et al. 1998),
- water management (Dupacova, Gaivoronski, Kos & Szantai 1991),
- agriculture (Shukla & Gupta 1989, Helgason & Wallace 1991),
- telecommunications (Laguna 1998, Tomasgard, Dye, Wallace, Audestad, Stougie & van der Vlerk 1998),
- design and optimization of chemical processing systems (Liu & Sahinidis 1996, Clay & Grossmann 1997, Acevedo & Pistikopoulos 1998, Gupta & Maranas 2000),
- finance (Kallberg, White & Ziemba 1982, Mulvey & Vladimirou 1992, Dert 1995, Carino & Ziemba 1998, Kouwenberg & Zenios 2001).

Finally, applications of fuzzy programming have included production planning (Inuiguchi, Sakawa & Kum 1994), transportation problems (Chanas, Kolodziejczyk & Machaj 1984, Bit, Bisswal & Alam 1993b, Bit, Bisswal & Alam 1993a, Chalam 1994), water supply planning (Slowinski 1986), forest management (Pickenss & Hof 1991), capacity expansion (Liu & Sahinidis 1997b), and bank management (Lai & Hwang 1993b, Lai & Hwang 1993a).

State-of-the-Art in Computations

Stochastic programming problems are much more difficult than their deterministic counterparts. Yet, significant progress has been made towards their exact and approximate solution.

Exact solution of deterministic equivalents of stochastic linear programs relies on decomposition. In a recent review paper, Birge (1997) reports the exact solution, on parallel computers, of stochastic linear programs with up to one million variables in their deterministic equivalents. Much larger problems are typically solvable by sampling-based rather than decomposition methods. Impressive computational results on a computational grid are reported by Linderoth, Shapiro & Wright (2002) on stochastic linear programs with up to 10^{81} scenarios that were solved using sample-average approximations.

Much smaller stochastic programs have been reported solved for the integer case. Exact solutions have been recently obtained for relatively small problems as reported by Ahmed et al. (2000). We refer the reader to Verweij, Ahmed, Kleywegt, Nemhauser & Shapiro (2002) for a recent application of sampling-based methods to a stochastic routing problem with 2^{1694} scenarios within an estimated 1% of optimality.

Finally, we refer the reader to the *Stochastic Programming Web Site* (2002) for links to software as well as test problem collections for stochastic programming.

Developments by the Process Systems Engineering Community

The process systems engineering community has long been involved in the development of tools for the solution of design and operational problems under uncertainty. These efforts have been motivated by applications and, in many cases, yielded general-purpose algorithms. In this section, we review four of these developments.

Aggregation-Disaggregation Algorithm for Two-Stage Stochastic Linear Programming

Clay & Grossmann (1997) address the two-stage stochastic linear programming program with discrete probability distributions for the uncertain parameters. In recognition of the fact that the complexity of the problem stems from the large number of scenarios of uncertainty, these authors propose an aggregation of the probability space followed by successive disaggregation. Lower and upper bounds on the optimal objective function of the original problem were then developed over partition elements of the probability space. A sensitivity analysis was also developed for guiding the disaggregation process. The algorithm was applied to stochastic planing models from the process industries and was demonstrated to require only few partitions for the bounds to converge. Problems with millions of rows and columns in the deterministic equivalent were successfully solved with this approach.

Multiparametric Programming Approach for Mixed-Integer Quadratic Programming

When the number of uncertain variables is relatively small, it is possible to obtain closed-form solutions of optimization problems in terms of the values of the uncertain parameters. Dua, Bozinis & Pistikopoulos (2002) obtain such solutions for mixed-integer quadratic programs with a few uncertain parameters. The basic idea is to utilize parametric nonlinear programming tools to systematically characterize the space of parameters by a set of regions of optimality. The algorithm developed by these authors was applied to model predictive and hybrid control problems.

