FEASIBILITY ANALYSIS AND OPTIMAL DESIGN USING A CHANCE CONSTRAINED PROGRAMMING FRAMEWORK

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Abstract

We propose a new framework to address feasibility analysis and optimal design problems under uncertainty. This approach is based on nonlinear chance constrained programming. The feasibility analysis problem is defined as the maximization of the achievable confidence level of satisfying all constraints for a given design. The solution can provide clear information about the dependence of the reliability of a nonlinear convex system on the values of the design variables. This offers *a priori* the feasible region for the design optimization problem. A feature of this approach is that for a design with a 100% confidence level the solution does not depend on the distribution of the uncertain variables. Moreover, the critical constraint which cuts off the largest part of the design space can be identified so that, if necessary, a decision can be made to relax this constraint to achieve a meaningful design. The chance constrained program will be relaxed to a single level nonlinear optimization problem (NLP). The scope of this approach is demonstrated with a nonlinear design problem.

Keywords

Optimal design, Uncertainty, Feasibility analysis, Chance constrained programming, NLP.

Introduction

Many studies on feasibility analysis and optimal design under uncertainty have recently been conducted. These investigations are motivated by the fact that almost all processes are subject to uncertainties. The *a priori* information about the impact of these uncertainties on process operations is desired so that a feasible as well as optimal design can be achieved. This remains as a challenging task for process systems engineering.

Following the definition of feasibility test and flexibility index, optimal design problems can be formulated with fixed flexibility (Halemane and Grossmann, 1983) or variable flexibility (Grossmann and Morari, 1983). A nested two-stage approach was proposed to solve the optimization problem (Pistikopoulos and Ierapetritou, 1995; Rooney and Biegler, 2001). The first stage (design stage) requires the solution of a multi-period problem constructed from discrete points in the intervals of the uncertain variables. The second stage (feasibility stage) tests the feasibility of the fixed design given by the first stage, over all ranges of the uncertain variables. The iteration between the first and the second stage demands expensive computational efforts.

In this work, we propose a novel framework for feasibility analysis and design optimization under uncertainty. This framework is based on the method of chance constrained programming for convex systems (Prékopa, 1995). The feasibility analysis problem is defined as maximization of the achievable confidence level of satisfying all constraints for a given design. The feature of this approach is that the chance constrained program can be relaxed to a single level (instead of multilevel) convex nonlinear optimization problem. If the resulting confidence level is 100%, there will be at least one set of operation variables that can be chosen during plant operation, such that, for every possible realization of the uncertain variables, all of the constraints will be satisfied. Otherwise, if the resulting confidence level is less than 100%, the given design will be infeasible.

This approach can employs the available probability density function (PDF) of the uncertain variables. Solutions in the whole design space provide clear information about the dependence of the reliability of a nonlinear convex system on the values of the design variables. On the other hand, if specific intervals of the uncertain variables are only available, one most prominent distribution (e.g. normal distribution) over the intervals will be assumed for the probability computation. Since the feasibility is determined only by the criterion of a 100% confidence level which is independent of the PDF of the uncertain variables, this assumption has no impact on the result of the feasibility analysis. In this case, the results over the whole design space offer *a priori* the feasible region for the design optimization problem. Moreover, the results make it possible to identify the critical constraint which will cut off the largest part of the design space. This is important information for the designer to relax the critical constraint, if allowable, to gain a meaningful design. An example is used to demonstrate the scope of the proposed approach.

Feasibility Analysis with Chance Constraints

Problem formulation

Consider feasibility analysis for a process with a given design that can be generally described as

$$\mathbf{h}(\mathbf{x},\mathbf{u},\boldsymbol{\theta},\hat{\mathbf{d}}) = \mathbf{0} \tag{1}$$

$$\mathbf{g}(\mathbf{x},\mathbf{u},\boldsymbol{\theta},\hat{\mathbf{d}}) \le \mathbf{0} \tag{2}$$

where $\mathbf{x} \in X \subseteq \Re^{n_\mathbf{x}}$ are the state variables, $\mathbf{u} \in U \subseteq \Re^{n_\mathbf{u}}$ are the control variables, $\boldsymbol{\theta} \in \Theta \subseteq \Re^{K}$ are the uncertain variables, and $\mathbf{d} \in D \subseteq \Re^{n_\mathbf{d}}$ are design variables. $\hat{\mathbf{d}}$ is a given point in the design space. $\mathbf{h} \subseteq \Re^{n_\mathbf{x}}$ are the model equations being able to describe the process over the whole spaces of *all* of the variables. $\mathbf{g} \subseteq \Re^{L}$ are the inequality constraints which must be satisfied during the process operation. Note that the dimension of $\mathbf{x}$ is always the same as the dimension of $\mathbf{h}$, and thus the state variables $\mathbf{x}$ can be implicitly eliminated by solving the equation system $\mathbf{h}$, through a simulation step. It means that we have in effect the following inequality constraints

