MIMO H-INFINITY CONTROLLER DESIGN FOR SIMULTANEOUS GUARANTEED INPUT AND OUTPUT STABILITY MARGINS

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Abstract

We consider completely controllable and observable plant described by:
Plant: $\dot{x} = Ax + Bu, \quad y = Cx$;
controller: $u(s) = K(s)y(s)$.
Find the transfer matrix of the output controller K(s), such that the following frequency inequalities are valid:

$$[I + W^y(-j\omega)]^T [I + W^y(j\omega)] \geq r^2 I,$$
$$[I + W^u(-j\omega)]^T [I + W^u(j\omega)] \geq r^2 I,$$

where W^y and W^u are, respectively, output and control open-loop transfer matrices. The controller has the following form:

$$\dot{x}_c = A_c x_c + B_c y, \quad u = C_c x_c + D_c y, \quad x_c \in R^{n_c}$$

with the associated transfer matrix

$$K(s) = C_c(sI - A_c)^{-1} B_c + D_c.$$

To obtain a numerical solution to the problem, the method of linear matrix inequalities (LMI) is used.

1 Introduction

In [3], output optimal controller design procedures where proposed using H_∞ methods. They guarantee the given radius of stability margins w.r.t. the physical output or input of the plant. However, large stability margin w.r.t. output does not necessarily ensure that of the input and vice versa$ an example of such a situation with H_∞ state controller is given below. This paper proposes a method that provides the given radius simultaneously w.r.t. output and input.

Considered is a completely controllable and observable plant described by $m_2 \times m$ transfer matrix $W_0(s)$. The objective is to find the transfer matrix of an output stabilizing controller $K(s)$ such that for all $\omega \in [0, \infty)$, the following circular frequency inequality holds

$$[I + W(-j\omega)]^T [I + W(j\omega)] \geq r^2 I, \quad (1)$$

where $0 < r \leq 1$ is the given radius of stability margins [3], and $W(s)$ is the open-loop transfer matrix of the system, which is broken at input ($W = -KW_0$), or output ($W = -W_0 K$). In other words, inequality (1) must be satisfied in both cases above.

We show that such a problem can be reduced to a standart H_∞-optimization problem; however, in contrast to the results in [3], only sufficient nature of the corresponding conditions can be obtained.

The numerical solution is convenient for implementation using state space approach on the basis of linear matrix inequalities and the MATLAB LMI Control Toolbox [6]. An illustrative example involves multivariable controller design for a multi-engine electrical actuator.

Example 1. [4] H-infinity state controller. We consider a time-invariant system described by

Plant: $\dot{x} = Ax + B_1 w + B_2 u, \quad y = x, \quad z_1 = Cx$ (2)

Controller: $u = Kx = -B_2^T Px$, (3)

where $P \geq 0$ is a solution of the algebraic Riccati equation (ARE) [5,7]

$$A^T P + PA + \gamma^{-2} PB_1 B_1^T P - PB_2 B_2^T P = -C^T QC \quad (4)$$

where γ is a scalar such that a solution $P \geq 0$ of ARE (4) exists. Let $B_1 = B_2$, then we introduce a number $\alpha = \gamma^{-2} - 1$ and rewrite ARE (4) in the following form

$$A^T P + PA + \alpha PB_1 B_1^T P = -C^T QC. \quad (5)$$

We consider system (2) with controller (3), where $\alpha = 10^{-2} - 10^{-6}$ and $Q = 1$ and

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}, \; B_2 = [0 \quad 1]^T, \; C = [-3 \quad 1].$$

It is easy to check that the solution of ARE (5) and the controller matrix (3) have the form

$$P = 10^3 \begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix} > 0, \quad K = 10^3 [3 \quad -1].$$

Consider open-loop transfer matrix (at the plant input) of system (2), (3):

$$W(s) = -K(sI - A)^{-1} B_2 ;$$

it has the following form:

$$W(s) = \frac{10^3 s}{s^2 + 5s + 4}$$

and satisfies the passivity condition $\operatorname{Re} W(j\omega) \geq 0$, $\omega \in [0, \infty)$. For this regulator, the gain margin is infinite and the phase margin exceeds 90^0.

Suppose that the element a_{11} of matrix A has the form $A_{11} = -3 + \varepsilon$, where $\varepsilon > 0$ is a small number. In this case, the characteristic polynomial of the closed-loop system is

$$D(s) = s^2 + (1005 - \varepsilon) + 4 - 1002\varepsilon .$$

The system is unstable if $\varepsilon > 4/1002$. Thus, in spite of large gain and phase margins, the H_∞ suboptimal system may not be robust.

In this example, it is easy to demonstrate the nearness to instability by breaking the loop at the plant output, for example, with respect to the variable x_1 (the first input of controller). In this case, the open-loop transfer function has the form

$$W_1 = \frac{-3000}{s^2 + 1005s + 3004}$$

and gains and phase margins are very low.