Exact Branch-And-Bound Algorithm for Two-Stage Stochastic Integer Programming

Ahmed et al. (2000) address a general class of two-stage stochastic integer programs with integer recourse and discrete distributions. By restating the problem in terms of the so-called *tender variables*, the discontinuities associated with the value function of the second-stage integer problem become orthogonal to the variable axes. The authors then develop a branch-and-bound algorithm to solve this problem. This scheme departs from previous literature in that it avoids explicit enumeration of the search space while guaranteeing finiteness.

An Approximation Scheme for Multistage Stochastic Integer Programs for Capacity Expansion

In a recent line of research, Liu & Sahinidis (1997a), Ahmed & Sahinidis (2000), Ahmed & Sahinidis (2002), and Furman & Sahinidis (2002) proposed linear programming based heuristics for operational and design problems in process systems engineering. The distinguishing feature of this solution paradigm is the analytical derivation of bounds for the quality of the solutions obtained by heuristics that run in polynomial time. In particular, Ahmed & Sahinidis (2002) address the approximate solution of large-scale multistage stochastic integer programs arising from capacity expansion in the process industries. The presence of integer variables in every stage makes this problem very challenging. Through a suitable rounding of the linear programming relaxation and bundling of the capacity expansion decisions, these authors obtain a feasible integer solution to this problem. Through a probabilistic analysis, the authors prove that the optimality gap of the solution thus obtained almost surely vanishes asymptotically as the number of stages increases. Computational experience demonstrates that the proposed approach yields near-optimal solutions even for small problem sizes.

Connections Between Global Optimization and Optimization under Uncertainty

The purpose of this section is to demonstrate that many optimization programs under uncertainty are very difficult to solve as they correspond to multi-extremal nonlinear optimization problems even when this is not directly apparent, as is the case in seemingly linear formulations. For this purpose, we present two examples, the first from stochastic integer programming and the second from probabilistic programming. Other classes of stochastic programs that give rise to multiextremal global optimization problems are possibilistic programs (Liu & Sahinidis 1997b), stochastic programs with decision-dependent uncertainties (Ahmed 2002), and robust stochastic programs (King et al. 1997).

Two-Stage Stochastic Integer Programming

Consider the following example of a two-stage stochastic integer program from Schultz et al. (1998):

$$\begin{aligned} \text{EX1}: \quad \min \quad & -1.5x_1 - 4x_2 + \sum_{s=1}^{4} p^s Q^s(x_1, x_2) \\ \text{s.t.} \quad & 0 \leq x_1 \leq 5 \\ & 0 \leq x_2 \leq 5 \end{aligned}$$

where $Q^s(x_1, x_2)$ is defined as the optimal objective function value of:

$$\begin{aligned} \min \quad & -16y_1 - 19y_2 - 23y_3 - 28y_4 \\ \text{s.t.} \quad & 2y_1 + 3y_2 + 4y_3 + 5y_4 \leq \omega_1^s - \frac{1}{3}x_1 - \frac{2}{3}x_2 \\ & 6y_1 + y_2 + 3y_3 + 2y_4 \leq \omega_2^s - \frac{2}{3}x_1 - \frac{1}{3}x_2 \\ & y_i \in \{0, 1\} \qquad i = 1, \ldots, 4 \end{aligned}$$

and $(\omega_1, \omega_2) \in \{5, 15\} \times \{5, 15\}$ with uniform probability (*i.e.*, $p^s = \frac{1}{4}$ for $s = 1, \ldots, 4$).

There are only two degrees of freedom in this example: the first-stage variables x_1 and x_2. Once their values are specified, the y-variables are determined by solving the second-stage integer optimization problem. This allows us to plot the objective function value of EX1 in the space of x_1-x_2 (Figure 1 from Ahmed (2002)). The objective is piecewise polyhedral with several local minima. The unique global minimum is at (0,3).