$$\mathbf{g}(\mathbf{u},\boldsymbol{\theta},\hat{\mathbf{d}}) \le \mathbf{0} \tag{3}$$

to be evaluated for the feasibility analysis. In this study, convex systems are considered, i.e. the components in $\mathbf{g}(\mathbf{u},\boldsymbol{\theta},\hat{\mathbf{d}})$ are convex functions. The feasibility analysis of the design $\hat{\mathbf{d}}$ is to check if there exists a set of controls $\mathbf{u} \in U$ with which all inequality constraints in (3) can be satisfied, under *any* possible realization of the uncertain variables $\boldsymbol{\theta}$. Unlike the definition of the feasibility test and flexibility index, we define the following stochastic optimization problem under *single* chance constraints

$$\begin{aligned} &\max_{\mathbf{u},\alpha} \quad \alpha \\ &\text{s.t.} \quad P_l\left\{g_l(\mathbf{u},\boldsymbol{\theta},\hat{\mathbf{d}}) \le \mathbf{0}\right\} \ge \alpha \qquad l = 1,\cdots,L \end{aligned} \tag{4}$$

where α is the confidence level (i.e. reliability of being feasible) to hold each constrain in (3), which is a variable to be maximized. $P_l \; (l = 1,\cdots,L)$ is the probability measure for constraint *l*. The aim of this problem is, for a given design, to search for the set of controls $\mathbf{u}^*$ with which the maximum confidence level, $\alpha^{\max}$, of satisfying all inequality constraints under any possible realization of the uncertain variables, is achieved. It should be noted that $\alpha^{\max}$ corresponds to the least value of the probabilities of holding the individual constraints. At the solution point, $\mathbf{u} = \mathbf{u}^*$, there may be one or more constraints which are active, i.e.

$$P_j\left\{g_j(\mathbf{u}^*,\boldsymbol{\theta},\hat{\mathbf{d}}) \le \mathbf{0}\right\} = \alpha^{\max} \qquad j \in l \tag{5}$$

And the rest of the constraints will have their probability levels larger than $\alpha^{\max}$. If $\alpha^{\max} = 1$, it means all constraints will be satisfied with a 100% confidence level and thus the given design $\hat{\mathbf{d}}$ is feasible. If $\alpha^{\max} < 1$, then $\hat{\mathbf{d}}$ should not be considered as a candidate for the design, since otherwise there is some probability of violating the constraints.

For different processes, the *a priori* knowledge about the uncertain variables $\boldsymbol{\theta}$ may be very different. In most previous studies, it has been assumed that the uncertain variables have probable values over certain intervals, i.e. $\boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U]$, where $\boldsymbol{\theta}^L$ and $\boldsymbol{\theta}^U$ are the known lower and upper bound, respectively. Even in the case of having the knowledge of uncertain variables with normal distributions (with expected values $\boldsymbol{\theta}^{\text{ex}}$ and variances $\boldsymbol{\sigma}$), they have also been treated as intervals with e.g. $\boldsymbol{\theta} \in [\boldsymbol{\theta}^{\text{ex}} - 4\boldsymbol{\sigma}, \boldsymbol{\theta}^{\text{ex}} + 4\boldsymbol{\sigma}]$ (Pistikopoulos and Ierapetritou, 1995; Bansal et al., 2002). In this way, the available distribution information is not fully utilized.

Using the method of chance constrained programming, the probability density function $\rho(\boldsymbol{\theta})$, if available, can be employed for the feasibility analysis. The solution of problem (4), $\alpha^{\max}$, represents the reliability of satisfying the constraints by the given design $\hat{\mathbf{d}}$. Since $\alpha^{\max}$ is a function of $\hat{\mathbf{d}}$, all solutions in the design space $\mathbf{d} \in D$ will provide clear information about the dependence of the reliability of the system on the values of the design variables.

On the other hand, in the case that uncertain variables are only known in some certain intervals, to use the chance constrained programming framework, we can assume that they have one most prominent distribution (e.g. normal or uniform) over the known intervals $\boldsymbol{\theta} \in [\boldsymbol{\theta}^L, \boldsymbol{\theta}^U]$. Since feasibility can only be determined by the criterion that the confidence level is 100%, which is independent of the PDF of the uncertain variables, this assumption has no impact on the result of the feasibility analysis. In this case, the result over the whole design space provides the feasible region for the design optimization problem.

