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Optimal sizing of production units for goods subject to stochastic demand

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Abstract

This paper presents a general framework to handle the optimal sizing of single-stage multi-product units incorporating demand uncertainty. This problem consists on the determination of the type, number of units and their capacities, chosen from a set of standard equipment sizes. The objective function is the profit expectation maximization, involving five components: i. revenue; ii. production cost; iii. equipment depreciation cost; iv. storage cost; v. cost due to non fulfillment of the demand target. The multiple dimension integral representation of the expectation is approximated through cubature formulae. With linear process models, an overall MILP problem formulation is obtained. The approach is applied to the design of the furnace section of a ceramic tile industrial plant, where different products are obtained.

Keywords: Process design, Sizing, Demand uncertainty, Cubature formulae.

1. Introduction

A large number of process industries involve a complex network of factors. Quite often the market dynamics significantly influence the overall product mix and their corresponding service levels. Optimal process design for these situations must explicitly incorporate the demand uncertainty, giving rise to process design problems under uncertainty. Three different types of strategies are found in literature, to address these problems: a. the scenario-based approach, in which the uncertainty is represented by a set of scenarios [1]; b. the parametric-based approach, characterized by parametrical solution of the optimization problem over the domain of uncertainty [1]; c. the stochastic-based approach, in which the uncertainties are described by probability functions, and sampling procedures [2] or integration formulae [3] are used to approximate the expectation operators. For a detailed analysis of stochastic optimization see the review of Sahinidis [4]. Stochastic features have been incorporated into several types of Chemical Engineering problems ranging from process scheduling to process design, capacity expansion and supply chain optimization. Here we introduce a stochastic-based approach for sizing the equipment that should be installed in multiproduct plants, where the demand of the product mix is subject to uncertainty. The aim is to determine the type and number of such units, their capacities and the corresponding operation plan, through maximization of the profit expectation.

2. General Formulation

The basic assumptions for the formulation given below are: i. The process diagram has just one processing stage. ii. Only one time period is considered in the economic evalutation of the profit obtained. iii. Uncertainty is considered in the demand levels for each product, but not on the corresponding prices. iv. The distribution functions are assumed to be independent for each parameter. Rather than fundamental assumptions, these hypotheses are made essentially for simplicity of the resulting mathematical model.
We denote by $N \in \{1,...,n\}$ the set of possible equipment options that can be allocated, and $P \in \{1,..., p\}$ the set of existing products. The major discrete decision variables are $y \equiv \{y_j, j \in N\}$, where $y_j \in \{0,1\}$ describes the allocation of units of type j. The continuous variables are $x \equiv \{x_i, i \in P\}$, the production level for each product, and $s \equiv \{s_i, i \in P\}$, the quantity sold of each product. Model parameters are $v \equiv \{v_j, j \in N\}$, the capacity factors for each possible type of equipment, and $\theta \equiv \{\theta_i, i \in P\}$, the demand level for each product. It is assumed that each θ_i is uncertain and modeled by a probability distribution function (PDF) $J(\theta_i)$. Additional variables in the model are $a \equiv \{a_i, i \in P\}$ the amount of product i overprocessed, and $b \equiv \{b_i, i \in P\}$, the demand *not met* for product i. These are both positive or null quantities defined by $a_i = \max\{0, x_i - \theta_i\}$ and $b_i = \max\{0, \theta_i - x_i\}$. The objective function is represented by $L(\theta) = S(\theta) - C_o - C_d - C_s(\theta) - C_b(\theta)$ where the revenue term is represented by $S(\theta)$, the operation cost by C_o, the equipment depreciation cost by C_d, the storage cost as $C_s(\theta)$, and the cost of producing below the demand and subsequently breaking the contracts with the customers as $C_b(\theta)$. Given the above definitions, the objective function is given as:

$$S = \sum_{i \in P} r_i s_i \ , \ C_o = \sum_{i \in P} c_{oi} x_i \ , \ C_d = \sum_{j \in N} c_{dj} y_j \ , \ C_s = \sum_{i \in P} c_{si} a_i \ , \ C_b = \sum_{i \in P} c_{bi} b_i \quad (1)$$

where r_i, c_{oi}, c_{dj}, c_{si} and c_{bi} are the corresponding cost coefficients.