2 Problem statement

Consider completely controllable and observable plant of in state space:

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{6}$$

where $x \in R^n$ are the plant states; $u \in R^m$ is the control input; $y \in R^{m_2}$ is measurable output ($m_2 = m$). The associate transfer matrix from $u(s)$ to $y(s)$ has the form

$$W(s) = C(sI - A)^{-1} B, \quad y(s) = W(s)u(s). \tag{7}$$

Problem. Find the transfer matrix of the output controller $K(s)$:

$$u(s) = K(s)y(s), \tag{8}$$

such that the following frequency inequalities are valid

$$[I + W^y(-j\omega)]^T [I + W^y(j\omega)] \geq r^2 I , \tag{9}$$

$$[I + W^u(-j\omega)]^T [I + W^u(j\omega)] \geq r^2 I . \tag{10}$$

Here r is a be given ($0 < r < 1$) number; $W^y(s) = -W(s)K(s)$ is the open-loop transfer matrix (at the plant physical output) of system (6), (8); $W^u(s) = -K(s)W(s)$ is the open-loop transfer matrix (at the plant physical input) of system (6), (8). By analogy with [1], the quantity r shall be referred to as radius of stability margins. If the frequency inequalities (9) and (10) are satisfied, we say w.r.t that the system has radius of stability margins r the plant input and output. Let's explain physical meaning of this concept.

In case of a single-input, single-output (SISO) ($m = m_2 = 1$) the fulfilment of inequalities (9), (10) means [1] that the frequency plot $W^u(j\omega) = W^y(j\omega)$ does not intersect a circle of radius r with center an the critical point (-1, $j0$). In the case, phase and gain stability margins can be determined from r; see below.

For MIMO systems, frequency inequalities (9), (10) have the following physical interpretation [8, 9]: for each of the physical outputs y_i (inputs u_i), the gains l_i with nominal value 1 given by can be changed independently over the intervals

$$\frac{1}{1+r} < l_i < \frac{1}{1-r}, \quad i = \overline{1, m_2} \quad (i = \overline{1, m}), \tag{11}$$

without loss of stability, see [8]. Intervals (11) are referred to as gain multiloop stability margins it physical outputs (inputs) of the plant.

The phase multiloop stability margins are introduced in a more abstract way. In this case, transfer functions $l_i(s) = e^{j\psi_i}$ can be incorporated into each of output (input) channels without loss of stability. The quantities ψ_i above are pure shifts with zero nominal values and the following limits:

$$|\psi_i| < \arccos\left(1 - \frac{r^2}{2}\right), \quad i = \overline{1, m_2} \quad (i = \overline{1, m}), \tag{12}$$

see [8].

3 Reduction to the standard problem H-infinity optimization

We show that the problem of ensuring the given radius of stability margins r in (9), (10) or its maximization (over $0 < r \leq 1$) can be reduced to standard H-infinity optimization. With this purpose we shall solve the following problem

$$\|T_{zw}\|_\infty \leq \gamma, \quad \gamma = r^{-1}, \tag{13}$$

where the matrix T_{zw} links the extended vector of controlled variables $z = [z_1^T \quad z_2^T]^T$ to the vector of dummy disturbances $w = [w_1^T \quad w_2^T]^T$ (see figure 1).

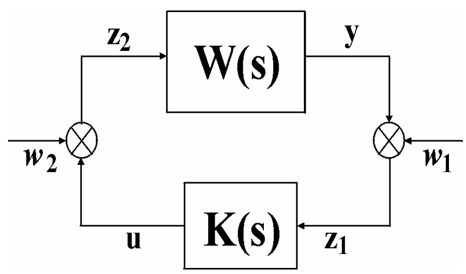

Figure 1: Control system configuration.

We show that the fulfilment of inequality (13) implies the fulfilment of inequalities (9), (10). We rewrite (13) as

$$\left\| \begin{matrix} T_{z_1 w_1} & T_{z_1 w_2} \\ T_{z_2 w_1} & T_{z_2 w_2} \end{matrix} \right\|_\infty \le \gamma ,$$

and if this inequality is satisfied, then the following inequalities are also satisfied:

$$\left\| T_{z_1 w_1} \right\|_\infty \le \gamma, \quad \left\| T_{z_1 w_2} \right\|_\infty \le \gamma, \tag{14}$$

$$\left\| T_{z_2 w_1} \right\|_\infty \le \gamma, \quad \left\| T_{z_2 w_2} \right\|_\infty \le \gamma, \tag{15}$$

The 1st and the 4st of inequalities (14), (15) can be rewritten in the form

$$T^T_{z_1 w_1}(-j\omega) T_{z_1 w_1}(j\omega) \le \gamma^2 I , \tag{16}$$

$$T_{z_1 w_1}(s) = [I + W^y(s)]^{-1}$$

$$T^T_{z_2 w_2}(-j\omega) T_{z_2 w_2}(j\omega) \le \gamma^2 I . \tag{17}$$

$$T_{z_2 w_2}(s) = [I + W^u(s)]^{-1}$$

Inequalities (16) and (17) are equivalent to frequency circular inequalities (9) and (10), respectively, where $\gamma = r^{-1}$.