The Solution Approach

To solve the chance constrained problem (4), it has to be transformed into an equivalent deterministic NLP problem. The essential challenge here lies in the computation of the probabilities of holding the constraints as well as the gradients of the probability functions. A solution approach to nonlinear problems with one single probabilistic constraint is proposed by Wendt et al. (2002). This approach is suitable for solving the feasibility analysis problems. The basic idea of this method is to map the output distribution to that of the uncertain input variables.

One of the uncertain variables θ_s that has a monotone relation with the constraint function g_l is selected. This can be done by analyzing the physical relations between y^C and $\boldsymbol{\theta}$. Due to this monotony, the required boundary of g_l (i.e. *zero*) in the output region corresponds to a boundary value θ_s^{Lim} for θ_s in the input region. The boundary value can be computed by solving the model equation system $\mathbf{h}$ based on the given design $\hat{\mathbf{d}}$ and given decision variables $\mathbf{u}$. Then the computation of a *single* probability of the output constraint can be transformed to a multivariate integration in the limited space of the uncertain inputs

$$P\left\{g_l(\mathbf{u},\boldsymbol{\theta},\hat{\mathbf{d}}) \le 0\right\} = P\left\{\theta_s \le \theta_s^{Lim},\ \theta_k \subseteq \Re^K, k \ne s\right\}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\xi_s^{Lim}} \cdots \int_{-\infty}^{\infty} \rho(\boldsymbol{\theta}) d\theta_1 \cdots d\theta_s \cdots d\theta_K \tag{6}$$

where the $\rho(\boldsymbol{\theta})$ is the unified distribution function of $\boldsymbol{\theta}$. Based on this probability computation, we propose a sequential NLP approach to solve the problem (4) for the given design $\hat{\mathbf{d}}$. An NLP solver is then used to optimize the variables $\mathbf{u}$ and α. The principle of the probability computation can be described with Fig. 1. Suppose we have the monotony that g_l is proportional to θ_s. Because of the uncertain variables $\boldsymbol{\theta}$, the values of different sets of controls ($\mathbf{u}^{(a)}, \mathbf{u}^{(b)}, \mathbf{u}^{(c)}$) may result in three different distributions of the constrained outputs (e.g. $g_l^{(a)}, g_l^{(b)}, g_l^{(c)}$) with $P\left\{g_l^{(a)} \le 0\right\} = 1$, $P\left\{g_l^{(b)} \le 0\right\} < 1$, and $P\left\{g_l^{(c)} \le 0\right\} << 1$. Due to the monotony, the limiting values ($\theta_s^{Lim(a)}, \theta_s^{Lim(b)}, \theta_s^{Lim(c)}$) of the uncertain variable θ_s can be determined based on the model equations and thus the corresponding probabilities of holding the constraint can be computed. Note that this solution strategy is independent of the distribution of the uncertain input variables, i.e. it can be applied to any distribution, if the multivariate integration in (6) can be carried out.

As mentioned before, to evaluate the feasibility of a given design, we only need to assess the value of $\alpha^{\max}$; if it is 1.0, then the design is feasible, otherwise it is infeasible. This decision is independent of the distribution of the uncertain variables. Thus we can assume they have *normal distributions* to obtain the decision. Collocation on finite elements for correlated uncertain variables with normal distributions is used for the multivariate integration. That is, the range of each uncertain variable is divided into certain intervals and in each interval the PDF is approximated by orthogonal polynomials based on internal collocation points. The gradients required by the NLP solver can be computed simultaneously during the integration as well. For solving problem (4), all single chance constraints and their gradients to the controls have to be computed.

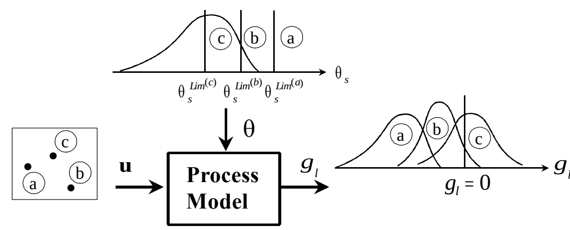

Figure 1. Probability computation of a single constraint.