To model the over/under production scenarios, a set of discrete variables $z_i \in \{0,1\}$ is introduced. If product i is overproduced $(x_i - \theta_i > 0)$ then $z_i = 1$, if it is underproduced $(x_i - \theta_i < 0)$ then $z_i = 0$; if $x_i = \theta_i$, z_i can assume any of the previous values. We also define $l_i = x_i - \theta_i$, a real variable that represents the level of production of product i. To relate l_i with z_i we add to the formulation the set of constraints $M(z_i - 1) \leq l_i \leq Mz_i$. The formulation then takes the form:

$$\max_{s,x,y,z} E_\theta\left[S(\theta) - C_o - C_d - C_s(\theta) - C_b(\theta)\right] = \max E_\theta\left[L(\theta)\right] \quad (2.a)$$

s.t.

$$h(x, y, v, \theta) = 0\,, \quad (2.b)$$

$$g(x, y, v, \theta) \leq 0 \quad (2.c)$$

$$l = x - \theta\,, \quad (2.d)$$

$$M(z_i - 1) \leq l_i \leq Mz_i \quad (2.e)$$

$$a_i = \max\{0, x_i - \theta_i\}\,, \quad (2.f)$$

$$b_i = \max\{0, \theta_i - x_i\} \quad (2.g)$$

$$s_i = \min\{x_i, \theta_i\} \quad (2.h)$$

$$s, x \geq 0\,, \qquad y_j, z_i \in \{0,1\}$$

where E_θ stands for the expectation operator. In Equations (2.b) and (2.c) the functions $h(\bullet)$ and $g(\bullet)$ are sets of equalities and inequalities modeling the process operation, including feasible operating levels. The expectation is represented by the p-dimension integral:

$$E_\theta\left[L(\theta)\right] = \int_{\theta_1}\int_{\theta_2} \cdots \int_{\theta_p} L(\theta) J_p(\theta_p) \cdots J_2(\theta_2) J_1(\theta_1) \mathrm{d}\theta_p \cdots \mathrm{d}\theta_2 \mathrm{d}\theta_1 \quad (3)$$

This formulation exploits the concept of robustness applied to process design, since it enables to produce more than the demand requirement with penalties arising in the form of storage costs. Production below the demand is also possible, with penalties due to contract unfulfillment and subsequent impact on the organization image. This approach is related with the multiobjective framework proposed by Goyal and Ierapetritou [5], that introduces a constraint to model customers satisfaction in case the target demand are met. The integral (7) is calculated employing cubature formulae based on the rules of Stroud [7], already used by Bernardo *et al.* with the same purpose [3], since they proved to be more accurate than the best sampling algorithms, namely the Hamersley sequence sampling introduced by Diwekar and Kalagnanam [8].

The evaluation of equations (4) and (5) can be simplified, noting that the variables a_i, b_i and s_i are only used in the objective function, and appear in sumations of negative and positive terms, respectively. For instance, the definition of a_i in equation (4) can be replaced by the set of inequalities

$$a_i \leq Mz_i\,,\ a_i \geq x_i - \theta_i\,,\ a_i \geq 0\,,\ \text{with}\ C_s = \sum_{i \in P} C_{si} a_i \tag{4}$$

since the term with C_s is negative in the objective. Similarly for b_i, we have

$$b_i \leq M(1 - z_i)\,,\ b_i \geq \theta_i - x_i\,,\ b_i \geq 0\,,\ \text{with}\ C_b = \sum_{i \in P} C_{bi} b_i \tag{5}$$

where M is an upper bound on the quantities produced. Alternative formulations to produce tighter relaxations of the constraints involving M could also be used [10, 11], although they were not found necessary in the examples considered. Finally for s_i, we need to consider both the situations where the demand is met and not. Since we are maximizing the objective, the sales term $S(\theta)$ can be computed by (1), with the constraints (6) added to the formulation.

$$s_i \leq x_i\,,\ s_i \leq \theta_i \tag{6}$$

This set of constraints allows an easier evaluation of the formulation given above. The problem so formulated originates an MILP, if the model equations and constraints (3) have a linear form.

3. Application

The framework introduced in section 2 is applied to the design of the furnaces section of a continuous ceramic tile production unit. The aim is to determine the number of furnace units to install, their dimension, and the optimal production plans for a set of discrete scenarios derived from cubature points used in integral calculation of the profit expectation in the domain θ. The unit produces three types of tiles with different dimensions but all of square form. Table 1 lists the product mix, prices, costs and demand features. The storage cost is due warehouse space, and the operation cost accounts for the cost of producing one square meter of tile. Since the heat required to process all types of tiles is equal, the operation cost is 9.08 €/m^2 for all products, and the storage cost is 0.501 €/m^2.week, independent on the characteristics of the units to install. The uncertainty of demand of each product is modeled by independent normal distributions captured from the market.