Now the problem of finding the desired transfer matrix $K(s)$ (such that (9) and (10) are satisfied) can be formulated as an H-infinity optimal control problem.

Problem 1. Let a robust performance bound $\gamma \ge 1$ be given. Find a controller transfer matrix (8) such that inequalities (9) and (10) be fulfilled.

Problem 2. Find a controller transfer matrix (8) such that inequalities (9) and (10) be satisfied with minimum possible $\gamma = \gamma_0$.

If problems 1 and 2 are solved the radius of stability margins is given by $r = \gamma^{-1}$, and from inequalities (11), (12), the boundaries on guaranteed gain and phase multiloop stability margins at physical input and output can be found.

The block diagram on fig. 1 corresponds to the following equations

$$y = Wz_2, \quad z_2 = u + w_2, \quad u = Kz_1, \quad z_1 = y + w_1 . \tag{18}$$

Equations (18) the following standard form adopted in H-infinity control theory [5, 7]:

$$z = G_{11} w + G_{12} u, \quad y = G_{21} w + G_{22} u, \quad u = Ky, \tag{19}$$

where G_{ij} ($i, j = 1,2$) are the elements of the transfer matrix $G(s)$ of augmented plant

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}.$$

Block diagram associated with equations (19) is given in fig.2.

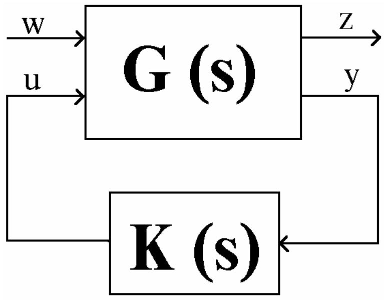

Figure 2: The block diagram of augmented plant.

It is easy to obtain transfer matrixes G_{ij} ($i, j = 1,2$) from (18). Indeed, by trivial algebraic manipulations, equation (18) reduces to

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} I & W \\ 0 & I \end{bmatrix} w + \begin{bmatrix} W \\ I \end{bmatrix} u ,$$

whence it is possible to obtain matrixes G_{11} and G_{12}. From (18) it also follows that the input signal of the controller is the controlled variable z_1 and, thus, z_1 and y represent the same signal on figure 2 (y on a figure 2 and y in equations (18) denote different signals!). From the above, equation (19) takes the following form:

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} I & W \\ 0 & I \end{bmatrix} w + \begin{bmatrix} W \\ I \end{bmatrix} u ,$$

$$y = z_1 = [I \quad W] w + W u ,$$

$$u = Ky = Kz_1 . \tag{20}$$

From (22) it is possible to find matrixes G_{21} and G_{22}:

$$G_{11} = \begin{bmatrix} I & W \\ 0 & I \end{bmatrix}, \quad G_{12} = [W \quad I]^T ,$$

$$G_{21} = [I \quad W], \quad G_{22} = W , \tag{21}$$

where W is the plant transfer matrix (7). Hence, the original problem is reduced to the standard problem of disturbance attenuation in system (19), (21) with disturbance w, controlled variables z and performance specification (13); in other words, to problems 1 and 2 which are usually considered in H_∞.

4 Numerical solution

Numerical solution can be performed by different methods. From the computational complexity the point of view, frequency or polynomial methods concede state-space method [7]. This approach requires the equations (19), (21) be in the state-space form [5]

$$\begin{aligned} \dot{x} &= Ax + \widetilde{B}_1 w + \widetilde{B}_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u , \\ y &= C_2 x + D_{21} w + D_{22} u , \end{aligned} \tag{22}$$

where, using the form (20) it is easy to establish that

$$\widetilde{B}_1 = [B \quad 0], \quad \widetilde{B}_2 = B, \quad C_1 = C_2 = C, \tag{23}$$

$$D_{11} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \; D_{12} = [0 \quad I]^T, \qquad (24)$$
$$D_{21} = [I \quad 0], \; D_{22} = 0,$$

where the matrixes A, B, C are taken from (6).

However, solution by state-space methods using 2-Riccati approach requires the fulfilment of some conditions on the matrices in (22). In particular,

$$D_{11} = 0, \; D_{22} = 0, \; D_{21} = [0 \quad I] \neq 0,$$
$$D_{12}^T = [0 \quad I] \neq 0, \; D_{12}^T C_1 = 0, \; \widetilde{B}_1 D_{12}^T \neq 0,$$
$$D_{12}^T = [0 \quad I] \neq 0, \; D_{12}^T C_1 = 0, \; \widetilde{B}_1 D_{12}^T = 0$$

not all of these conditions are satisfied in our case.

Therefore, for the solution of the given problem we use the LMI approach [2], which is free from the limitations indicated above. Namely, we take the controller in the following form:

$$\dot{x}_c = A_c x_c + B_c y, \; u = C_c x_c + D_c y, \quad x_c \in R^{n_c}.$$

From here, we obtain the required transfer matrix of controller (8):

$$K(s) = C_c (sI - A_c)^{-1} B_c + D_c.$$

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