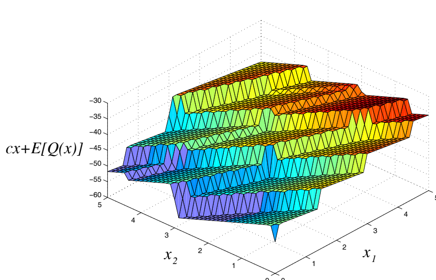

Figure 1: Value function of EX1

Probabilistic Programming

Consider the following probabilistic program in terms of two variables and two probabilistic constraints:

$$\begin{aligned} \text{EX2}: \quad \max \quad & c^t x \\ \text{s.t.} \quad & P\begin{pmatrix} x_1 + x_2 \geq b_1 \\ x_1 + 3x_2 \geq b_2 \end{pmatrix} \geq 0.5 \\ & x_1 \geq 0, x_2 \geq 0, \end{aligned}$$

where b_1 and b_2 are dependent random variables with $P(b_1 = 2, b_2 = 4) = 0.5$ and $P(b_1 = 3, b_2 = 0) = 0.5$. Clearly, any (x_1, x_2) satisfying $x_1 + x_2 \geq 2$ and $x_1 + 3x_2 \geq 4$ is feasible to EX2. Let this polyhedral set be denoted by $P1$. Similarly, let $P2$ denote the polyhedral set of points satisfying $x_1 + x_2 \geq 3$ and $x_1 + 3x_2 \geq 0$. The union of $P1$ and $P2$ provides the complete feasible set of EX2. This union is not convex as shown in Figure 2.

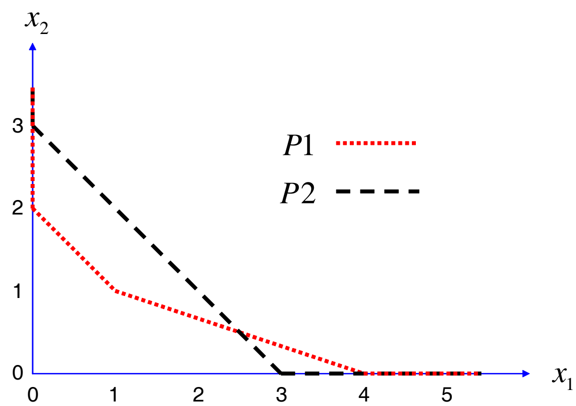

Figure 2: Feasible set of EX2

Conclusions

Several modeling frameworks have been proposed in the literature for optimization under uncertainty. Along with them, a variety of algorithms have been developed and used successfully in many applications.

The current state-of-the-art in this field allows approximate solution of very large-scale problems with sampling-based methods. Exact solution of deterministic equivalents is much harder and requires the use of advanced computer architectures.

There are several challenges in the area of optimization under uncertainty. Here, we mention a few:

- There is a notable need for systematic comparisons between the different modeling philosophies. A small step in this direction has been taken by Liu & Sahinidis (1996) who compared stochastic programming and fuzzy programming as applied to chemical process planning.

- While significant progress has been made towards the solution of two-stage stochastic programs, the multi-stage case represents a significant challenge. In the case of stochastic integer programming with integer variables in stages other than the first, this represents a conceptual in addition to computational challenge. Deeper understanding of problem structure and properties is required in order to devise applicable algorithms.

- Contrary to the linear case, the integer and nonlinear cases have received limited attention. Computational results abound for the linear case but are somewhat limited for the integer and nonlinear cases. It appears unlikely that general-purpose algorithms will solve such problems exactly. Instead, we expect to see the development of problem-specific approximation schemes for integer and nonlinear problems such as the asymptotically optimal approximation scheme recently proposed by Ahmed & Sahinidis (2002) for capacity expansion of chemical processes.

- As the previous section illustrated, there are several opportunities for the development and application of global optimization algorithms to solve optimization problems under uncertainty.

Acknowledgments

I am grateful to my former students M. L. Liu and S. Ahmed who introduced me to many of the topics discussed in this paper. Partial financial support from the National Science Foundation under award DMI 01-15166 is also gratefully acknowledged.

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