Computation results

To demonstrate the proposed approach, the example given by Bansal et al. (2002) is considered. After elimination of the state variables, the system is described with the following set of inequalities:

$$g_1 = 0.08u^2 - \theta_1 - \frac{1}{20}\theta_2 + \frac{1}{5}d_1 - 13 \le 0 \tag{7}$$

$$g_2 = -u - \frac{1}{3}\theta_1^{1/2} + \frac{1}{20}d_2 + \frac{1}{5}d_1 + 11\frac{1}{3} \le 0 \tag{8}$$

$$g_3 = \exp(0.21u) + \theta_1 + \frac{1}{20}\theta_2 - \frac{1}{5}d_1 - \frac{1}{20}d_2 - 11 \le 0 \tag{9}$$

The uncertain variables both have nominal values of 3 and expected deviations ± 1, i.e. $2 \le \theta_i \le 4$, $i = 1, 2$. For a given design, $\hat{d}_1, \hat{d}_2$, we determine the feasibility of the system in three ways. First, the feasibility of a design can be directly determined by analyzing the inequalities (7)-(9), i.e. the given design will be feasible, if a value of *u* can be found to satisfy all three inequalities. This algebraic analysis leads to the result that the design will be either feasible or infeasible. Fig. 2 shows the feasible region of the system.

Second, we assume both uncertain variables have uniform distributions. The corresponding problem in the form of (4) is formulated and solved with the proposed approach. Fig. 3 shows the feasible regions with different probability levels of holding the constraints. It can be seen that the feasible region will shrink, if the required confidence level is increased. With the 100% reliability, the feasible region is identical with that shown in Fig. 2.

Third, both uncertain variables are assumed to be normally distributed. The corresponding chance constrained problem is solved and the result is shown in Fig. 4. In comparison to Fig. 3, the feasible region with same confidence levels will be different due to the different distributions of the uncertain variables. But at the 100% confidence level, both kinds of distributions lead to the same feasible region.

Fig. 5 illustrates the achievable maximum probability in relation to the value of design variables, with the uncertain variables uniformly distributed. It can clearly indicate the dependence of the reliability of a design.

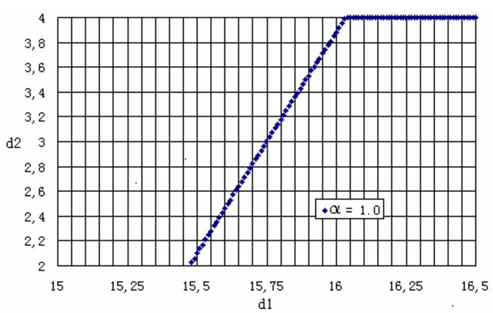

Figure 2. Feasible region of the problem through algebraic analysis.

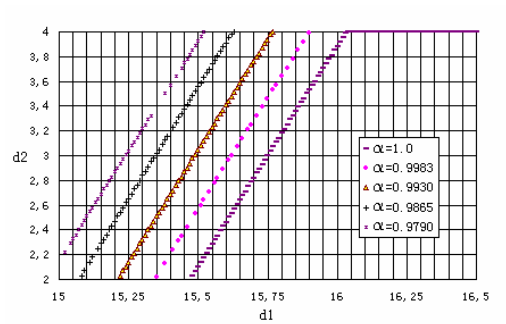

Figure 3. Feasible region with uniform distribution of uncertain variables.

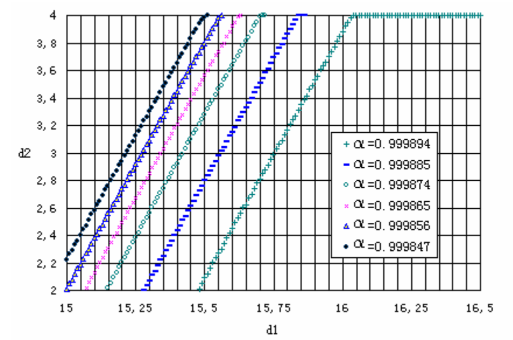

Figure 4. Feasible region with normal distribution of uncertain variables.

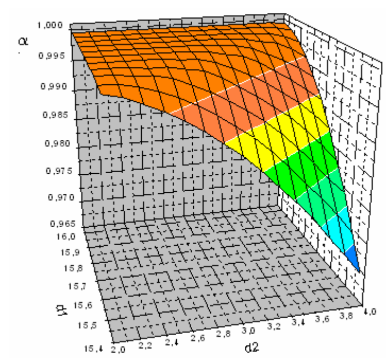

Figure 5. Probability distribution (confidence level) corresponding to different designs.

Conclusions

A new framework was proposed to analyze the feasibility of design problems with uncertainty. This is made by chance constrained programming which leads to a single-stage computational approach. A feasible design will be ensured by a 100% probability of holding the process constraints. The feasible region can be identified so that the designer can chose a decision according to specific design criteria. In particular, an optimal design can be gained based on the results of feasibility analysis, i.e. a chance constrained optimization can be carried out if a feasible region is available. Correlations between uncertain variables have not been considered in the current work. Moreover, convexity analysis of this framework presents a challenge future work.

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