The furnaces available in the market are of discrete length and width, and it is possible to choose any combination of sizes listed in Table 2. The depreciation cost is established assuming that the equipment has a life time of 9.5 years [9]. The process model is presented as following:

$$\pi_{i,j,k} = \left\lfloor \frac{w_j - 2\varepsilon}{az_i} \right\rfloor az_i r_k \frac{24}{\tau} 7 \quad (7)$$

$$\Pi_{m,k,j,i,t} = \varsigma_{m,k,j,i,t} \pi_{i,j,k} \quad (8)$$

$$\alpha_{i,t} = \sum_{m \in N_3} \sum_{k \in N_2} \sum_{j \in N_1} \Pi_{m,k,j,i,t} \quad (9)$$

$$y_{m,k,j} \geq \sum_{i \in P} \varsigma_{m,k,j,i,t} \quad (10)$$

$$0 \leq \sum_{i \in P} \varsigma_{m,k,j,i,t} \leq 1 \quad (11)$$

$$\sum_{m \in N_3} y_{m,k,j} \leq 1 \quad (12)$$

where $N_1 \in \{1,...,n_1\}$ is the set of lengths of furnaces, $N_2 \in \{1,...,n_2\}$ the set of widths, $w \equiv \{w_j, j \in N_2\}$ the furnace width, $r \equiv \{r_k, k \in N_1\}$ the furnace length, $az \equiv \{az_i, i \in P\}$ the size of tiles, $T \in \{1,...,t\}$ the set of discretization points, $\Pi \equiv \{\Pi_{m,k,j,i,l}, m \in N_3, k \in N_1, j \in N_2, i \in P, t \in T\}$ the productions in each unit, $\varsigma \equiv \{\varsigma_{m,k,j,i,l}, m \in N_3, k \in N_1, j \in N_2, i \in P, t \in T\}$ the fraction of time used to produce each product in each unit, $\pi \equiv \{\pi_{k,j,i}, k \in N_1, j \in N_2, i \in P\}$ the production capacity of each unit, $N_3 \in \{1,...,n_3\}$ the set of units with the same dimensions, $\alpha \equiv \{\alpha_{i,t}, i \in P, t \in T\}$ the production level, designated in the general formulation as $x_i, i \in P$, $\lfloor \bullet \rfloor$ stands for the floor (int) operator, ε for the distance between furnace walls and tile, and τ stands for the time each square meter of tile is inside the furnaces (0.83 h) independently on the length of the furnaces, causing the velocity of displacement inside furnaces of different lengths to be different.

Table 1. Products, prices, costs and demand representation.

Product	Dimension (cm)	Price and costs (€/m^2)		Demand Uncertainty (m^2/week)
		r	C_b	
1	20×20	12.0	12.15	$N(2400,100)$*
2	31×31	15.0	15.15	$N(2500,100)$
3	50×50	20.0	20.15	$N(1600,50)$

* $N(\mu,\sigma)$ - normal distribution with average μ and standard deviation σ.

Table 2. Furnace sizes in the market and depreciation cost (€/week).

Width (m)	Length (m) 65	70	80	90
1.6	397	416	452	502
1.8	434	448	502	548

The expectation integral is evaluated based on 4 points in each dimension that leads to a full grid comprising 64 points. The optimization problem comprising

8328 equations and 10650 variables is solved with GAMS/CPLEX to a relative tolerance of 10^{-6}. The optimal solution is presented in Table 3, requiring 2.47 CPU s in a Windows XP Pentium IV based platform, and leading to a revenue of 32,922 € per week.

Table 3. Optimal solution (number of units).

Furnaces width (m)	Furnaces length (m)			
	65	70	80	90
1.6	-	-	-	-
1.8	1	-	-	1

4. Conclusion

This paper presents a general framework for the optimal design of single-stage production units devoted to process goods subject to stochastic demand. The optimal design is achieved through the maximization of the profit expectation employing cubature formulae to evaluate the multi-dimension integral. The original problem is reformulated to fall in the MILP class, with linear process models, aiming to exploit the guarantee of existence of solutions and the robustness and efficiency of the algorithms available. The formulation was applied to the design of the furnace section of a ceramic tile plant producing three different products, showing excellent efficiency properties and appealing characteristics to handle linear process models of much larger size. It is also applicable to non-linear processes, although the complexity of the solution of the resulting MINLPs might introduce more significant problems limitations.

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